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On Spines of Knot Spaces

by

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Summary. We give examples of knot spaces which have a common spine but are not homeomorphic.

We shall work in the PL category. Let $L_1 \subset S^3$ be the square knot and let $L_2 \subset S^3$ be the granny knot. Take a tubular neighbourhood $T_j \subset S^3$ of $L_j (j = 1, 2)$ and let $N_j = S^3 - \operatorname{int} T_j$. Then the two compact 3-manifolds with boundary N_1 and N_2 are not homeomorphic because for example, the knots L_1 and L_2 have different signatures [3; Example (8.E.15)]. Nevertheless, it was conjectured in [2; Conj. 2] that the two knot spaces possess the same spine. We confirm this conjecture by proving the following more general theorem (using $A \# B$ to denote the oriented connected sum of knots A and B [3; pp. 281–282]):

THEOREM. *Let $K_{p,q} \subset S^3$ be the (p, q)-torus knot $(p \geq 2, q \geq 2, (p, q) = 1)$ and let $L \subset S^3$ be an arbitrary knot. Let $M_1 \subset S^3$ be the knot space of the knot $K_{p,q} \# L$ and $M_2 \subset S^3$ — the knot space of the knot $K_{p,-q} \# L$. Then there exist a compact 2-dimensional polyhedron P and PL embeddings $\varphi_j: P \to \operatorname{int} M_j$, $j = 1$, 2, such that $\varphi_j(P)$ is a spine of M_j, i.e., $M_j - \varphi_j(P) \underset{PL}{\cong} \partial M_j \times [0, 1)$.*

COROLLARY. *Let $K_{p,q} \subset S^3$ be the (p, q)-torus knot $(p \geq 2, q \geq 2, (p, q) = 1)$ and let $M_1 \subset S^3$ (resp., $M_2 \subset S^3$) be the knot space of $K_{p,q} \# K_{p,q}$ (resp., $K_{p,q} \# K_{p,-q}$). Then the knot spaces M_1 and M_2 are not homeomorphic but they do possess homeomorphic spines.*

Proof. Immediately by our theorem there follows that M_1 and M_2 have homeomorphic spines. To verify the other assertion, observe that $K_{p,q}$ is not ambiently isotopic to $K_{p,-q}$. Since the complement of a composite knot determines the knot's type [1] by the unique decomposition theorem for knots [4] it follows that $M_1 \neq M_2$.

REMARK. Our theorem provides examples of knots whose complements are not homeomorphic but have homeomorphic spines. This phenomenon is only

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possible for *composite* knots because of W. Whitten's Rigidity Theorem [5]: prime knots $K \subset S^3$ with isomorphic groups have homeomorphic complements.

Proof of the Theorem. A spine $\Sigma_{p,q} \subset S^3$ of the torus knot $K_{p,q} \subset S^3$ can be described as follows: $\Sigma_{p,q} = (S^1 \times [0, 1])/\sim$, where the equivalence relation $\sim$ is defined as follows: for every $z \in S^1 = \{e^{2\pi i t} | 0 \leqslant t < 1\}$,

$$(z, 0) \sim (z \cdot e^{2\pi i/p}, 0) \quad \text{and} \quad (z, 1) \sim (z \cdot e^{2\pi i/q}, 1)$$

i.e., $\Sigma_{p,q}$ is the space obtained from the annulus $S^1 \times [0, 1]$ by the Z_p-action on the $S^1 \times \{0\}$ boundary component and the Z_q-action on the $S^1 \times \{1\}$ boundary component. Note that $\Sigma_{p,q} = \Sigma_{p,-q}$.

Let $J_{p,q} \subset \partial M_1$ (resp., $J_{p,-q} \subset \partial M_2$) be a meridian curve on the torus ∂M_1 (resp., ∂M_2). Let $\gamma_{p,q} \subset \Sigma_{p,q}$ (resp., $\gamma_{p,-q} \subset \Sigma_{p,-q}$) be the image of $J_{p,q}$ (resp., $J_{p,-q}$) under the (obvious) retraction of M_1 (resp., M_2) onto $\Sigma_{p,q}$ (resp., $\Sigma_{p,-q}$). Then:

$$\gamma_{p,q} = (\{1\}) \times [0, 1]) \cup \{[z, t] \in \Sigma_{p,q} | z = e^{2\pi i[(1-t) \cdot \frac{r}{p} + t \cdot \frac{s}{q}]}\}$$

where $[z, t] \in \Sigma_{p,q}$ is the equivalence class of the point $(z, t) \in S^1 \times [0, 1]$ under the identification $S^1 \times [0, 1] \to \Sigma_{p,q} = (S^1 \times [0, 1])/\sim$, and where $p > r > 0$, $s > q > 0$ are such that $rq \equiv 1 \pmod{p}$ and $sp \equiv 1 \pmod{q}$. Furthermore,

$$\gamma_{p,-q} = \{[z, t] \in \Sigma_{p,-q} | [\bar{z}, t] \in \gamma_{p,q}\}$$

where $\bar{z}$ denotes the conjugate of $z \in S^1 \subset C$. The map $f : \Sigma_{p,q} \to \Sigma_{p,-q}$, given by $f([z, t]) = [\bar{z}, t]$, for every $[z, t] \in \Sigma_{p,q}$, is clearly a level preserving homeomorphism and f takes $\gamma_{p,q}$ onto $\gamma_{p,-q}$. Consequently, the pairs $(\Sigma_{p,q}, \gamma_{p,q})$ and $(\Sigma_{p,-q}, \gamma_{p,-q})$ are equivalent.

Observe now that a spine of M_1 can be obtained by glueing a spine Σ_L of the knot spaces of L to $\Sigma_{p,q}$ along $\gamma_{p,q}$. Clearly, this procedure will yield the same compact polyhedron P if $(\Sigma_{p,q}, \gamma_{p,q})$ is replaced by $(\Sigma_{p,-q}, \gamma_{p,-q})$, i.e.,

$$\Sigma_L \bigcup_{\gamma_{p,q}} \Sigma_{p,q} = P = \Sigma_L \bigcup_{\gamma_{-p,q}} \Sigma_{p,-q}.$$

This completes the proof of our theorem.

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В. И. Р. Митчелл, И. Пшитыцки, Д. Реповш, **Спины узловых пространств**

Даются примеры узловых пространств, обладающих общим спином, но не являющихся гомеоморфными.