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Nonlinear algebraic systems with discontinuous terms

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ABSTRACT

Using a multiple critical points theorem for locally Lipschitz continuous functionals, we establish the existence of at least three distinct solutions for a parametric discrete differential inclusion problem involving a real symmetric and positive definite matrix. Applications to tridiagonal, fourth-order and partial difference inclusions are presented.

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1. Introduction

Let $T > 1$ be a positive integer and let $g_k : \mathbb{R} \to \mathbb{R}$ be locally essentially bounded functions, for every $k \in \mathbb{Z}[1,T] := \{1, 2, \dots, T\}$. In this paper we are interested in the existence of multiple solutions for the following discrete problem

$$\sum_{l=1}^{T} a_{kl} u_l \in \lambda [g_k^-(u_k), g_k^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T], \tag{$S^g_{A,\lambda}$}$$

where λ is a positive parameter, $A := (a_{ij})_{T\times T}$ is a real symmetric positive definite matrix and

$$g_k^-(t) := \lim_{\delta\to 0^+} \operatorname*{essinf}_{|\xi - t|<\delta} g_k(\xi), \qquad g_k^+(t) := \lim_{\delta\to 0^+} \operatorname*{esssup}_{|\xi - t|<\delta} g_k(\xi),$$

for every $k \in \mathbb{Z}[1,T]$. It is clear that if the functions g_k are continuous (instead of locally essentially bounded) problem ($S^g_{A,\lambda}$) becomes a more familiar nonlinear algebraic system

$$Au = \lambda g(u),$$

in which $u = (u_1, \dots, u_T)^t \in \mathbb{R}^T$ and $g(u) := (g_1(u_1), \dots, g_T(u_T))^t$.

A considerable number of problems, which are strictly connected both with boundary value differential problems and numerical simulations of some mathematical models arising from many research areas (e.g. biological, physical and computer science) can be formulated as special cases of nonlinear algebraic systems (see, for instance [1]).

However, to the best of our knowledge, for discrete difference inclusions there are only few papers involving the second-order difference operator. For instance, in [2], the existence of at least one solution was obtained via the set-valued mapping

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theory, while in [3], existence results for suitable second-order discrete discontinuous equations have been investigated by variational methods.

The aim of this paper is to establish a three solutions result (Theorem 3.1) for problem $(S^g_{\lambda,\lambda})$. Successively, in Corollary 3.1, the existence of at least two nontrivial solutions is obtained for sufficiently large values of the real parameter λ.

Our proof is based on a three critical points theorem for nondifferentiable functionals obtained by Bonanno and Marano (see Theorem 2.1 in Section 2). We also note that, very recently, Theorem 2.1 has been used in [4], in order to obtain an analogous version of Theorem 3.1 for Dirichlet inclusions involving the discrete p-Laplacian operator.

Due to the generality of problem $(S^g_{\lambda,\lambda})$ remarkable applications are achieved. For instance, in Section 4 we present some existence theorems concerning discrete inclusions involving certain tridiagonal matrices, fourth-order discrete inclusions and partial difference inclusions.

A special case of Corollary 3.1 reads as follows.

Theorem 1.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded positive function and consider the usual forward difference operator $\Delta u_{k-1} := u_k - u_{k-1}$. Assume that*

(h_2) $\limsup_{t\to 0^+} \frac{h(t)}{t} = 0$;

(h_3) $\limsup_{|t|\to\infty} \frac{h(t)}{t} < \frac{\lambda_1}{T}$, *where λ_1 is the first eigenvalue of the discrete problem*

$$\begin{cases} -\Delta^4 u_{k-2} = \lambda u_k, \quad \forall\, k \in \mathbb{Z}[1,T] \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0. \end{cases}$$

Then for every

$$\lambda > \frac{2}{T} \inf_{\delta>0} \frac{\delta^2}{\int_0^\delta h(t)dt},$$

the following fourth-order discrete inclusion

$$\begin{cases} -\Delta^4 u_{k-2} \in \lambda[h^-(u_k), h^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T] \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0. \end{cases} \tag{D^h_λ}$$

admits at least two nontrivial solutions.

Finally, we emphasize that our results are also new in the continuous setting. In this case, the existence and multiplicity of solutions was investigated in a large number of other papers under various assumptions (see for instance [5,1,6,7,3] and references therein). See also recent papers [8–11] for related topics.

The plan of the paper is as follows. In the next section we introduce our abstract framework. Successively, in Section 3, we show our multiplicity results.

In conclusion, concrete examples of applications of our abstract results are presented.

2. Abstract framework

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $J : X \to \mathbb{R}$ is called locally Lipschitz continuous if to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|J(z) - J(w)| \leq L_x \|z - w\|, \quad \forall\, z, w \in V_x.$$

If $x, z \in X$, we write $J^0(x; z)$ for the generalized directional derivative of J at the point x along the direction z, i.e.,

$$J^0(x; z) := \limsup_{w\to x,\, t\to 0^+} \frac{J(w + tz) - J(w)}{t}.$$

The generalized gradient of the function J in x, denoted by $\partial J(x)$, is the set

$$\partial J(x) := \left\{ x^* \in X^* : \langle x^*, z\rangle \leq J^0(x; z),\ \forall\, z \in X \right\}.$$

The basic properties of the generalized directional derivative and the generalized gradient were studied in [12,13]. We recall that if J is continuously Gâteaux differentiable at u, then J is locally Lipschitz at u and $\partial J(u) = \{J'(u)\}$, where $J'(u)$ stands for the first derivative of J at u.

Further, a point u is called a (generalized) critical point of the locally Lipschitz continuous function J if $0_{X^*} \in \partial J(u)$, i.e.

$$J^0(u; z) \geq 0,$$

for every $z \in X$. Clearly, if J is a continuously Gâteaux differentiable at u, then u becomes a (classical) critical point of J, that is $J'(u) = 0_{X^*}$.

For an exhaustive overview on the non-smooth calculus we mention the excellent monograph [14]. Further, we cite a very recent book [15] as a general reference on this subject.

The main tool will be the following abstract critical point theorem for locally Lipschitz continuous functions on finite dimensional Banach spaces that can be derived from [16, Theorem 3.6].

Theorem 2.1. *Let X be a finite dimensional real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals such that $\Phi(0_X) = \Psi(0_X) = 0$ and assume that the following conditions are satisfied:*

(a_1) *there exist $r > 0$ and $\bar{u} \in X$, with $\Phi(\bar{u}) > r$, such that:*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})};$$

(a_2) *for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}\right[$ the functional $J_\lambda := \Phi - \lambda\Psi$ is coercive, that is*

$$\lim_{\|u\| \to \infty} (\Phi(u) - \lambda\Psi(u)) = +\infty.$$

Then for each $\lambda \in \Lambda_r$, the functional J_λ has at least three distinct critical points in X.

We also cite the quoted papers of Ricceri [17–19] for some related topics in the smooth context.

Here, as the ambient space X, we consider the T-dimensional Banach space $\mathbb{R}^T$ endowed by the norm

$$\|u\| := \left(\sum_{k=1}^{T} u_k^2\right)^{1/2},$$

induced by the standard Euclidean inner product $\langle u, v\rangle_X := \sum_{k=1}^{T} u_k v_k$.

Set $\mathfrak{X}_T$ to be the class of all symmetric and positive definite matrices of order T. Further, we denote by $\lambda_1, \ldots, \lambda_T$ the eigenvalues of A ordered as follows $0 < \lambda_1 \le \cdots \le \lambda_T$.

It is well-known that if $A \in \mathfrak{X}_T$, for every $u \in X$, then one has

$$\lambda_1 \|u\|^2 \le u^t A u \le \lambda_T \|u\|^2, \tag{1}$$

and

$$\|u\|_\infty \le \frac{1}{\sqrt{\lambda_1}} (u^t A u)^{1/2}, \tag{2}$$

where $\|u\|_\infty := \max_{k \in \mathbb{Z}[1,T]} |u_k|$.

For every $u \in X$, put

$$\Phi(u) := \frac{u^t A u}{2}, \qquad \Psi(u) := \sum_{k=1}^{T} G_k(u_k), \quad \text{and} \quad J_\lambda(u) := \Phi(u) - \lambda\Psi(u), \tag{3}$$

where $G_k(t) := \int_0^t g_k(\xi)\,d\xi$, for every $(k, t) \in \mathbb{Z}[1, T] \times \mathbb{R}$.

It is easy to verify that Φ is continuously Gâteaux differentiable, while Ψ is locally Lipschitz continuous.

Proposition 2.1. *Fix $\lambda > 0$ and assume that $u \in X$ is a critical point of the functional $J_\lambda := \Phi - \lambda\Psi$. Then u is a solution of problem ($S^g_{A,\lambda}$).*

Proof. If u is a critical point of J_λ, bearing in mind of [13, Propositions 2.3.1 and 2.3.3], it follows that

$$\Phi'(u)(z) \le \Psi^0(u; z) \le \lambda \left(\sum_{k=1}^{T} G_k^0(u_k; z_k)\right), \tag{4}$$

for every $z \in X$. Moreover,

$$\Phi'(u)(z) = \frac{\langle \nabla(u^t A u), z\rangle_X}{2}, \tag{5}$$

for every $z \in X$. For every $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1,T]$, by putting in (4) the vector $z = \xi e_k$, where e_k are the canonical unit vectors of X, and taking in mind (5), we get

$$\left\langle \sum_{l=1}^{T} a_{kl}u_l, \xi \right\rangle_{\mathbb{R}} = \Phi'(u)(z) \leq \lambda G_k^0(u_k;\xi),$$

namely

$$\sum_{l=1}^{T} a_{kl}u_l \in \lambda \partial G_k(u_k).$$

Finally, since it is well-known that

$$\partial G_k(u_k) = [g_k^-(u_k), g_k^+(u_k)],$$

for every $k \in \mathbb{Z}[1,T]$ (see for instance [13, Example 2.2.5]) it follows that

$$\sum_{l=1}^{T} a_{kl}u_l \in \lambda[g_k^-(u_k), g_k^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T].$$

Therefore our assertion is proved. □

3. Main results

In this section we present our multiplicity results for problem $(S^g_{A,\lambda})$ which can be deduced by using Theorem 2.1.

Theorem 3.1. *Let $g_k : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function for every $k \in \mathbb{Z}[1,T]$. Assume that*

(g_1) *there exist positive constants γ and δ, with*

$$\delta > \left(\frac{\lambda_1}{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}\right)^{1/2} \gamma,$$

such that

$$\frac{\sum_{k=1}^{T} \sup_{|\xi|\leq\gamma} G_k(\xi)}{\gamma^2} < \frac{\lambda_1}{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}} \frac{\sum_{k=1}^{T} G_k(\delta)}{\delta^2};$$

(g_2) $\limsup_{|\xi|\to\infty} \frac{G_k(\xi)}{\xi^2} < \frac{\lambda_1}{2}$, $\forall\, k \in \mathbb{Z}[1,T]$.

Then, for every λ belonging to

$$\Lambda := \left]\frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}{2} \frac{\delta^2}{\sum_{k=1}^{T} G_k(\delta)}, \frac{\lambda_1}{2} \frac{\gamma^2}{\sum_{k=1}^{T} \sup_{|\xi|\leq\gamma} G_k(\xi)}\right[,$$

problem $(S^g_{A,\lambda})$ admits at least three solutions.

Proof. Fix $\lambda \in \Lambda$ and let Φ, Ψ, J_λ as indicated in (3). Since the critical points of J_λ are the solutions of problem $(S^g_{A,\lambda})$, our aim is to apply Theorem 2.1 to function J_λ. Therefore, put

$$r := \frac{\lambda_1}{2}\gamma^2,$$

and let us denote

$$A_k := \{u \in X : |u_k| \leq \gamma\},$$

for every $k \in \mathbb{Z}[1,T]$.
Thanks to (2), it follows that

$$\{u \in X : u^t A u \leq 2r\} \subseteq \{u \in X : \|u\|_\infty \leq \gamma\} \subseteq A_k,$$

for every $k \in \mathbb{Z}[1, T]$. Hence

$$\chi(r) := \frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} \leq \frac{\sup_{\|u\|_\infty \leq \gamma} \sum_{k=1}^{T} G_k(u_k)}{r}$$
$$\leq \frac{\sum_{k=1}^{T} \sup_{u \in A_k} G_k(u_k)}{r} = \frac{2}{\lambda_1} \frac{\sum_{k=1}^{T} \sup_{|\xi| \leq \gamma} G_k(\xi)}{\gamma^2}. \tag{6}$$

Now consider $\bar{u} \in X$ such that $\bar{u}_k := \delta$, for every $k \in \mathbb{Z}[1, T]$, and observe that, since

$$\delta > \left(\frac{\lambda_1}{\mathrm{Tr}(A) + 2 \sum_{i<j} a_{ij}} \right)^{1/2} \gamma,$$

one has $\Phi(\bar{u}) > r$. Further,

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} = \frac{2}{\mathrm{Tr}(A) + 2 \sum_{i<j} a_{ij}} \frac{\sum_{k=1}^{T} G_k(\delta)}{\delta^2}. \tag{7}$$

At this point, taking into account relations (6) and (7), condition (g_2) implies that

$$\chi(r) \leq \frac{2}{\mathrm{Tr}(A) + 2 \sum_{i<j} a_{ij}} \frac{\sum_{k=1}^{T} G_k(\delta)}{\delta^2}.$$

Therefore, hypothesis (a_1) holds and $\Lambda \subseteq \Lambda_r$.
By (g_2) there are constants $\epsilon \in]0, \lambda_1/2[$ and $\sigma > 0$ such that

$$\frac{\int_0^\xi g_k(t)dt}{\xi^2} < \frac{\lambda_1}{2} - \epsilon, \tag{8}$$

for every $|\xi| \geq \sigma$ and $k \in \mathbb{Z}[1, T]$. Let us put

$$M_1 := \max_{(k,\xi) \in \mathbb{Z}[1,T] \times [-\sigma, \sigma]} \int_0^\xi g_k(t)dt. \tag{9}$$

At this point note that, for every $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1, T]$, one has

$$\int_0^\xi g_k(t)dt \leq M_1 + M_2 \xi^2,$$

where

$$M_2 := \frac{\lambda_1}{2} - \epsilon.$$

Moreover, the following inequality holds

$$J_\lambda(u) \geq \frac{u^t A u}{2} - \sum_{k=1}^{T} \left[M_1 + M_2 u_k^2 \right], \quad \forall\, u \in X.$$

Hence,

$$J_\lambda(u) \geq \frac{u^t A u}{2} - M_2 \|u\|^2 - T M_1, \quad \forall\, u \in X,$$

and, by relation (1), one has

$$J_\lambda(u) \geq \epsilon \|u\|^2 - T M_1, \quad \forall\, u \in X, \tag{10}$$

which clearly shows that

$$\lim_{\|u\|\to\infty} J_\lambda(u) = +\infty. \tag{11}$$

So, the assumptions of Theorem 2.1 are satisfied and our conclusions follow from Proposition 2.1. □

Remark 3.1. We point out that very recently, Theorem 3.1 has been exploited in [4], in order to obtain an analogous version of Theorem 3.1 for Dirichlet inclusions involving the discrete p-Laplacian operator. A related variational approach has been also adopted, studying difference equations (in the continuous case) in papers [20,21] as well as in [22,23] proving multiplicity results for nonlinear algebraic systems.

A direct consequence of Theorem 3.1 reads as follows.

Corollary 3.1. *Let $\alpha : \mathbb{Z}[1,T] \to \mathbb{R}$ be a nonnegative (not identically zero) function and let $h : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded map. Assume that*

(h_1) *there exists $\delta > 0$ such that $h(t) > 0$ for every $0 < |t| < \delta$;*
(h_2) $\limsup_{t\to 0^+} \frac{h(t)}{t} = 0$;
(h_3) $\limsup_{|t|\to\infty} \frac{h(t)}{t} < \frac{\lambda_1}{\sum_{k=1}^{T}\alpha_k}$.

Then for every

$$\lambda > \frac{\mathrm{Tr}(A)/2 + \sum_{i<j} a_{ij}}{\sum_{k=1}^{T}\alpha_k} \frac{\delta^2}{\int_0^\delta h(t)dt},$$

the following discrete problem

$$\sum_{l=1}^{T} a_{kl}u_l \in \lambda\alpha_k[h^-(u_k), h^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T], \tag{$S^{\alpha,h}_{A,\lambda}$}$$

admits at least two nontrivial solutions.

Proof. Our aim is to apply Theorem 3.1 to problem ($S^{\alpha,h}_{A,\lambda}$). Hence, let us put $H(\xi) := \int_0^\xi h(t)dt$ for every $\xi \in \mathbb{R}$. By (h_1) one has $H(\delta) > 0$ and

$$\sup_{|\xi|\le\gamma} H(\xi) = H(\gamma),$$

for every $\gamma \in (0,\delta]$. Further, fix

$$\lambda > \frac{\mathrm{Tr}(A)/2 + \sum_{i<j} a_{ij}}{\sum_{k=1}^{T}\alpha_k} \frac{\delta^2}{H(\delta)}. \tag{12}$$

By (h_2) there exists $\bar\gamma \in (0,\delta)$ such that

$$h(t) < \frac{\lambda_1 t}{\lambda\left(\sum_{k=1}^{T}\alpha_k\right)},$$

for every $t \in]0,\bar\gamma[$. Hence we obtain

$$\sup_{t\in(0,\bar\gamma)} \frac{H(t)}{t^2} \le \frac{\lambda_1}{2\lambda\left(\sum_{k=1}^{T}\alpha_k\right)}.$$

On the other hand, bearing in mind (12), one has

$$\left(\frac{\lambda_1}{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}\right) \frac{\sum_{k=1}^{T}\alpha_k H(\delta)}{\delta^2} > \frac{\lambda_1}{2\lambda}.$$

Taking

$$0 < \gamma < \min\left\{\bar{\gamma}, \left(\frac{\lambda_1}{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}\right)^{-1/2} \delta\right\},$$

and considering the above relations, one has

$$\frac{\left(\sum_{k=1}^{T} \alpha_k\right) \sup_{|\xi|\le\gamma} H(\xi)}{\gamma^2} \le \frac{\lambda_1}{2\lambda} < \left(\frac{\lambda_1}{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}\right) \frac{\sum_{k=1}^{T} \alpha_k H(\delta)}{\delta^2}. \tag{13}$$

Thus, condition (g_1) in Theorem $(S^{\alpha,h}_{A,\lambda})$ is verified.

Now, we proceed by proving that condition (h_3) implies (g_2). Indeed, by (h_3), there are constants $\epsilon' \in \left]0, \lambda_1 / \left(\sum_{k=1}^{T} \alpha_k\right)\right[$, and $\sigma > 0$ such that

$$\frac{h(t)}{t} < \frac{\lambda_1}{\sum_{k=1}^{T} \alpha_k} - \epsilon',$$

for every $|t| \ge \sigma$. Since h is a locally essentially bounded function, we also have

$$M := \operatorname*{esssup}_{t\in[-\sigma,\sigma]} |h(t)| < +\infty.$$

Therefore, if $\xi \ge \sigma$, one has

$$\int_0^{\xi} h(t)dt = \int_0^{\sigma} h(t)dt + \int_{\sigma}^{\xi} h(t)dt \le M\sigma + \frac{1}{2}\left(\lambda_1 \Big/ \left(\sum_{k=1}^{T} \alpha_k\right) - \epsilon'\right)\xi^2,$$

while, for $\xi \le -\sigma$, it follows that

$$\int_0^{\xi} h(t)dt = -\left[\int_{\xi}^{-\sigma} h(t)dt + \int_{-\sigma}^{0} h(t)dt\right] \le M\sigma + \frac{1}{2}\left(\lambda_1 \Big/ \left(\sum_{k=1}^{T} \alpha_k\right) - \epsilon'\right)\xi^2.$$

Consequently,

$$\int_0^{\xi} h(t)dt \le M\sigma + \frac{1}{2}\left(\lambda_1 \Big/ \left(\sum_{k=1}^{T} \alpha_k\right) - \epsilon'\right)\xi^2, \quad \forall\, \xi \in \mathbb{R}. \tag{14}$$

Hence, by using (14), one has

$$\limsup_{|\xi|\to\infty} \frac{\int_0^{\xi} \alpha_k h(t)dt}{\xi^2} = \alpha_k \limsup_{|\xi|\to\infty} \frac{H(\xi)}{\xi^2} \le \frac{1}{2}\left(\sum_{k=1}^{T} \alpha_k\right)\left(\lambda_1 \Big/ \left(\sum_{k=1}^{T} \alpha_k\right) - \epsilon'\right) < \frac{\lambda_1}{2}.$$

So, it is clear that condition (g_2) holds, too. Finally, we achieve the conclusion by applying Theorem 3.1, taking into account that condition (13) ensures that

$$\lambda \in \left]\frac{\mathrm{Tr}(A)/2 + \sum_{i<j} a_{ij}}{\sum_{k=1}^{T} \alpha_k} \frac{\delta^2}{H(\delta)}, \frac{\lambda_1}{2} \frac{\gamma^2}{\left(\sum_{k=1}^{T} \alpha_k\right) \sup_{|\xi|\le\gamma} H(\xi)}\right[.$$

Thus, our claim holds and the proof is completed. □

Remark 3.2. As pointed out in the Introduction, in recent years, some related results on the existence of multiple solutions for algebraic systems of the form

$$Au = \lambda g(u), \tag{15}$$

were obtained by several authors (see the papers [24–26]).

We also emphasize for completeness that Theorem 3.1 and Corollary 3.1 are mutually independent with respect to results and methods contained in the cited works also in the continuous setting, as can be easily seen by direct computations.

Remark 3.3. In the smooth case, Corollary 3.1 (and even the main abstract result) can be obtained directly from the celebrated Pucci–Serrin result (see [15, Theorem 1.13]); namely, for enough large parameters, a solution appears as a global minimum point for the energy functional (with negative energy level) while a second solution as a mountain pass one (with positive energy level). See, for instance, the recent paper [27, Theorem 3.1] where, under suitable assumptions at zero and at infinity on the nonlinearity g, has been proved that, for every

$$\lambda > \left(\frac{\mathrm{Tr}(A) + 2\sum_{i<j} a_{ij}}{2}\right)\left(\max_{\xi\neq 0}\frac{\sum_{k=1}^{T} G_k(\xi)}{\xi^2}\right)^{-1},$$

problem (15) has at least two distinct and nontrivial solutions.

4. Applications

In this section we present some important cases of Theorem 3.1. More precisely, we consider our previous result for some classes of discrete problems that appear in several technical applications.

4.1. Tridiagonal inclusions

Let $T \geq 1$ and $(a, b) \in \mathbb{R}^- \times \mathbb{R}^+$ be such that

$$\cos\left(\frac{\pi}{T+1}\right) < -\frac{b}{2a}. \tag{16}$$

Set

$$\mathrm{Trid}_T(a,b,a) = \begin{pmatrix} b & a & 0 & \cdots & 0 \\ a & b & a & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & a & b & a \\ 0 & \cdots & 0 & a & b \end{pmatrix}_{T\times T},$$

and consider the following discrete problem

$$L_{\mathrm{Trid}}(u) \in \lambda[j_k^-(u_k), j_k^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T], \tag{S_λ^j}$$

where

$$L_{\mathrm{Trid}}(u) := \begin{cases} bu_1 + au_2 \\ au_{k-1} + bu_k + au_{k+1}, & \forall\, k \in \{2,\dots,T-1\} \\ au_{T-1} + bu_T, \end{cases}$$

the functions $j_k : \mathbb{R} \to \mathbb{R}$ are assumed to be locally essentially bounded, and λ is a positive parameter. At this point, observing that $\mathrm{Trid}_T(a,b,a)$ is a symmetric and positive definite matrix whose first eigenvalue is given by

$$\lambda_1 = b + 2a\cos\left(\frac{\pi}{T+1}\right),$$

see [28, Example 9; p.179], we obtain the following multiplicity result.

Theorem 4.1. *Assume that*

(g_1) *there exist positive constants γ and δ, with*

$$\delta > \left(\frac{b + 2a\cos(\pi/(T+1))}{bT + 2a(T-1)}\right)^{1/2}\gamma,$$

such that

$$\frac{\sum_{k=1}^{T}\sup_{|\xi|\leq\gamma}\int_0^\xi j_k(t)dt}{\gamma^2} < \frac{b + 2a\cos(\pi/(T+1))}{bT + 2a(T-1)}\,\frac{\sum_{k=1}^{T}\int_0^\delta j_k(t)dt}{\delta^2};$$

(g_2) $\limsup_{|\xi|\to\infty}\frac{\int_0^\xi j_k(t)dt}{\xi^2} < \frac{b+2a\cos(\pi/(T+1))}{2}$, $\forall\, k \in \mathbb{Z}[1,T]$.

Then for every λ belonging to

$$\Lambda := \left]\frac{bT+2a(T-1)}{2}\frac{\delta^2}{\sum_{k=1}^{T}\int_0^{\delta} j_k(t)dt}, \frac{b+2a\cos(\pi/(T+1))}{2}\frac{\gamma^2}{\sum_{k=1}^{T}\sup_{|\xi|\le\gamma}\int_0^{\xi} j_k(t)dt}\right[,$$

problem (S_λ^j) admits at least three solutions.

The above result can be applied to second-order difference inclusions. Indeed, it is well-known that the matrix

$$\mathrm{Trid}_T(-1,2,-1) := \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{T\times T} \in \mathfrak{X}_T,$$

is associated to the second-order discrete boundary value problem

$$\begin{cases} -\Delta^2 u_{k-1} \in \lambda[j_k^-(u_k), j_k^+(u_k)], \quad \forall k \in \mathbb{Z}[1,T] \\ u_0 = u_{T+1} = 0, \end{cases} \tag{$\widetilde{S}_\lambda^j$}$$

where $\Delta^2 u_{k-1} := \Delta(\Delta u_{k-1})$, and, as usual, $\Delta u_{k-1} := u_k - u_{k-1}$ denotes the forward difference operator.

4.2. Fourth-order difference inclusions

Boundary value problems involving fourth-order difference inclusions such as

$$\begin{cases} -\Delta^4 u_{k-2} \in \lambda[g_k^-(u_k), g_k^+(u_k)], \quad \forall\, k \in \mathbb{Z}[1,T] \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0, \end{cases} \tag{D_λ^g}$$

can also be expressed as problem $(S_{A,\lambda}^g)$, where A is the real symmetric and positive definite matrix of the form

$$A := \begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix} \in \mathfrak{X}_T.$$

In this case, let λ_1 be the first eigenvalue of A. Assuming that the following condition holds:

(g_1') *there exist two positive constants γ and δ, with*

$$\delta > \frac{\sqrt{\lambda_1}}{2}\gamma,$$

such that

$$\frac{\sum_{k=1}^{T}\sup_{|\xi|\le\gamma} G_k(\xi)}{\gamma^2} < \frac{\lambda_1}{4}\frac{\sum_{k=1}^{T} G_k(\delta)}{\delta^2},$$

in addition to (g_2), one has that, for every λ belonging to

$$\Lambda := \left]\frac{2\delta^2}{\sum_{k=1}^{T} G_k(\delta)}, \frac{\lambda_1}{2}\frac{\gamma^2}{\sum_{k=1}^{T}\sup_{|\xi|\le\gamma} G_k(\xi)}\right[,$$

problem (D_λ^g) admits at least three solutions.

4.3. Partial difference inclusions

Nonlinear systems of the form $(S^g_{A,\lambda})$ arise in many applications as the boundary value problems for common boundary value problems involving partial difference equations. For instance, here we consider the following problem, namely (E^f_λ), given as follows

$$4u(i,j) - u(i+1,j) - u(i-1,j) - u(i,j+1) - u(i,j-1) \in \lambda[f^-_{(i,j)}(u(i,j)), f^+_{(i,j)}(u(i,j))]$$

for every $(i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]$, with boundary conditions

$$u(i,0) = u(i,n+1) = 0, \quad \forall\, i \in \mathbb{Z}[1,m],$$
$$u(0,j) = u(m+1,j) = 0, \quad \forall\, j \in \mathbb{Z}[1,n],$$

where every $f_{(i,j)} : \mathbb{R} \to \mathbb{R}$ denotes a locally essentially bounded function and λ is a positive real parameter.

Let $z : \mathbb{Z}[1,m] \times \mathbb{Z}[1,n] \to \mathbb{Z}[1,mn]$ be the bijection defined by $z(i,j) := i + m(j-1)$, for every $(i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]$. Let us denote $w_k := u(z^{-1}(k))$ and $g_k(w_k) := f_{z^{-1}(k)}(w_k)$, for every $k \in \mathbb{Z}[1,mn]$. With the above notations, problem (E^f_λ) can be written as a nonlinear algebraic inclusion of the form

$$\sum_{l=1}^{T} b_{kl} w_l \in \lambda[g^-_k(w_k), g^+_k(w_k)], \quad \forall\, k \in \mathbb{Z}[1,mn], \tag{$S^g_{B,\lambda}$}$$

where B is given by

$$B := (b_{ij}) = \begin{pmatrix} L & -I_m & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -I_m & L & -I_m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -I_m & L & -I_m & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & L & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & L & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_m & L & -I_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I_m & L & -I_m \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -I_m & L \end{pmatrix} \in \mathfrak{X}_{mn},$$

in which L is the matrix of order m defined by

$$L := \begin{pmatrix} 4 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 4 \end{pmatrix},$$

and I_m is the identity matrix. Denoting by λ_B the first eigenvalue of B, we obtain the next result.

Theorem 4.2. *Assume that*

(g_1) *there exist two positive constants γ and δ, with*

$$\delta > \left(\frac{\lambda_B}{2(m+n)}\right)^{1/2} \gamma,$$

such that

$$\frac{\sum_{k=1}^{mn} \sup_{|\xi| \le \gamma} G_k(\xi)}{\gamma^2} < \frac{\lambda_B}{2(m+n)} \frac{\sum_{k=1}^{mn} G_k(\delta)}{\delta^2};$$

(g_2) $\limsup_{|\xi| \to \infty} \frac{\int_0^\xi g_k(t)\,dt}{\xi^2} < \frac{\lambda_B}{2}$, $\forall\, k \in \mathbb{Z}[1,mn]$.

Then for every λ belonging to

$$\Lambda := \left[(m+n)\frac{\delta^2}{\sum_{k=1}^{mn} G_k(\delta)}, \frac{\lambda_B}{2}\frac{\gamma^2}{\sum_{k=1}^{mn} \sup_{|\xi|\leq\gamma} G_k(\xi)} \right[,$$

problem $(S^g_{B,\lambda})$ admits at least three solutions.

Remark 4.1. We observe that in [29], Ji and Yang studied the structure of the spectrum of problem (E^f_λ) by investigating the existence of a positive eigenvector corresponding to the smallest positive eigenvalue λ_B. In conclusion, we refer the reader to the paper of Galewski and Orpel [30] for some multiplicity results on discrete partial difference equations as well as to the monograph of Cheng [31] for their discrete geometrical interpretation.

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