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DIRECT LIMIT TOPOLOGIES IN THE CATEGORIES OF TOPOLOGICAL GROUPS AND OF UNIFORM SPACES

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Abstract. Given an increasing sequence (G_n) of topological groups, we study the topologies of the direct limits of the sequence (G_n) in the categories of topological groups and of uniform spaces and find conditions under which these two direct limit topologies coincide.

1. Introduction.

Given a tower

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

of topological groups, we shall study in this paper the topological structure of the direct limit $\text{g-}\varinjlim G_n$ of the tower (G_n) in the category of topological groups. By definition, $\text{g-}\varinjlim G_n$ is the union $G = \bigcup_{n\in\omega} G_n$ endowed with the strongest (not necessarily Hausdorff) topology that turns G into a topological group and makes the identity inclusions $G_n \to G$, $n \in \omega$, continuous. Here $\omega = \{0, 1, 2, \dots\}$ stands for the set of finite ordinals.

Besides the topology of $\text{g-}\varinjlim G_n$, the union $G = \bigcup_{n\in\omega} G_n$ carries the topology of the direct limit $\text{t-}\varinjlim G_n$ of the tower $(G_n)_{n\in\omega}$ in the category of topological spaces. The topology of $\text{t-}\varinjlim G_n$ is the strongest topology on G making the identity inclusions $G_n \to G$, $n \in \omega$, continuous.

The definitions of the direct limits $\text{g-}\varinjlim G_n$ and $\text{t-}\varinjlim G_n$ imply that the identity map

$$\text{t-}\varinjlim G_n \to \text{g-}\varinjlim G_n$$

is continuous. This map is a homeomorphism if and only if $\text{t-}\varinjlim G_n$ is a topological group. It was observed in [2] and [16] that the group operation on $G = \text{t-}\varinjlim G_n$ is not necessarily continuous with respect to the topology $\text{t-}\varinjlim G_n$. Moreover, if each group G_n, $n \in \omega$, is metrizable and closed in G_{n+1}, then the topological direct limit $\text{t-}\varinjlim G_n$ is a topological group if and only if either all groups G_n are locally compact or some group G_n is open in all groups G_m, $m \geq n$ (see [6] or [18]).

Thus in many interesting cases (in particular, those considered in [7], [10], [11], [12], [13]), the topology of $\text{g-}\varinjlim G_n$ differs from the topology of the topological direct limit $\text{t-}\varinjlim G_n$. However, in contrast with the topology of $\text{t-}\varinjlim G_n$ which has an explicit description (as the family of all subsets $U \subset \bigcup_{n\in\omega} G_n$ that have open traces $U \cap G_n$ on all spaces G_n) the topological structure of the direct limit $\text{g-}\varinjlim G_n$ is not so clear. The problem of explicit

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description of the topological structure of the direct limit g-$\varinjlim G_n$ was discussed in [8], [11], [12], [13], [14], [16].

In this paper we shall show that, under certain conditions on a tower of topological groups $(G_n)_{n\in\omega}$, the topology of the direct limit g-$\varinjlim G_n$ coincides with one (or all) of four simply described topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overset{\rightleftarrows}{\tau}$ or $\overleftrightarrow{\tau}$ on the group $G = \bigcup_{n\in\omega} G_n$. These topologies will be considered in Section 2. In Sections 3 and 4 we shall study two properties (PTA and the balanced property) of a tower of topological groups $(G_n)_{n\in\omega}$ which imply that the topology of g-$\varinjlim G_n$ coincides with the topology $\overset{\rightleftarrows}{\tau}$, which is the strongest among the four topologies on G. In Section 5 we shall define another (bi-balanced) property of the tower (G_n) which guarantees that the topology of g-$\varinjlim G_n$ coincides with the topology $\overleftrightarrow{\tau}$, which is the weakest among the four topologies on G. In Section 9 we shall reveal the uniform nature of the topologies $\overrightarrow{\tau}$ and $\overleftarrow{\tau}$ and show that they coincide with the topologies of the uniform direct limits u-$\varinjlim G_n^{\mathrm{L}}$ and u-$\varinjlim G_n^{R}$ of the groups G_n endowed with the left and right uniformities. In Section 10 we shall sum up the results obtained in this paper and we shall pose some open problems.

2. The semitopological groups $\overrightarrow{G}$, $\overleftarrow{G}$, $\overleftrightarrow{G}$ and $\overset{\rightleftarrows}{G}$. In this section, given a tower of topological groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots,$$

we define four topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$ or $\overset{\rightleftarrows}{\tau}$ on the group $G = \bigcup_{n\in\omega} G_n$.

Given a sequence of subsets $(U_n)_{n\in\omega}$ of the group G, consider their directed products in G:

$$\overrightarrow{\prod_{n\in\omega}} U_n = \bigcup_{m\in\omega} \overrightarrow{\prod_{0\le n\le m}} U_n \quad \text{where} \quad \overrightarrow{\prod_{k\le n\le m}} U_n = U_k U_{k+1}\cdots U_m\,,$$

$$\overleftarrow{\prod_{n\in\omega}} U_n = \bigcup_{m\in\omega} \overleftarrow{\prod_{0\le n\le m}} U_n \quad \text{where} \quad \overleftarrow{\prod_{k\le n\le m}} U_n = U_m \cdots U_{k+1} U_k\,,$$

$$\overleftrightarrow{\prod_{n\in\omega}} U_n = \bigcup_{m\in\omega} \overleftrightarrow{\prod_{0\le n\le m}} U_n \quad \text{where} \quad \overleftrightarrow{\prod_{k\le n\le m}} U_n = U_m \cdots U_k U_k \cdots U_m\,.$$

Observe that

$$\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)^{-1} = \overleftarrow{\prod_{n\in\omega}} U_n^{-1} \text{ and } \overleftrightarrow{\prod_{n\in\omega}} U_n = \Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big)\cdot\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)$$

provided that each set U_n contains the neutral element e of the group G.

In each topological group G_n fix a base $\mathcal{B}_n$ of open symmetric neighborhoods $U = U^{-1} \subset G_n$ of the neutral element e.

The topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$ and $\overset{\rightleftarrows}{\tau}$ on the group $G = \bigcup_{n\in\omega} G_n$ are generated by the bases:

$$\overrightarrow{\mathcal{B}} = \Big\{\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)\cdot x\,;\ x \in G,\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\Big\},$$

$$\overleftarrow{\mathcal{B}} = \Big\{x\cdot\Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big)\,;\ x \in G,\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\Big\},$$

$$\overset{\leftrightarrow}{\mathcal{B}} = \big\{x \cdot \big(\overleftrightarrow{\prod_{n\in\omega}} U_n\big) \cdot y\,;\ x, y \in G,\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\}\,,$$
$$\overset{\rightleftarrows}{\mathcal{B}} = \big\{x\big(\overleftarrow{\prod_{n\in\omega}} U_n\big) \cap \big(\overrightarrow{\prod_{n\in\omega}} U_n\big)y\,;\ x, y \in G,\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\}\,.$$

By $\vec{G}$, $\overset{\leftarrow}{G}$, $\overset{\leftrightarrow}{G}$, $\overset{\rightleftarrows}{G}$ we denote the groups G endowed with the topologies $\overset{\rightarrow}{\tau}$, $\overset{\leftarrow}{\tau}$, $\overset{\leftrightarrow}{\tau}$, $\overset{\rightleftarrows}{\tau}$, respectively. It is easy to check that $\vec{G}$, $\overset{\leftarrow}{G}$, $\overset{\leftrightarrow}{G}$, $\overset{\rightleftarrows}{G}$ are semitopological groups having the families

$$\overset{\rightarrow}{\mathcal{B}}_e = \big\{\overrightarrow{\prod_{n\in\omega}} U_n\,;\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\}\,,$$
$$\overset{\leftarrow}{\mathcal{B}}_e = \big\{\overleftarrow{\prod_{n\in\omega}} U_n\,;\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\}\,,$$
$$\overset{\leftrightarrow}{\mathcal{B}}_e = \big\{\overleftrightarrow{\prod_{n\in\omega}} U_n\,;\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\} = \{U^{-1}U\,;\ U \in \overset{\rightarrow}{\mathcal{B}}_e\}\,,$$
$$\overset{\rightleftarrows}{\mathcal{B}}_e = \big\{\big(\overrightarrow{\prod_{n\in\omega}} U_n\big) \cap \big(\overleftarrow{\prod_{n\in\omega}} U_n\big)\,;\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\big\} = \{U \cap U^{-1}\,;\ U \in \overset{\rightarrow}{\mathcal{B}}_e\}$$

as neighborhood bases at the identity e. Since the inversion $(\cdot)^{-1} : G \to G$ is continuous with respect to the topologies $\overset{\leftrightarrow}{\tau}$ or $\overset{\rightleftarrows}{\tau}$, the semitopological groups $\overset{\leftrightarrow}{G}$ and $\overset{\rightleftarrows}{G}$ are quasitopological groups.

We recall that a group H endowed with a topology is said to be

- a *semitopological group* if the binary operation $H \times H \to H$, $(x, y) \mapsto xy$, is separately continuous;
- a *quasitopological group* if H is a semitopological group with continuous inversion $(\cdot)^{-1} : H \to H$, $(\cdot)^{-1} : x \mapsto x^{-1}$.

Now we see that for any tower $(G_n)_{n\in\omega}$ of topological groups we get the following five semitopological groups linked by continuous identity homomorphisms:

$$\begin{tikzcd}
 & \overset{\leftarrow}{G} \arrow[dr] & & \\
\overset{\rightleftarrows}{G} \arrow[ur] \arrow[dr] & & \overset{\leftrightarrow}{G} \arrow[r] & \text{g-}\varinjlim G_n\,. \\
 & \vec{G} \arrow[ur] & &
\end{tikzcd}$$

The continuity of the final map in the diagram is not trivial:

PROPOSITION 2.1. *The identity map $\overset{\leftrightarrow}{G} \to \text{g-}\varinjlim G_n$ is continuous.*

PROOF. Since $\overset{\leftrightarrow}{G}$ and $\text{g-}\varinjlim G_n$ are semitopological groups, it suffices to prove the continuity of the identity map $\overset{\leftrightarrow}{G} \to \text{g-}\varinjlim G_n$ at the neutral element e.

Given a neighborhood $U \subset \text{g-}\varinjlim G_n$ of e, find an open neighborhood $V \subset \text{g-}\varinjlim G_n$ of e such that $V^{-1}V \subset U$. Such a neighborhood exists because $\text{g-}\varinjlim G_n$ is a topological group.

By induction, construct a sequence of open symmetric neighborhoods $V_n \subset \text{g-}\varinjlim G_n$ of e such that $V_0^2 \subset V$ and $V_{n+1}^2 \subset V_n$ for all $n \in \omega$. By induction on $m \in \omega$ we shall prove the inclusion

$$\Big(\overrightarrow{\prod_{0\le n<m}} V_n\Big)\cdot V_m^2 \subset V\,. \tag{1}$$

For $m = 0$ this inclusion holds according to the choice of V_0. Assuming that for some m the inclusion is true, observe that

$$\Big(\overrightarrow{\prod_{0\le n\le m}} V_n\Big)\cdot V_{m+1}^2 = \Big(\overrightarrow{\prod_{0\le n<m}} V_n\Big)\cdot V_m V_{m+1}^2 \subset \Big(\overrightarrow{\prod_{0\le n<m}} V_n\Big)\cdot V_m V_m \subset V$$

by the inductive hypothesis. Then

$$\overrightarrow{\prod_{n\in\omega}} V_n = \bigcup_{m\in\omega}\overrightarrow{\prod_{0\le n\le m}} V_n \subset V\,.$$

For every $n \in \omega$ find a basic neighborhood $W_n \in \mathcal{B}_n$ in the group G_n such that $W_n \subset V_n$ and observe that $\overrightarrow{\prod}_{n\in\omega} W_n \subset \overrightarrow{\prod}_{n\in\omega} V_n \subset V$, and hence

$$\overleftrightarrow{\mathcal{B}}_e \ni \overleftrightarrow{\prod_{n\in w}} W_n = \Big(\overrightarrow{\prod_{n\in\omega}} W_n\Big)^{-1}\cdot\overrightarrow{\prod_{n\in\omega}} W_n \subset V^{-1}V \subset U$$

witnessing the continuity of the identity map $\overleftrightarrow{G} \to \text{g-}\varinjlim G_n$ at e. □

One may ask about conditions guaranteeing that the semitopological groups $\overrightarrow{G}$, $\overleftarrow{G}$, $\overset{\rightleftarrows}{G}$ or $\overleftrightarrow{G}$ are topological groups.

THEOREM 2.2. *The following conditions* (1) *through* (5) *are equivalent*:

(1) $\overrightarrow{G}$ *is a topological group*;

(2) $\overleftarrow{G}$ *is a topological group*;

(3) $\overset{\rightleftarrows}{G}$ *is a topological group*;

(4) *the identity map* $\overleftrightarrow{G} \to \overset{\rightleftarrows}{G}$ *is continuous*;

(5) *the identity map* $\overset{\rightleftarrows}{G} \to \text{g-}\varinjlim G_n$ *is a homeomorphism.*

The equivalent conditions (1) *through* (5) *imply the following two equivalent conditions*:

(6) $\overleftrightarrow{G}$ *is a topological group*;

(7) *the identity map* $\overleftrightarrow{G} \to \text{g-}\varinjlim G_n$ *is a homeomorphism.*

PROOF. (1) ⇒ (2) Assume that $\overrightarrow{G}$ is a topological group. Then the identity map $\overrightarrow{G} \to \overleftarrow{G}$ is continuous because each basic neighborhood $\overleftarrow{\prod}_{n\in\omega} U_n \in \overleftarrow{\mathcal{B}}_e$ of e in $\overleftarrow{G}$ is open in $\overrightarrow{G}$, being the inversion $(\overrightarrow{\prod}_{n\in\omega} U_n)^{-1}$ of the basic neighborhood $\overrightarrow{\prod}_{n\in\omega} U_n \in \overrightarrow{\mathcal{B}}_e$ of e in the topological group $\overrightarrow{G}$. By the same reason, the identity map $\overrightarrow{G} \to \overleftarrow{G}$ is open. Consequently, the topologies $\overrightarrow{\tau}$ and $\overleftarrow{\tau}$ on G coincide, and hence $\overleftarrow{G}$ is a topological group.

The implication (2) ⇒ (1) can be proved similarly.

(1) $\Rightarrow$ (3) If $\overrightarrow{G}$ is a topological group, then $\overrightarrow{\tau} = \overleftarrow{\tau}$ and then $\overset{\rightleftarrows}{\tau} = \overrightarrow{\tau} = \overleftarrow{\tau}$, by the definition of the topology $\overset{\rightleftarrows}{\tau}$. Consequently, $\overset{\rightleftarrows}{G} = \overrightarrow{G}$ is a topological group.

(3) $\Rightarrow$ (5) If $\overset{\rightleftarrows}{G}$ is a topological group, then the identity map g-$\varinjlim G_n \to \overset{\rightleftarrows}{G}$ is continuous by the definition of g-$\varinjlim G_n$ because all the identity homomorphisms $G_n \to \overset{\rightleftarrows}{G}$, $n \in \omega$, are continuous. Since the inverse (identity) map $\overset{\rightleftarrows}{G} \to$ g-$\varinjlim G_n$ is always continuous by Proposition 2.1, it is a homeomorphism.

(5) $\Rightarrow$ (4) If the identity map $\overset{\rightleftarrows}{G} \to$ g-$\varinjlim G_n$ is a homeomorphism, then the identity map $\overleftrightarrow{G} \to \overset{\rightleftarrows}{G}$ is continuous as the composition of two continuous maps $\overleftrightarrow{G} \to$ g-$\varinjlim G_n \to \overset{\rightleftarrows}{G}$.

(4) $\Rightarrow$ (1) Assume that the identity map $\overleftrightarrow{G} \to \overset{\rightleftarrows}{G}$ is a homeomorphism. Then the identity maps between the semitopological groups $\overset{\rightleftarrows}{G}, \overrightarrow{G}, \overleftarrow{G}, \overleftrightarrow{G}$ are homeomorphisms. Consequently, $\overrightarrow{G}$ is a quasitopological group because so is $\overset{\rightleftarrows}{G}$ or $\overleftrightarrow{G}$. Furthermore, $\overrightarrow{G}$ is a topological group since the multiplication map $\overrightarrow{G} \times \overrightarrow{G} \to \overrightarrow{G}$, $(x, y) \mapsto xy$, is continuous as the map $\overleftarrow{G} \times \overrightarrow{G} \to \overleftrightarrow{G}$.

(5) $\Rightarrow$ (7) If the identity map $\overset{\rightleftarrows}{G} \to$ g-$\varinjlim G_n$ is a homeomorphism, then the identity map g-$\varinjlim G_n \to \overleftrightarrow{G}$ is continuous as the composition of two continuous maps g-$\varinjlim G_n \to \overset{\rightleftarrows}{G} \to \overleftrightarrow{G}$. The inverse (identity) map $\overleftrightarrow{G} \to$ g-$\varinjlim G_n$ is continuous by Proposition 2.1.

(7) $\Rightarrow$ (6) If the identity map $\overleftrightarrow{G} \to$ g-$\varinjlim G_n$ is a homeomorphism, then $\overleftrightarrow{G}$ is a topological group because so is g-$\varinjlim G_n$.

(6) $\Rightarrow$ (7) In this case, g-$\varinjlim G_n \to \overleftrightarrow{G}$ is continuous and hence a homeomorphism. $\square$

REMARK 2.3. The topology $\overleftrightarrow{\tau}$ on the union $G = \bigcup_{n\in\omega} G_n$ of a tower of topological groups (G_n) was introduced in [16] and called the *bamboo-shoot* topology. This topology was later discussed in [8], [11], [12], [13], [14].

3. The Passing Through Assumption. In this section we shall discuss implications of PTA, the Passing Through Assumption, introduced by Tatsuuma, Shimomura, and Hirai in [16].

DEFINITION 3.1. A tower of topological groups $(G_n)_{n\in\omega}$ is said to satisfy PTA if each group G_n has a neighborhood base $\mathcal{B}_n$ at the identity e, consisting of open symmetric neighborhoods $U \subset G_n$ such that for every $m \geq n$ and every neighborhood $V \subset G_m$ of e there is a neighborhood $W \subset G_m$ of e such that $WU \subset UV$.

It was proved in [14] and [16] that for a tower of topological groups $(G_n)_{n\in\omega}$ satisfying PTA, the semitopological group $\overleftrightarrow{G}$ is a topological group, which can be identified with the direct limit g-$\varinjlim G_n$.

The following theorem says a bit more:

THEOREM 3.2. *If a tower of topological groups* $(G_n)_{n\in\omega}$ *satisfies* PTA, *then the semitopological group* $\overrightarrow{G}$ *is a topological group and hence the conditions* (1) *through* (7) *of Theorem* 2.2 *hold. In particular, the topology of* $\text{g-}\varinjlim G_n$ *coincides with any of the topologies* $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overrightarrow{\overleftarrow{\tau}}$, $\overleftrightarrow{\tau}$.

PROOF. Since the tower $(G_n)_{n\in\omega}$ satisfies PTA, each topological group G_n admits a neighborhood base $\mathcal{B}_n$ described in Definition 3.1.

In order to show that the semitopological group $\overrightarrow{G}$ is a topological group, it suffices to check the continuity of the multiplication and of the inversion at the neutral element e.

In order to prove the continuity of the multiplication at e, we shall show that for every neighborhood $\overrightarrow{\prod}_{n\in\omega} W_n \in \overrightarrow{\mathcal{B}}_e$, there exists a neighborhood $\overrightarrow{\prod}_{n\in\omega} V_n \in \overrightarrow{\mathcal{B}}_e$ such that $\big(\overrightarrow{\prod}_{n\in\omega} V_n\big)^2 \subset \big(\overrightarrow{\prod}_{n\in\omega} W_n\big)$.

For every $n \in \omega$ take a neighborhood $U_n \in \mathcal{B}_n$ with $U_nU_n \subset W_n$. Putting $V_n^{(0)} = U_n$ and using PTA, for every $0 < i \le n$ we can find inductively a neighborhood $V_n^{(i)} \in \mathcal{B}_n$ such that

- $V_n^{(i)} \subset U_n$;
- $V_n^{(i)} U_{n-i} \subset U_{n-i} V_n^{(i-1)}$.

Observe that for $i = n - k$ the latter inclusion yields

$$V_n^{(n-k)} U_k \subset U_k V_n^{(n-k-1)}. \tag{2}$$

We claim that $\big(\overrightarrow{\prod}_{n\in\omega} V_n^{(n)}\big)^2 \subset \big(\overrightarrow{\prod}_{n\in\omega} W_n\big)$. Since $V_n^{(n)} \subset U_n$, this inclusion follows if we check that

$$\overrightarrow{\prod_{n\le m}} V_n^{(n)} \cdot \overrightarrow{\prod_{n\le m}} U_n \subset \overrightarrow{\prod_{n\le m}} W_n \tag{3}$$

for every $m > 0$.

For every non-negative integer $k \le m + 1$ consider the subset

$$\Pi_k = \overrightarrow{\prod_{0\le n<k}} W_n \cdot \overrightarrow{\prod_{k\le n\le m}} V_n^{(n-k)} \cdot \overrightarrow{\prod_{k\le n\le m}} U_n$$

of the group G_m. Observe that (3) is equivalent to the inclusion $\Pi_0 \subset \Pi_{m+1}$. Hence it suffices to check that $\Pi_k \subset \Pi_{k+1}$ for every $k \le m$.

By induction on k we can deduce from (2) the inclusion

$$\Big(\overrightarrow{\prod_{k<n\le m}} V_n^{(n-k)}\Big) \cdot U_k \subset U_k \cdot \overrightarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}. \tag{4}$$

This inclusion combined with $V_k^{(0)} U_k = U_k U_k \subset W_k$ yields the desired inclusion as

$$\begin{aligned}\Pi_k &= \overrightarrow{\prod_{0\le n<k}} W_n \cdot \overrightarrow{\prod_{k\le n\le m}} V_n^{(n-k)} \cdot \overrightarrow{\prod_{k\le n\le m}} U_n \\ &= \big(\overrightarrow{\prod_{0\le n<k}} W_n\big) \cdot V_k^{(0)} \cdot \big(\overrightarrow{\prod_{k<n\le m}} V_n^{(n-k)}\big) \cdot U_k \cdot \overrightarrow{\prod_{k<n\le m}} U_n \\ &\subset \big(\overrightarrow{\prod_{0\le n<k}} W_n\big) \cdot V_k^{(0)} \cdot \big(U_k \cdot \overrightarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}\big) \cdot \overrightarrow{\prod_{k<n\le m}} U_n \\ &\subset \big(\overrightarrow{\prod_{0\le n<k}} W_n\big) \cdot W_k \cdot \overrightarrow{\prod_{k<n\le m}} V_n^{(n-k-1)} \cdot \overrightarrow{\prod_{k<n\le m}} U_n = \Pi_{k+1}.\end{aligned}$$

Next, we verify the continuity of the inversion at e. For a given set $\overrightarrow{\prod}_{n\in\omega} U_n \in \vec{\mathcal{B}}_e$, we need to find a set $\overrightarrow{\prod}_{n\in\omega} V_n \in \vec{\mathcal{B}}_e$ with $\big(\overrightarrow{\prod}_{n\in\omega} V_n\big)^{-1} \subset \overrightarrow{\prod}_{n\in\omega} U_n$.

For every $n\in\omega$ put $V_n^{(0)} = U_n$ and using PTA, for every $0 < i \le n$ choose a neighborhood $V_n^{(i)} \in \mathcal{B}_n$ such that $V_n^{(i)} U_{n-i} \subset U_{n-i} V_n^{(i-1)}$. Thus defined sets satisfy the inclusions

$$V_n^{(n-k)} U_k \subset U_k V_n^{(n-k-1)}, \quad 0 \le k < n. \tag{5}$$

We claim that

$$\big(\overrightarrow{\prod_{n\in\omega}} V_n^{(n)}\big)^{-1} \subset \overrightarrow{\prod_{n\in\omega}} U_n.$$

It suffices to check that

$$\big(\overrightarrow{\prod_{n\le m}} V_n^{(n)}\big)^{-1} = \overleftarrow{\prod_{n\le m}} V_n^{(n)} \subset \overrightarrow{\prod_{n\le m}} U_n \tag{6}$$

for all $m\in\omega$. The left-hand equality follows from the symmetry of the neighborhoods $V_n^{(n)} \in \mathcal{B}_n$.

For the proof of the right-hand inclusion, for every $k \le m+1$ consider the subset

$$\Pi_k = \overrightarrow{\prod_{0\le n<k}} U_n \cdot \overleftarrow{\prod_{k\le n\le m}} V_n^{(n-k)}$$

of the group G_m, and observe that (6) is equivalent to the inclusion $\Pi_0 \subset \Pi_{m+1}$. So it suffices to check that $\Pi_k \subset \Pi_{k+1}$ for every $k \le m+1$.

By induction on $k \le m+1$ we can show that (5) implies

$$\big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k)}\big) \cdot U_k \subset U_k \cdot \overleftarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}.$$

Now the inclusion $\Pi_k \subset \Pi_{k+1}$ can be seen as

$$\begin{aligned}\Pi_k &= \overrightarrow{\prod_{0\le n<k}} U_n \cdot \overleftarrow{\prod_{k\le n\le m}} V_n^{(n-k)} = \overrightarrow{\prod_{0\le n<k}} U_n \cdot \big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k)}\big) \cdot V_k^{(0)} \\ &= \overrightarrow{\prod_{0\le n<k}} U_n \cdot \big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k)}\big) \cdot U_k \subset \big(\overrightarrow{\prod_{0\le n<k}} U_n\big) \cdot U_k \cdot \overleftarrow{\prod_{k<n\le m}} V_n^{(n-k-1)} = \Pi_{k+1}.\end{aligned}$$

□

4. Balanced triples and towers of groups. In this section we introduce another condition guaranteeing that the topology of the direct limit g-$\varinjlim G_n$ of a tower of topological groups $(G_n)_{n\in\omega}$ coincides with the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overrightarrow{\overleftarrow{\tau}}$ and $\overleftrightarrow{\tau}$.

Let us observe that a tower of topological groups $(G_n)_{n\in\omega}$ satisfies PTA if each group G_n, $n\in\omega$, is *balanced*. The latter means that G_n has a neighborhood base at e consisting of G-invariant neighborhoods (see [1, p. 69]).

We define a subset $U\subset G$ of a group G to be *H-invariant* for a subgroup $H\subset G$ if $xUx^{-1}=U$ for all $x\in H$. Observe that for any subset $U\subset G$ the set

$$\sqrt{U}^H=\{x\in G;\ x^H\subset U\}$$

is the largest H-invariant subset of U. Here $x^H=\{hxh^{-1};\ h\in H\}$ stands for the H-conjugacy class of a point $x\in G$.

Observe that a topological group G is balanced if and only if, for every neighborhood $U\subset G$ of e, the set $\sqrt{U}^G$ is a neighborhood of e.

DEFINITION 4.1. A triple (G,Γ,H) of topological groups $H\subset\Gamma\subset G$ is called *balanced* if for any neighborhoods $V\subset\Gamma$ and $U\subset G$ of the neutral element e of G the product $V\cdot\sqrt{U}^H$ is a neighborhood of e in G.

A tower of topological groups $(G_n)_{n\in\omega}$ is called *balanced* if each triple (G_{n+2},G_{n+1},G_n), $n\in\omega$, is balanced.

THEOREM 4.2. *If a tower of topological groups $(G_n)_{n\in\omega}$ is balanced, then the semitopological group $\overrightarrow{G}$ is a topological group and hence all the conditions* (1) *through* (7) *of Theorem* 2.2 *hold. In particular the topology of* g-$\varinjlim G_n$ *coincides with the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overrightarrow{\overleftarrow{\tau}}$, $\overleftrightarrow{\tau}$.*

PROOF. In order to show that the semitopological group $\overrightarrow{G}$ is a topological group, it suffices to check the continuity of the multiplication and of the inversion at the neutral element e.

In order to check the continuity of multiplication at e, fix a neighborhood $\overrightarrow{\prod}_{n\in\omega}U_n\in\overrightarrow{\mathcal{B}}_e$. For every $n\in\omega$, find a symmetric neighborhood W_n of e in the group G_n such that $W_n\cdot W_n\subset U_n$, and let

$$Z_n=\sqrt{W_n}^{G_{n-2}}=\{x\in G_n;\ x^{G_{n-2}}\subset W_n\}$$

be the largest G_{n-2}-invariant subset of W_n (here we assume that $G_k=\{e\}$ for $k<0$).

Let $V_0=U_0\cap W_1$ and $V_1\subset G_1$ be a symmetric neighborhood of e such that $V_1^2\subset W_1$. Next, for each $n\geq 2$, inductively choose a neighborhood $V_n\subset G_n$ so that

(a) $V_n^2\subset V_{n-1}\cdot Z_n$, and

(b) $V_n\subset W_{n+1}$.

The condition (a) can be satisfied because the triple (G_n,G_{n-1},G_{n-2}) is balanced according to our hypothesis.

We claim $\big(\overrightarrow{\prod}_{n\in\omega} V_n\big)^2 \subset \overrightarrow{\prod}_{n\in\omega} U_n$. This inclusion will follow if we check that

$$\big(\overrightarrow{\prod_{n\le m}} V_n\big)\cdot\big(\overrightarrow{\prod_{n\le m}} V_n\big) \subset \overrightarrow{\prod_{n\le m+1}} U_n$$

for every $m \ge 2$.

For every $1 \le k \le m$ consider the subset

$$\varPi_k = \big(\overrightarrow{\prod_{n\le m-k}} V_n\big)\cdot V_{m-k+1}^2\cdot\big(\overrightarrow{\prod_{n\le m-k}} V_n\big)\cdot\big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\big)\cdot V_m.$$

We claim that $\varPi_k \subset \varPi_{k+1}$. Indeed,

$$\begin{aligned}
\varPi_k &= \big(\overrightarrow{\prod_{n\le m-k}} V_n\big)\cdot V_{m-k+1}^2\cdot\big(\overrightarrow{\prod_{n\le m-k}} V_n\big)\cdot\big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\big)\cdot V_m\\
&\subset \big(\overrightarrow{\prod_{n\le m-k}} V_n\big)\cdot V_{m-k}\cdot Z_{m-k+1}\cdot\big(\overrightarrow{\prod_{n<m-k}} V_n\big)\cdot V_{m-k}\cdot\big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\big)\cdot V_m\\
&= \big(\overrightarrow{\prod_{n<m-k}} V_n\big)\cdot V_{m-k}^2\cdot\big(\overrightarrow{\prod_{n<m-k}} V_n\big)\cdot Z_{m-k+1}\cdot V_{m-k}\cdot\big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\big)\cdot V_m\\
&= \big(\overrightarrow{\prod_{n<m-k}} V_n\big)\cdot V_{m-k}^2\cdot\big(\overrightarrow{\prod_{n<m-k}} V_n\big)\cdot\big(\overrightarrow{\prod_{m-k\le n<m}} Z_{n+1}V_n\big)\cdot V_m = \varPi_{k+1}.
\end{aligned}$$

Now we see that

$$\begin{aligned}
\big(\overrightarrow{\prod_{i\le m}} V_i\big)^2 &\subset \big(\overrightarrow{\prod_{i\le m-1}} V_i\big)\cdot V_m^2\cdot\big(\overrightarrow{\prod_{i\le m}} V_i\big) = \varPi_1 \subset \varPi_m = V_0V_1^2V_0\cdot\big(\overrightarrow{\prod_{0<n<m}} Z_{n+1}V_n\big)V_m\\
&\subset U_0W_1W_1\cdot\big(\overrightarrow{\prod_{0<n<m}} W_{n+1}W_{n+1}\big)W_{m+1} \subset U_0U_1\big(\overrightarrow{\prod_{0<n<m}} U_{n+1}\big)\cdot U_{m+1}\\
&= \overrightarrow{\prod_{n\le m+1}} U_n\,.
\end{aligned}$$

Now we check that the inversion is continuous at e with respect to the topology $\overrightarrow{\tau}$. Given any basic set $\overrightarrow{\prod}_{n\in\omega} W_n \in \overrightarrow{\mathcal{B}}_e$, we need to find a basic set $\overrightarrow{\prod}_{n\in\omega} U_n \in \overrightarrow{\mathcal{B}}_e$ such that $\big(\overrightarrow{\prod}_{n\in\omega} U_n\big)^{-1} \subset \overrightarrow{\prod}_{n\in\omega} W_n$.

For every $n \in \omega$ let $Z_{n+2} = \sqrt{W_{n+2}}^{G_n}$ be the largest G_n-invariant subset of W_{n+2}. For each non-negative number $n < 2$ pick a symmetric neighborhood $V_n \subset G_n$ such that $V_n^2 \subset W_n$. For $n \ge 2$, inductively choose a symmetric neighborhood $V_n \subset G_n$ of e such that $V_n^2 \subset W_n \cap (V_{n-1}\cdot Z_n)$. Such a neighborhood V_n exists by the balanced property of the triple (G_n, G_{n-1}, G_{n-2}). Finally, for every $n \in \omega$ put $U_n = V_n \cap V_{n+1}$.

We claim that $\big(\overrightarrow{\prod}_{n\in\omega} U_n\big)^{-1} \subset \overrightarrow{\prod}_{n\in\omega} W_n$. This inclusion will follow if we check that $\overleftarrow{\prod}_{n<m} U_n \subset \overrightarrow{\prod}_{n\le m} W_n$ for every $m \in \omega$. By induction we shall prove that

$$V_m\cdot\overleftarrow{\prod_{n<m}} U_n \subset \overrightarrow{\prod_{n\le m}} W_n \tag{7}$$

for every $m \in \boldsymbol{N}$.

For $m = 1$ the inclusion (7) is true since $V_1 U_0 \subset V_1^2 \subset W_1 \subset W_0 W_1$. Assume that the inclusion (7) has been proved for some $m = k \geq 1$. Then

$$\begin{aligned} V_{m+1} \cdot \overleftarrow{\prod_{n \leq m}} U_n \subset V_{m+1} \cdot U_m \cdot \overleftarrow{\prod_{n<m}} U_n \subset V_{m+1}^2 \cdot \overleftarrow{\prod_{n<m}} U_n \subset V_m Z_{m+1} \overleftarrow{\prod_{n<m}} U_n \\ = V_m \cdot \Big(\overleftarrow{\prod_{n<m}} U_n \Big) \cdot Z_{m+1} \subset \Big(\overrightarrow{\prod_{n \leq m}} W_n \Big) \cdot Z_{m+1} \subset \overrightarrow{\prod_{n \leq m+1}} W_n, \end{aligned}$$

which means that the inclusion (7) holds for $m = k + 1$. □

5. Bi-balanced triples and towers of groups. In this section we introduce the bi-balanced property of a tower (G_n), which is weaker than the balanced property and implies that the semitopological group $\overleftrightarrow{G}$ is a topological group.

DEFINITION 5.1. A triple (G, Γ, H) of topological groups $H \subset \Gamma \subset G$ is called *bi-balanced* if for any neighborhoods $V \subset \Gamma$ and $U \subset G$ of the neutral element e of G the product $\sqrt{U}^H \cdot V \cdot \sqrt{U}^H$ is a neighborhood of e in G.

A tower of topological groups $(G_n)_{n\in\omega}$ is called *bi-balanced* if each triple (G_{n+2}, G_{n+1}, G_n), $n \in \omega$, is bi-balanced.

THEOREM 5.2. *If a tower of topological groups $(G_n)_{n\in\omega}$ is bi-balanced, then the identity map $\overleftrightarrow{G} \to$ g-$\varinjlim G_n$ is a homeomorphism and hence the topology of* g-$\varinjlim G_n$ *coincides with the topology $\overleftrightarrow{\tau}$.*

PROOF. By Theorem 2.2, it suffices to show that $\overleftrightarrow{G}$ is a topological group. Since $\overleftrightarrow{G}$ is a quasitopological group, it suffices to check the continuity of multiplication at the neutral element. Given a basic neighborhood $\overleftrightarrow{\prod}_{n\in\omega} W_n \in \overleftrightarrow{\mathcal{B}}_e$, we should find a neighborhood $\overleftrightarrow{\prod}_{n\in\omega} V_n \in \overleftrightarrow{\mathcal{B}}_e$ such that $\big(\overleftrightarrow{\prod}_{n\in\omega} V_n \big)^2 \subset \overleftrightarrow{\prod}_{n\in\omega} W_n$.

For every $n \in \omega$, find a symmetric neighborhood $U_n \subset G_n$ of e such that $U_n^2 \subset W_n$ and let $Z_n = \sqrt{U_n}^{G_{n-2}}$ be the maximal G_{n-2}-invariant subset of U_n (here we assume that $G_k = \{e\}$ for $k < 0$). Let $\widetilde{U}_0 = W_0$ and inductively for every $n \in \mathbf{N}$ choose a symmetric neighborhood $\widetilde{U}_n \subset U_n$ of e such that $\widetilde{U}_n^3 \subset Z_n \widetilde{U}_{n-1} Z_n$. The choice of the neighborhood $\widetilde{U}_n$ is possible because the set $Z_n \widetilde{U}_{n-1} Z_n$ is a neighborhood of e in the group G_n by the bi-balanced property of the triple (G_n, G_{n-1}, G_{n-2}). Finally, for every $n \in \omega$ let $V_n = G_n \cap \widetilde{U}_{n+1}$.

We claim that $\overleftrightarrow{\prod}_{n\in\omega} V_n$ is the required neighborhood with $\big(\overleftrightarrow{\prod}_{n\in\omega} V_n \big)^2 \subset \overleftrightarrow{\prod}_{n\in\omega} W_n$. This inclusion will follow as soon as we check that

$$\Big(\overleftrightarrow{\prod_{n<m}} V_n \Big)^2 \subset \overleftrightarrow{\prod_{n \leq m}} W_n \tag{8}$$

for all $m \in \omega$.

For $m = 0$ this inclusion is trivial. Assume that $m > 0$. For every non-negative $k < m$ consider the subset

$$\varPi_k = \Big(\overleftarrow{\prod_{k\le n<m}} V_n Z_{n+1}\Big) \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \widetilde{U}_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{k\le n<m}} Z_{n+1} V_n\Big)$$

of the group G_m. The following chain of inclusions guarantees that $\varPi_{k+1} \subset \varPi_k$:

$$\begin{aligned}
\varPi_{k+1} &= \Big(\overleftarrow{\prod_{k<n<m}} V_n Z_{n+1}\Big) \cdot \Big(\overleftrightarrow{\prod_{n\le k}} V_n\Big) \cdot \widetilde{U}_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n\le k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{k<n<m}} Z_{n+1} V_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{k<n<m}} V_n Z_{n+1}\Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot V_k \cdot \widetilde{U}_{k+1} \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot V_k \cdot \Big(\overrightarrow{\prod_{k<n<m}} Z_{n+1} V_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{k<n<m}} V_n Z_{n+1}\Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \widetilde{U}^3_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot V_k \cdot \Big(\overrightarrow{\prod_{k<n<m}} Z_{n+1} V_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{k<n<m}} V_n Z_{n+1}\Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot Z_{k+1} \cdot \widetilde{U}_k \cdot Z_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot V_k \cdot \Big(\overrightarrow{\prod_{k<n<m}} Z_{n+1} V_n\Big)\\
&= \Big(\overleftarrow{\prod_{k<n<m}} V_n Z_{n+1}\Big) \cdot V_k \cdot Z_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \widetilde{U}_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot Z_{k+1} \cdot V_k \cdot \Big(\overrightarrow{\prod_{k<n<m}} Z_{n+1} V_n\Big)\\
&= \Big(\overleftarrow{\prod_{k\le n<m}} V_n Z_{n+1}\Big) \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \widetilde{U}_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{k\le n<m}} Z_{n+1} V_n\Big) = \varPi_k\,.
\end{aligned}$$

Now we see that

$$\begin{aligned}
\Big(\overleftrightarrow{\prod_{n<m}} V_n\Big)^2 &\subset \Big(\overleftrightarrow{\prod_{n<m}} V_n\Big) \cdot \widetilde{U}_m \cdot \Big(\overleftrightarrow{\prod_{n<m}} V_n\Big) = \varPi_m \subset \varPi_0 = \Big(\overleftarrow{\prod_{n<m}} V_n Z_{n+1}\Big) \cdot \widetilde{U}_0 \cdot \Big(\overrightarrow{\prod_{n<m}} Z_{n+1} V_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{n<m}} U^2_{n+1}\Big) \cdot \widetilde{U}_0 \cdot \Big(\overrightarrow{\prod_{n<m}} U^2_{n+1}\Big) \subset \Big(\overleftarrow{\prod_{n<m}} W_{n+1}\Big) \cdot W_0^2 \cdot \Big(\overrightarrow{\prod_{n<m}} W_{n+1}\Big) = \overleftrightarrow{\prod_{n\le m}} W_m.
\end{aligned}$$

□

6. The independence of PTA and the balanced property. Looking at Theorems 3.2 and 4.2 (which have the same conclusion) the reader can ask about the interplay between PTA and the balanced property. These two properties are independent.

First we present an example of a tower of topological groups $(G_n)_{n\in\omega}$ that is balanced but does not satisfy PTA.

Let $G = \mathcal{H}_c(\boldsymbol{R})$ be the group of all homeomorphisms $h : \boldsymbol{R} \to \boldsymbol{R}$ having compact support $\mathrm{supp}(h) = \mathrm{cl}_{\boldsymbol{R}}\{x \in \boldsymbol{R}\,;\ h(x) \neq x\}$.

The homeomorphism group $G = \mathcal{H}_c(\boldsymbol{R})$ is endowed with the Whitney topology whose base at a homeomorphism $h \in \mathcal{H}_c(\boldsymbol{R})$ consists of the sets

$$B(h, \varepsilon) = \{f \in \mathcal{H}_c(\boldsymbol{R})\,;\ |f - h| < \varepsilon\}$$

where $\varepsilon : \boldsymbol{R} \to (0, 1)$ runs over continuous positive functions on the real line.

It is well-known that the Whitney topology turns the homeomorphism group $G = \mathcal{H}_c(\boldsymbol{R})$ into a topological group (see e.g., [4]). This group can be written as the countable union $G = \bigcup_{n\in\omega} G_n$ of the closed subgroups

$$G_n = \big\{h \in G\,;\ \mathrm{supp}(h) \subset [-n, n]\big\}\,.$$

Each subgroup G_n can be identified with the group $\mathcal{H}_+(\boldsymbol{I}_n)$ of orientation-preserving homeomorphisms of the closed interval $\boldsymbol{I}_n = [-n, n]$. The Whitney topology of the group G induces on each subgroup G_n the compact-open topology, generated by the sup-metric $\|f - h\| = \sup_{x \in \boldsymbol{R}} |f(x) - h(x)|$.

In the following theorem we shall show that the topology of the direct limit $\text{g-}\varinjlim G_n$ coincides with the topology $\overrightarrow{\tau}$ on G but the tower $(G_n)_{n\in\omega}$ does not satisfy PTA. This answers [12, Problem 17.3] of Glöckner.

THEOREM 6.1. (1) *The tower of the homeomorphism groups* $(\mathcal{H}_+(\boldsymbol{I}_n))_{n\in\omega}$ *is balanced.*

(2) *The tower* $(\mathcal{H}_+(\boldsymbol{I}_n))_{n\in\omega}$ *does not satisfy* PTA.

(3) *The Whitney topology on* $\mathcal{H}_c(\boldsymbol{R})$ *coincides with the topologies* $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overset{\rightleftarrows}{\tau}$, $\overleftrightarrow{\tau}$ *and those topologies coincide with the topology of the direct limit* $\text{g-}\varinjlim \mathcal{H}_+(\boldsymbol{I}_n)$.

PROOF. Let $G = \mathcal{H}_c(\boldsymbol{R})$ and $G_n = \mathcal{H}_+(\boldsymbol{I}_n) \subset G$ for $n \in \omega$. For a constant $\varepsilon > 0$ the ε-ball $B(\mathrm{id}_{\boldsymbol{R}}, \varepsilon) = \{f \in \mathcal{H}_c(\boldsymbol{R})\,;\ \|f - \mathrm{id}\| < \varepsilon\} \subset G$ centered at the identity homeomorphism $\mathrm{id}_{\boldsymbol{R}}$ will be denoted by $B(\varepsilon)$.

1. We need to show that for every $n \in \omega$ the triple (G_{n+2}, G_{n+1}, G_n) is balanced. This will follow if we check that for every neighborhood $U \subset G_{n+1}$ of the identity homeomorphism $\mathrm{id}_{\boldsymbol{R}}$ and any neighborhood $W \subset G_{n+2}$ of $\mathrm{id}_{\boldsymbol{R}}$ the set $U \cdot \sqrt{W}^{G_n}$ is a neighborhood of $\mathrm{id}_{\boldsymbol{R}}$ in G_{n+2}. Since the Whitney topology on the subgroup $G_{n+2} = \mathcal{H}_+(\boldsymbol{I}_{n+2})$ is generated by the sup-metric, the neighborhood $W \subset G_{n+2}$ contains the ε-ball $G_{n+2} \cap B(\varepsilon)$ for some positive constant $\varepsilon < 1$. The constant ε can be chosen so small that $G_{n+1} \cap B(\varepsilon) \subset U$.

Consider the closed subgroup

$$H = \{h \in G_{n+2}\,;\ \mathrm{supp}(h) \subset \boldsymbol{I}_{n+2} \setminus \boldsymbol{I}_n\}$$

of G_{n+2} and observe that $W \cap H \subset \sqrt{W}^{G_n}$. Now it suffices to check that $U \cdot (W \cap H)$ contains the ball $G_{n+2} \cap B(\varepsilon/2)$. Take any homeomorphism $h \in G_{n+2} \cap B(\varepsilon/2)$ and observe that h maps the interval $\boldsymbol{I}_n = [-n, n]$ into the interval $[-n - \varepsilon/2, n + \varepsilon/2]$. Then, we can consider the homeomorphism $g \in G_{n+1}$, which is equal to h on the interval $\boldsymbol{I}_n$ and is linear on the intervals $[n, n + 1]$ and $[-n - 1, -n]$. It is clear that $\|g - \mathrm{id}\| \le \|h - \mathrm{id}\| < \varepsilon/2$ and $\|g^{-1} - \mathrm{id}\| = \|g - \mathrm{id}\|$. Let $f = h \circ g^{-1} \in G_{n+2}$. The equality $g|\boldsymbol{I}_n = h|\boldsymbol{I}_n$ implies $f|\boldsymbol{I}_n = \mathrm{id}|\boldsymbol{I}_n$ and thus $f \in H$. It follows that

$$\|f - \mathrm{id}\| = \|h \circ g^{-1} - \mathrm{id}\| \le \|h \circ g^{-1} - g^{-1}\| + \|g^{-1} - \mathrm{id}\| < \varepsilon/2 + \varepsilon/2 = \varepsilon\,.$$

Now we see that the elements $g \in G_{n+1} \cap B(\varepsilon/2) \subset U$ and $f = g^{-1} \circ h \in H \cap B(\varepsilon) \subset \sqrt{W}^{G_n}$ yield $h = g \circ f \subset U \cdot \sqrt{W}^{G_n}$, which establishes the required inclusion $G_{n+2} \cap B(\varepsilon/2) \subset U \cdot \sqrt{W}^{G_n}$.

2. Assuming that the tower $(G_n)_{n\in\omega}$ satisfies PTA, we can find a neighborhood $U \subset G_1$ of $\mathrm{id}_{\boldsymbol{R}}$ such that for every neighborhood $V \subset G_2$ there is a neighborhood $W \subset G_2$ such that $WU \subset UV$.

Find $\varepsilon \in (0,1)$ such that $U \supset G_1 \cap B(\varepsilon)$. Then for the neighborhood $V = G_2 \cap B(\varepsilon)$ there is a neighborhood $W \subset G_2$ of $\mathrm{id}_{\boldsymbol{R}}$ with $WU \subset UV$. Find a positive constant $\delta < \varepsilon$ with $G_2 \cap B(\delta) \subset W$. It follows that $(G_2 \cap B(\delta)) \cdot (G_1 \cap B(\varepsilon)) \subset WU \subset UV \subset G_1 \cdot (G_2 \cap B(\varepsilon))$ and, after inversion, $(G_1 \cap B(\varepsilon)) \cdot (G_2 \cap B(\delta)) \subset (G_2 \cap B(\varepsilon)) \cdot G_1$. Take a homeomorphisms $f \in G_2 \cap B(\delta)$ such that $f(1) = 1 - \delta/2$ and a homeomorphism $g \in G_1 \cap B(\varepsilon)$ such that $g(1 - \delta/2) = 1 - \varepsilon$. Then $g \circ f(1) = 1 - \varepsilon$ which is not possible since $g \circ f \in (G_2 \cap B(\varepsilon)) \cdot G_1 \subset \{h \in G_2;\, |h(1) - 1| < \varepsilon\}$.

3. Since the tower $(G_n)_{n\in\omega}$ is balanced, the topology of $\text{g-}\varinjlim G_n$ coincides with the topologies $\overset{\rightarrow}{\tau}$, $\overset{\leftarrow}{\tau}$, $\overset{\rightleftarrows}{\tau}$, and $\overset{\leftrightarrow}{\tau}$ according to Theorem 4.2. Since $\text{g-}\varinjlim G_n$ carries the strongest group topology inducing the original topology on each group G_n, we conclude that the Whitney topology is weaker than the topology $\overset{\rightarrow}{\tau}$. In order to show that these two topologies coincide, it suffices to check that each basic neighborhood $\overrightarrow{\prod}_{n\in\omega} U_n$ of e in the topology $\overset{\rightarrow}{\tau}$ is a neighborhood of $\mathrm{id}_{\boldsymbol{R}}$ in the Whitney topology. Here, for every $n \in \omega$, U_n is an open symmetric neighborhood of $\mathrm{id}_{\boldsymbol{R}}$ in the group $G_n = \mathcal{H}_+(\boldsymbol{I}_n)$. Since the Whitney topology on the subgroup G_n is generated by the sup-metric, we can find a positive constant $\varepsilon_n < 1/2$ such that $U_n \supset G_n \cap B(\varepsilon_n)$.

Choose a continuous function $\varepsilon : \boldsymbol{R} \to (0, 1/2)$ such that

$$\sup\{\varepsilon(x)\,;\ x \in \boldsymbol{I}_n \setminus \boldsymbol{I}_{n-4}\} < \varepsilon_n/2 \quad \text{for all } n \in \omega\,. \tag{9}$$

Here we assume that $\boldsymbol{I}_k = \emptyset$ for all negative k.

The function ε determines the neighborhood $B(\varepsilon) = \{h \in G\,;\ |h - \mathrm{id}_{\boldsymbol{R}}| < \varepsilon\}$ of the identity map $\mathrm{id}_{\boldsymbol{R}}$ in the Whitney topology.

We claim that $B(\varepsilon) \subset \overrightarrow{\prod}_{n\in\omega} U_n$. Fix any homeomorphism $h \in B(\varepsilon)$ and for every $n \in \boldsymbol{N}$ consider the homeomorphism $h_n \in G_n$ such that $h_n|\boldsymbol{I}_{n-1} = h|\boldsymbol{I}_{n-1}$ and h_n is linear on the intervals $[n, n+1]$ and $[-n-1, -n]$. For $n \le 0$ we put $h_n = \mathrm{id}_{\boldsymbol{R}}$. It is clear that $h_m = h$ for some $m \in \boldsymbol{N}$.

For every $n \in \omega$ consider the homeomorphism $g_n = h_{n-1}^{-1} \circ h_n \in G_n$. Then $h = h_m = \overrightarrow{\prod}_{n\le m} g_n$. It remains to prove that each homeomorphism g_n belongs to the neighborhood U_n. This will follow if we check that $|g_n(x) - x| < \varepsilon_n$ for any $x \ne g_n(x)$.

Since $h_{n-1}|\boldsymbol{I}_{n-2} = h_n|\boldsymbol{I}_{n-2} = h|\boldsymbol{I}_{n-2}$, we conclude that x is in $\boldsymbol{I}_n \setminus \boldsymbol{I}_{n-2}$. It follows from $h_n \in G_n \cap B(1/2)$ that the point $y = h_n(x)$ belongs to the set $\boldsymbol{I}_n \setminus \boldsymbol{I}_{n-3}$. Since $h_{n-1} \in G_{n-1} \cap B(1/2)$, the point $z = h_{n-1}^{-1}(y)$ belongs to $\boldsymbol{I}_n \setminus \boldsymbol{I}_{n-4}$.

We claim that

$$|h_{n-1}(z) - z| < \varepsilon_n/2\,. \tag{10}$$

If $z \in \boldsymbol{I}_{n-2} \setminus \boldsymbol{I}_{n-4}$, then $|h_{n-1}(z) - z| = |h(z) - z| < \varepsilon(z) \le \varepsilon_n/2$ by the condition (9) from the definition of the function ε. If $z \in \boldsymbol{I}_{n-1} \setminus \boldsymbol{I}_{n-2}$, then the linearity of h_{n-1} on the two intervals composing the set $\boldsymbol{I}_{n-1} \setminus \boldsymbol{I}_{n-2}$ implies that

$$|h_{n-1}(z) - z| \le \max_{t\in\partial\boldsymbol{I}_{n-2}} |h(t) - t| < \max_{t\in\partial\boldsymbol{I}_{n-2}} \varepsilon(t) \le \varepsilon_n/2$$

by the definition of the function ε. Here $\partial \boldsymbol{I}_k = \{k, -k\}$ stands for the boundary of the interval $\boldsymbol{I}_k = [-k, k]$ in $\boldsymbol{R}$.

By a similar argument we can prove the inequality

$$|h_n(x) - x| < \varepsilon_n/2\,. \tag{11}$$

Unifying (10) and (11) we obtain the desired inequality:

$$\begin{aligned}|g_n(x) - x| = |h_{n-1}^{-1} \circ h_n(x) - x| &\le |h_{n-1}^{-1} \circ h_n(x) - h_n(x)| + |h_n(x) - x|\\ &= |z - h_{n-1}(z)| + |h_n(x) - x| < \frac{1}{2}\varepsilon_n + \frac{1}{2}\varepsilon_n = \varepsilon_n.\end{aligned}$$

□

Next, we present an example of a tower $(G_n)_{n\in\omega}$ that satisfies PTA but is not bi-balanced.

EXAMPLE 6.2. Let $(e_n)_{n\in\omega}$ be an orthonormal basis of the separable Hilbert space l_2 and $\mathcal{B}(l_2)$ be the Banach algebra of bounded linear operators on l_2. For every $n \in \boldsymbol{N}$ let G_n be the subgroup of $\mathcal{B}(l_2)$ consisting of invertible linear operators $T : l_2 \to l_2$ such that

- $Te_0 \in (0, +\infty) \cdot e_0$;
- $Te_i \in e_i + \boldsymbol{R} \cdot e_0$ for all $1 \le i \le n$;
- $Te_i = e_i$ for all $i > n$.

The tower $(G_n)_{n\in\boldsymbol{N}}$ satisfies PTA because each group G_n is locally compact (see [16], [14]). On the other hand, for every $n \in \boldsymbol{N}$ the triple (G_n, G_{n+1}, G_{n+2}) is not bi-balanced. The reason is that, for the neighborhood $W = \{T \in G_{n+2}; Te_{n+2} \in e_{n+2} + (-1, 1)e_0\}$ of the identity id in G_{n+2}, the set $Z = \sqrt{W}^{G_n}$ lies in the subgroup G_{n+1}. Then for each neighborhood $V \subset G_{n+1}$, the product $ZVZ \subset G_{n+1}$ fails to be a neighborhood of id in the group G_{n+2}.

It is clear that each balanced triple of groups is bi-balanced. The converse implication is not true.

EXAMPLE 6.3. In the group $G = GL(3, \boldsymbol{R})$ of non-degenerated 3×3-matrices consider the subgroups

$$\Gamma = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}.$$

It is easy to check that the triple (G, Γ, H) is bi-balanced but not balanced.

This example can be generalized in order to obtain a natural example of a bi-balanced but not balanced tower of topological groups.

EXAMPLE 6.4. For every $n \in \boldsymbol{N}$ identify each matrix $A \in GL(n, \boldsymbol{R})$ with the matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL(n + 1, \boldsymbol{R})\,.$$

Thus we obtain an embedding $GL(n, \boldsymbol{R}) \hookrightarrow GL(n+1, \boldsymbol{R})$. It can be shown that the tower $\big(GL(n, \boldsymbol{R})\big)_{n \in N}$ satisfies PTA (because it consists of locally compact groups), is bi-balanced but not balanced.

7. Direct limits in the category of uniform spaces. In this section we shall discuss the notion of the direct limit in the category of uniform spaces and their uniformly continuous maps. In Section 9 we shall apply those results to show that for a tower $(G_n)_{n\in\omega}$ of topological groups the topologies $\overrightarrow{\tau}$ and $\overleftarrow{\tau}$ on the union $G = \bigcup_{n\in\omega} G_n$ are generated by uniformities of direct limits of the groups G_n endowed with the left and right uniformities.

Fundamenta of the theory of uniform spaces can be found in [9, Ch.8]. Uniformities on groups were thoroughly discussed in [15] and [1, §1.8]. In the sequel, for a uniform space X, we shall denote by $\mathcal{U}_X$ the uniformity of X.

Let

$$X_0 \to X_1 \to X_2 \to \cdots$$

be a sequence of uniform spaces and their injective uniformly continuous maps. We shall identify each space X_n with a subset of the uniform space X_{n+1}, carrying its own uniformity, which is stronger than that inherited from X_{n+1}. By the uniform direct limit u-$\varinjlim X_n$ of the sequence of uniform spaces $(X_n)_{n\in\omega}$ we understand the union $X = \bigcup_{n\in\omega} X_n$ endowed with the strongest (not necessarily separated) uniformity turning the identity inclusions $X_n \to X$, $n \in \omega$, into uniformly continuous maps.

A sequence

$$X_0 \to X_1 \to X_2 \to \cdots$$

of uniform spaces is called a *tower of uniform spaces* if each uniform space X_n is a subspace of the uniform space X_{n+1}, so the identity inclusion $X_n \to X_{n+1}$ is a uniform embedding.

The uniformity of the uniform direct limit u-$\varinjlim X_n$ of a tower $(X_n)_{n\in\omega}$ of uniform spaces was described in [5] with help of uniform pseudometrics.

Let us recall that a pseudometric on a uniform space Y is *uniform* if for every $\varepsilon > 0$ the set

$$\{d < \varepsilon\} := \{(x, y) \in Y\ ;\ d(x, y) < \varepsilon\}$$

belongs to the uniformity $\mathcal{U}_Y$ of Y. By [9, 8.1.10], the uniformity $\mathcal{U}_Y$ of a uniform space Y is generated by the family $\mathcal{PM}_Y$ of all uniform pseudometrics on Y in the sense that the sets $\{d < 1\}$, $d \in \mathcal{PM}_Y$, form a base of the uniformity $\mathcal{U}_Y$.

Let $(X_n)_{n\in\omega}$ be a tower of uniform spaces. A sequence of pseudometric $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{PM}_{X_n}$ is called *monotone* if $d_n \le d_{n+1}|X_n^2$ for every $n \in \omega$. Let

$$\bigwedge_{n\in\omega} \mathcal{PM}_{X_n} = \{(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{PM}_{X_n}\ ;\ (d_n)_{n\in\omega} \text{ is monotone}\}$$

be the subspace of Cartesian product, consisting of monotone sequences of uniform pseudometrics on the uniform spaces X_n.

A family $\mathcal{A} \subset \bigwedge_{n\in\omega} \mathcal{PM}_{X_n}$ is defined to be *adequate* if for each sequence of entourages $(U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}_{X_n}$ there is a monotone sequence of uniform pseudometrics $(d_n)_{n\in\omega} \in \mathcal{A}$ such that $\{d_n < 1\} \subset U_n$ for all $n \in \omega$.

The following proposition proved in [5] shows that adequate families exist.

PROPOSITION 7.1. *For any tower $(X_n)_{n\in\omega}$ of uniform spaces the family $\mathcal{A} = \bigwedge_{n\in\omega} \mathcal{PM}_{X_n}$ is adequate.*

For a point $x \in X = \bigcup_{n\in\omega} X_n$ let $|x| = \min\{n \in \omega\,;\ x \in X_n\}$ be the *height* of x in X. For two points $x, y \in X$ put $|x, y| = \max\{|x|, |y|\}$. Now we define a limit operator $\varinjlim$ assigning to each sequence of pseudometrics $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{PM}_{X_n}$ the pseudometric $d_\infty = \varinjlim d_n$ on X defined by the formula

$$d_\infty(x, y) = \inf\Big\{\sum_{i=1}^{m} d_{|x_{i-1},x_i|}(x_{i-1}, x_i)\,;\ x = x_0, x_1, \dots, x_n = y\Big\}.$$

In fact, the pseudometric $\varinjlim d_n$ is well-defined for any functions $d_n : X_n \times X_n \to [0, \infty)$, $n \in \omega$, such that $d_n(x, x) = 0$ and $d_n(x, y) = d_n(y, x)$ for all $x, y \in X_n$.

The following theorem proved in [5] describes the uniformity of uniform direct limits.

THEOREM 7.2. *For a tower of uniform spaces $(X_n)_{n\in\omega}$ and an adequate family $\mathcal{A} \subset \bigwedge_{n\in\omega} \mathcal{PM}_{X_n}$ the uniformity of the uniform direct limit* u-$\varinjlim X_n$ *is generated by the family of pseudometrics $\{\varinjlim d_n\,;\ (d_n)_{n\in\omega} \in \mathcal{A}\}$.*

Theorem 7.2 implies a simple description of the topology of the uniform limit u-$\varinjlim X_n$ also given in [5]. Given two subsets $U, V \subset X^2$ of the square of X, consider their composition (as relations):

$$A \circ B = \{(x, z) \in X^2\,;\ \text{ there is } y \in X \text{ such that } (x, y) \in A \text{ and } (y, z) \in B\ \}.$$

This operation can be extended to finite and infinite sequences of subsets $(A_n)_{n\in\omega}$ of X^2 by the formula

$$\sum_{n\ge k} A_n = \bigcup_{n\ge k} A_k \circ A_{k+1} \circ \cdots \circ A_n\,.$$

For a point x of a set X and a subset $U \subset X^2$ let $B(x, U) = \{y \in X\,;\ (x, y) \in U\}$ be the U-ball centered at x. We recall that for a point x of the union $X = \bigcup_{n\in\omega} X_n$ of a tower (X_n), we denote by $|x|$ the height $\min\{n \in \omega\,;\ x \in X_n\}$ of x in X.

THEOREM 7.3. *The topology of the uniform direct limit* u-$\varinjlim X_n$ *of a tower of uniform spaces (X_n) is generated by the base*

$$\mathcal{B} = \Big\{B\big(x; \sum_{n\ge|x|} U_n\big)\,;\ x \in X,\ (U_n)_{n\ge|x|} \in \prod_{n\ge|x|} \mathcal{U}_{X_n}\Big\}.$$

8. Uniformities on groups. In this section we discuss some natural uniformities on topological groups. For more information on this subject, see [15] and [1, §1.8].

Let us recall that each topological group G carries four natural uniformities:

(1) the *left uniformity* $\mathcal{U}^{\mathrm{L}}$, generated by the entourages $U^{\mathrm{L}} = \{(x, y) \in G\,;\ x \in yU\}$ where $U \in \mathcal{B}_e$;

(2) the *right uniformity* $\mathcal{U}^{\mathrm{R}}$, generated by the entourages $U^{\mathrm{R}} = \{(x, y) \in G\,;\ x \in Uy\}$ where $U \in \mathcal{B}_e$;

(3) the *two-sided uniformity* $\mathcal{U}^{\mathrm{LR}}$, generated by the entourages $U^{\mathrm{LR}} = \{(x, y) \in G\,;\ x \in yU \cap Uy\}$ with $U \in \mathcal{B}_e$;

(4) the *Roelcke uniformity* $\mathcal{U}^{\mathrm{RL}}$, generated by the entourages $U^{\mathrm{RL}} = \{(x, y) \in G\,;\ x \in UyU\}$ with $U \in \mathcal{B}_e$.

Here $\mathcal{B}_e$ stands for the family of open symmetric neighborhoods $U = U^{-1} \subset G$ of the neutral element e in the topological group G.

The group G endowed with the uniformity $\mathcal{U}^{\mathrm{L}}, \mathcal{U}^{\mathrm{R}}, \mathcal{U}^{\mathrm{LR}}$ or $\mathcal{U}^{\mathrm{RL}}$ will be denoted by G^{L}, G^{R}, G^{LR} or G^{RL}, respectively. It follows from the definition of those uniformities that the identity maps in the following diagram are uniformly continuous:

$$\begin{tikzcd} & G^{\mathrm{R}} \arrow[dr] & \\ G^{\mathrm{LR}} \arrow[ur] \arrow[dr] & & G^{\mathrm{RL}} \\ & G^{\mathrm{L}} \arrow[ur] & \end{tikzcd}$$

Any isomorphic topological embedding $H \hookrightarrow G$ of topological groups induces uniform embeddings

$$H^{\mathrm{L}} \hookrightarrow G^{\mathrm{L}}, \quad H^{\mathrm{R}} \hookrightarrow G^{\mathrm{R}}, \quad H^{\mathrm{LR}} \hookrightarrow G^{\mathrm{LR}}$$

of the corresponding uniform spaces (see [1, Proposition 1.8.4]). For the Roelcke uniformity the induced map $H^{\mathrm{RL}} \to G^{\mathrm{RL}}$ is merely uniformly continuous, but is not necessarily a uniform embedding (see [17]).

Let us observe that G^{LR}, G^{L}, G^{R}, G^{RL} are groups endowed with uniformities which are tightly connected with their algebraic structure.

By analogy with semitopological and quasitopological groups, let us define a group G endowed with a uniformity to be

- a *semiuniform group* if left and right shifts on G are uniformly continuous;
- a *quasiuniform group* if G is a semiuniform group with uniformly continuous inversion;
- a *uniform group* if G is a quasiuniform group with uniformly continuous multiplication $G \times G \to G$, $(x, y) \mapsto xy$.

The groups G^{L}, G^{R}, G^{LR}, G^{RL} are basic examples of groups endowed with a uniformity. Some elementary properties of those groups are presented in the following two propositions whose proofs are left to the interested reader (cf. [1, Corollary 1.8.16]).

PROPOSITION 8.1. *For any topological group G*

(1) *G^{L} and G^{R} are semiuniform topological groups*;

(2) G^{LR} *and* G^{RL} *are quasiuniform topological groups.*

PROPOSITION 8.2. *For a topological group G the following conditions are equivalent*:

(1) G^{L} *is a quasiuniform group*;
(2) G^{L} *is a uniform group*;
(3) G^{R} *is a quasiuniform group*;
(4) G^{R} *is a uniform group*;
(5) G^{LR} *is a uniform group*;
(6) G^{RL} *is a uniform group*;
(7) *the left and right uniformities on G coincide*;
(8) *the group G is balanced.*

9. The uniform structure of the semitopological groups $\overrightarrow{G}$, $\overleftarrow{G}$, and $\overset{\rightleftarrows}{G}$. Each tower of topological groups $(G_n)_{n\in\omega}$ induces four ascending sequences of uniform spaces $(G_n^{\mathrm{L}})_{n\in\omega}$, $(G_n^{\mathrm{R}})_{n\in\omega}$, $(G_n^{\mathrm{LR}})_{n\in\omega}$, $(G_n^{\mathrm{RL}})_{n\in\omega}$. The direct limits of these sequences in the category of uniform spaces are denoted by

$$\text{u-}\varinjlim G_n^{\mathrm{L}}, \quad \text{u-}\varinjlim G_n^{\mathrm{R}}, \quad \text{u-}\varinjlim G_n^{\mathrm{LR}}, \quad \text{and} \quad \text{u-}\varinjlim G_n^{\mathrm{RL}},$$

respectively.

These uniform spaces endowed with the group operation inherited from $G = \bigcup_{n\in\omega} G_n$ are semiuniform groups. The uniform continuity of the left and right shifts follows from the uniform continuity of the left and right shifts on the semiuniform groups G_n^{L}, G_n^{R}, G_n^{LR}, G_n^{RL}, $n \in \omega$. Moreover, the semiuniform groups u-$\varinjlim G_n^{\mathrm{LR}}$ and u-$\varinjlim G_n^{\mathrm{RL}}$ are quasiuniform because so are the groups G_n^{LR} and G_n^{RL}, $n \in \omega$.

The uniform continuity of the identity maps

$$\begin{tikzcd}
 & G_n^{\mathrm{R}} \arrow[dr] & & \\
G_n^{\mathrm{LR}} \arrow[ur] \arrow[dr] & & G_n^{\mathrm{RL}} \arrow[r] & (\text{g-}\varinjlim G_n)^{\mathrm{RL}} \\
 & G_n^{\mathrm{L}} \arrow[ur] & &
\end{tikzcd}$$

for all $n \in \omega$ implies the uniform continuity of the identity maps:

$$\begin{tikzcd}
 & \text{u-}\varinjlim G_n^{\mathrm{R}} \arrow[dr] & & \\
\text{u-}\varinjlim G_n^{\mathrm{LR}} \arrow[ur] \arrow[dr] & & \text{u-}\varinjlim G_n^{\mathrm{RL}} \arrow[r] & (\text{g-}\varinjlim G_n)^{\mathrm{RL}}. \\
 & \text{u-}\varinjlim G_n^{\mathrm{L}} \arrow[ur] & &
\end{tikzcd}$$

Theorems 7.2 and 7.3 imply the following description of the uniform and topological structure of the uniform limit u-$\varinjlim G_n^{\mathrm{L}}$.

THEOREM 9.1. *For a tower of topological groups* $(G_n)_{n\in\omega}$

(1) *the topology of the semiuniform group* u-$\varinjlim G_n^{\mathrm{L}}$ *coincides with the topology* $\vec{\tau}$ *on the group* $G = \bigcup_{n\in\omega} G_n$;

(2) *the uniformity of* u-$\varinjlim G_n^{\mathrm{L}}$ *is generated by the family of pseudometrics*

$$\{\varinjlim d_n\ ;\ (d_n)_{n\in\omega} \in \mathcal{A}\}$$

for any adequate family $\mathcal{A} \subset \bigwedge_{n\in\omega} \mathcal{PM}_{G_n}$.

This theorem allows us to identify the semitopological group $\vec{G}$ with the semiuniform group u-$\varinjlim G_n^{\mathrm{L}}$. In the same way we shall identify the semitopological group $\overleftarrow{G}$ with the semiuniform group u-$\varinjlim G_n^{\mathrm{R}}$.

The semitopological group $\overset{\rightleftarrows}{G}$ is a quasiuniform group with respect to the uniformity inherited from the product $\overleftarrow{G} \times \vec{G}$ by the diagonal embedding

$$\overset{\rightleftarrows}{G} \hookrightarrow \overleftarrow{G} \times \vec{G}\,, \quad x \mapsto (x, x)\,.$$

The uniform continuity of the identity maps

$$\text{u-}\varinjlim G_n^{\mathrm{LR}} \to \text{u-}\varinjlim G_n^{\mathrm{L}} = \vec{G} \quad \text{and} \quad \text{u-}\varinjlim G_n^{\mathrm{LR}} \to \text{u-}\varinjlim G_n^{\mathrm{R}} = \overleftarrow{G}$$

yields the uniform continuity of the identity map u-$\varinjlim G_n^{\mathrm{LR}} \to \overset{\rightleftarrows}{G}$.

Now we discuss the interplay between the semitopological group $\overleftrightarrow{G}$ and the semiuniform group u-$\varinjlim G_n^{\mathrm{RL}}$. Since the topological embedding $G_n^{\mathrm{RL}} \to G_{n+1}^{\mathrm{RL}}$ is not a uniform embedding in general, Theorems 7.2 and 7.3 cannot be applied to describe the uniform and topological structures of the uniform direct limit u-$\varinjlim G_n^{\mathrm{RL}}$. Hence, this case requires a special treatment.

Given a pseudometric d on a group H, let d^{-1} be the *mirror pseudometric* defined by

$$d^{-1}(x, y) = d(x^{-1}, y^{-1}) \quad \text{for } x, y \in H\,.$$

THEOREM 9.2. *For a tower of topological groups* $(G_n)_{n\in\omega}$,

(1) *the uniformity of the uniform direct limit* u-$\varinjlim G_n^{\mathrm{RL}}$ *is generated by the family of pseudometrics* $\{\varinjlim \min\{d_n, d_n^{-1}\}\ ;\ (d_n)_{n\in\omega} \in \bigwedge_{n\in\omega} \mathcal{PM}_{G_n^{\mathrm{L}}}\}$;

(2) *the uniformity of* u-$\varinjlim G_n^{\mathrm{RL}}$ *coincides with the strongest uniformity on the group* $G = \bigcup_{n\in\omega} G_n$ *such that the identity maps* $\vec{G} \to G$ *and* $\overleftarrow{G} \to G$ *are uniformly continuous*;

(3) *the identity map* $\overleftrightarrow{G} \to$ u-$\varinjlim G_n^{\mathrm{RL}}$ *is continuous*;

(4) *the identity map* $\overleftrightarrow{G} \to$ u-$\varinjlim G_n^{\mathrm{RL}}$ *is a homeomorphism if* $\overleftrightarrow{G}$ *is a topological group or if each identity inclusion* $G_n^{\mathrm{RL}} \to G_{n+1}^{\mathrm{RL}}$, $n \in \omega$, *is a uniform embedding*;

(5) u-$\varinjlim G_n^{\mathrm{RL}}$ *is a topological group if and only if the identity map* u-$\varinjlim G_n^{\mathrm{RL}} \to$ g-$\varinjlim G_n$ *is a homeomorphism.*

PROOF. (1) First we show that for any monotone sequence of pseudometrics $(d_n)_{n\in\omega} \in \bigwedge_{n\in\omega} \mathcal{PM}_{G_n^{\mathrm{L}}}$ the pseudometric $d_\infty = \varinjlim \min\{d_n, d_n^{-1}\}$ is uniform on $G = \text{u-}\varinjlim G_n^{\mathrm{RL}}$. For this it suffices to check that d_∞ is uniform on each quasiuniform group G_n^{RL}, $n \in \omega$.

Fix any $\varepsilon > 0$. Since each pseudometric $d_n \in \mathcal{PM}_{G_n^{\mathrm{L}}}$ is uniform with respect to the left uniformity on the topological group G_n, there is an open symmetric neighborhood $U_n \subset G_n$ of e such that $U_n^{\mathrm{L}} \subset \{d_n < \varepsilon/2\}$. After inversion, we get the inclusion $U_n^{\mathrm{R}} \subset \{d_n^{-1} < \varepsilon/2\}$. We claim that $U_n^{\mathrm{RL}} \subset \{d_\infty|G_n^2 < \varepsilon\}$. Take any points $(x, y) \in U_n^{\mathrm{RL}}$. Then $y = uxv$ for some $u, v \in U_n$. Consider the chain of points $x_0 = x$, $x_1 = ux$, $x_2 = uxv = y$ and observe that

$$\begin{aligned}
d_\infty(x, y) &\le \min\{d_{|x_0,x_1|}(x_0, x_1), d^{-1}_{|x_0,x_1|}(x_0, x_1)\} + \min\{d_{|x_1,x_2|}(x_1, x_2), d^{-1}_{|x_1,x_2|}(x_1, x_2)\} \\
&\le \min\{d_n(x, ux), d_n^{-1}(x, ux)\} + \min\{d_n(ux, uxv), d_n^{-1}(ux, uxv)\} \\
&\le d_n(x^{-1}, x^{-1}u^{-1}) + d_n(ux, uxv) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,
\end{aligned}$$

witnessing that the pseudometric d_∞ is uniform.

Now given any entourage $U \subset G^2$ that belongs to the uniformity of the space $\text{u-}\varinjlim G_n^{\mathrm{RL}}$, we shall find a monotone sequence $(d_n)_{n\in\omega} \in \bigwedge_{n\in\omega} \mathcal{PM}_{G_n^{\mathrm{L}}}$ such that $\{d_\infty < 1\} \subset U$ for the limit pseudometric $d_\infty = \varinjlim \min\{d_n, d_n^{-1}\}$.

By [9, 8.1.10], there is a uniform pseudometric d on $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ such that $\{d < 1\} \subset U$. Since the inversion is uniformly continuous on $\text{u-}\varinjlim G_n^{\mathrm{RL}}$, the pseudometric $\rho = \max\{d, d^{-1}\}$ on $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ is uniform. Now observe that for every $n \in \omega$ the pseudometric $d_n = \rho|(G_n^{\mathrm{RL}})^2$ on G_n^{RL} is uniform and the sequence $(d_n)_{n\in\omega}$ is monotone. The triangle inequality and the definition of the pseudometric $d_\infty = \varinjlim \min\{d_n, d_n^{-1}\} = \varinjlim d_n$ implies $d_\infty = \rho$. In this case we have $\{d_\infty < 1\} = \{\rho < 1\} \subset \{d < 1\} \subset U$.

(2) Let $\mathcal{U}$ be the largest uniformity on G such that the identity maps $\vec{G} \to (G, \mathcal{U})$ and $\overleftarrow{G} \to (G, \mathcal{U})$ are uniformly continuous. The definition of $\mathcal{U}$ implies that $\mathcal{U}$ coincides with its mirror uniformity $\mathcal{U}^-$ consisting of the sets $U^- = \{(x^{-1}, y^{-1})\,;\ (x, y) \in U\}$, $U \in \mathcal{U}$.

We need to show that $\mathcal{U}$ coincides with the uniformity of $\text{u-}\varinjlim G_n^{\mathrm{RL}}$. The uniform continuity of the identity maps from $\vec{G}$ and $\overleftarrow{G}$ into $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ implies that $\mathcal{U}$ is larger than the uniformity of $\text{u-}\varinjlim G_n^{\mathrm{RL}}$. It remains to prove that each entourage $U \in \mathcal{U}$ belongs to the uniformity of $\text{u-}\varinjlim G_n^{\mathrm{RL}}$. By [9, 8.1.10], there is a uniform pseudometric d on $(G, \mathcal{U})$ such that $\{d < 1\} \subset U$. Since the inversion on G is uniformly continuous with respect to the uniformity $\mathcal{U}$, the mirror pseudometric d^{-1} is uniform on $(G, \mathcal{U})$ and so is the pseudometric $\rho = \max\{d, d^{-1}\}$. Now we see that for each $n \in \omega$ the restriction $d_n = \rho|G_n^2$ belongs to the family $\mathcal{PM}_{G_n^{\mathrm{L}}}$ and is equal to its mirror pseudometric d_n^{-1}. The sequence $(d_n)_{n\in\omega}$ belongs to $\bigwedge_{n\in\omega} \mathcal{PM}_{G_n^{\mathrm{L}}}$, and the definition of the pseudometric d_∞ implies that $d_\infty = \rho$. By the first item, the pseudometric d_∞ is uniform on $\text{u-}\varinjlim G_n^{\mathrm{LR}}$ and consequently, the entourage

$$U \supset \{d < 1\} \supset \{\rho < 1\} = \{d_\infty < 1\}$$

belongs to the uniformity of the space $\text{u-}\varinjlim G_n^{\mathrm{RL}}$.

(3) Since $\overleftrightarrow{G}$ and $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ are semitopological groups, the continuity of the identity map $\overleftrightarrow{G} \to \text{u-}\varinjlim G_n^{\mathrm{RL}}$ is equivalent to its continuity at the neutral element e.

Given a neighborhood $O^{\mathrm{RL}}(e) \subset \text{u-}\varinjlim G_n^{\mathrm{RL}}$ of e, find a uniform pseudometric d on $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ such that $\{x \in G;\ d(x,e) < 1\} \subset O^{\mathrm{RL}}(e)$. For every $n \in \omega$ the identity map $G_n^{\mathrm{RL}} \to \text{u-}\varinjlim G_n^{\mathrm{RL}}$ is uniformly continuous, so we can find a symmetric neighborhood $U_n \subset G_n$ of e such that $U_n^{\mathrm{RL}} \subset \{d|G_n^2 < 1/2^{n+1}\}$. By the definition of the topology $\overleftrightarrow{\tau}$ of the quasitopological group $\overleftrightarrow{G}$, the set $\overleftrightarrow{\prod}_{n\in\omega} U_n$ is a neighborhood of e in $\overleftrightarrow{G}$. We claim that $\overleftrightarrow{\prod}_{n\in\omega} U_n$ is included in $O^{\mathrm{RL}}(e)$. Given any point $z \in \overleftrightarrow{\prod}_{n\in\omega} U_n$, find two points $x \in \overleftarrow{\prod}_{n\in\omega} U_n$ and $y \in \overrightarrow{\prod}_{n\in\omega} U_n$ with $z = xy$.

By the definition of the directed products $\overleftarrow{\prod}_{n\in\omega} U_n$ and $\overrightarrow{\prod}_{n\in\omega} U_n$, there are chains of points $e = x_0, x_1, \dots, x_m = x$ and $e = y_0, y_1, \dots, y_m$ in G such that $x_{i+1} \in U_i x_i$ and $y_{i+1} \in y_i U_i$ for all $i < m$. Now consider the chain $e = x_0y_0, x_1y_1, \dots, x_my_m = xy$ linking the points e and $z = xy$. Observe that for every $i < m$, $x_{i+1}y_{i+1} \in U_i x_i y_i U_i$ implies $(x_{i+1}y_{i+1}, x_iy_i) \in U_i^{\mathrm{RL}}$ and hence $d(x_{i+1}y_{i+1}, x_iy_i) < 1/2^{i+1}$ by the choice of the neighborhood U_i. Consequently,

$$d(e,xy) \le \sum_{i<m} d(x_iy_i, x_{i+1}y_{i+1}) \le \sum_{i<m} \frac{1}{2^{i+1}} < 1$$

and $xy \in O^{\mathrm{RL}}(e)$ by the choice of the pseudometric d.

(4) If $\overleftrightarrow{G}$ is a topological group, then the identity map $\overleftrightarrow{G} \to \text{g-}\varinjlim G_n$ is a homeomorphism by Theorem 2.2. By the preceding item, the identity map $\overleftrightarrow{G} \to \text{u-}\varinjlim G_n^{\mathrm{RL}}$ is continuous, and has continuous inverse, which is the composition of two continuous maps $\text{u-}\varinjlim G_n^{\mathrm{RL}} \to \text{g-}\varinjlim G_n \to \overleftrightarrow{G}$.

Assume that the identity maps $G_n^{\mathrm{RL}} \to G_{n+1}^{\mathrm{RL}}$, $n \in \omega$, are uniform embeddings. In this case Theorem 7.3 implies that the uniform direct limit $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ has the family

$$\Big\{B\big(e; \sum_{n\in\omega} U_n^{\mathrm{RL}}\big);\ (U_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{B}_n\Big\}$$

as a neighborhood base at e. Since each set $B(e; \sum_{n\in\omega} U_n^{\mathrm{RL}})$ coincides with $\overleftrightarrow{\prod}_{n\in\omega} U_n$, we see that the topologies of the semitopological groups $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ and $\overleftrightarrow{G}$ coincide at e and thus coincide everywhere.

(5) If $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ is a topological group, then the identity map $\text{g-}\varinjlim G_n \to \text{u-}\varinjlim G_n^{\mathrm{RL}}$ is continuous because its restrictions to the groups G_n are continuous. The inverse identity map $\text{u-}\varinjlim G_n^{\mathrm{RL}} \to \text{g-}\varinjlim G_n$ is continuous because it is uniformly continuous as the identity map into the topological group $(\text{g-}\varinjlim G_n)^{\mathrm{RL}}$ endowed with the Roelcke uniformity.

If the identity map $\text{u-}\varinjlim G_n^{\mathrm{RL}} \to \text{g-}\varinjlim G_n$ is a homeomorphism, then $\text{u-}\varinjlim G_n^{\mathrm{RL}}$ is a topological group because $\text{g-}\varinjlim G_n$ is a topological group. $\square$

10. Open problems. Summing up, we conclude that, for any tower of topological groups $(G_n)_{n\in\omega}$,

- the direct limit g-$\varinjlim G_n$ is a topological group,
- t-$\varinjlim G_n$ and $\overset{\leftrightarrow}{G}$ are quasitopological groups,
- $\vec{G} =$ u-$\varinjlim G_n^{\mathrm{L}}$ and $\overset{\leftarrow}{G} =$ u-$\varinjlim G_n^{\mathrm{R}}$ are semiuniform groups, and
- $\overset{\rightleftarrows}{G}$, u-$\varinjlim G_n^{\mathrm{LR}}$ and u-$\varinjlim G_n^{\mathrm{RL}}$ are quasiuniform groups,

having the union $G = \bigcup_{n\in\omega} G_n$ as their underlying group.

The interplay between these semitopological and semiuniform groups are described in the following diagram. A simple (resp. double) arrow indicates that the corresponding identity map is continuous (resp. uniformly continuous).

$$\begin{tikzcd}
 & & & \text{u-}\varinjlim G_n^{\mathrm{L}} \arrow[ddrr] \arrow[d, <<->>] & & & \\
 & & & \vec{G} \arrow[dr] & & & \\
\text{t-}\varinjlim G_n \arrow[r] & \text{u-}\varinjlim G_n^{\mathrm{LR}} \arrow[uurr] \arrow[ddrr] \arrow[r, ->>] & \overset{\rightleftarrows}{G} \arrow[ur] \arrow[dr] & & \overset{\leftrightarrow}{G} \arrow[r] & \text{u-}\varinjlim G_n^{\mathrm{RL}} \arrow[r] & \text{g-}\varinjlim G_n \\
 & & & \overset{\leftarrow}{G} \arrow[ur] & & & \\
 & & & \text{u-}\varinjlim G_n^{\mathrm{R}} \arrow[uurr] \arrow[u, <<->>] & & &
\end{tikzcd}$$

Under certain conditions on the tower $(G_n)_{n\in\omega}$ some of the identity maps in this diagram are homeomorphisms. In particular,

- t-$\varinjlim G_n \to$ g-$\varinjlim G_n$ is a homeomorphism if all topological groups G_n, $n \in \omega$, are locally compact [16];
- u-$\varinjlim G_n^{\mathrm{LR}} \to$ g-$\varinjlim G_n$ is a homeomorphism if all topological groups G_n, $n \in \omega$ are balanced [5];
- $\overset{\rightleftarrows}{G} \to$ g-$\varinjlim G_n$ is a homeomorphism if the tower $(G_n)_{n\in\omega}$ is balanced or satisfies PTA (Theorems 4.2 and 3.2);
- $\overset{\leftrightarrow}{G} \to$ g-$\varinjlim G_n$ is a homeomorphism if the tower $(G_n)_{n\in\omega}$ is bi-balanced (Theorem 5.2).

Nonetheless many open questions related to this diagram remain unsolved.

PROBLEM 10.1. Is the identity map $\overset{\rightleftarrows}{G} \to$ u-$\varinjlim G_n^{\mathrm{LR}}$ (uniformly) continuous?

PROBLEM 10.2. What can be said about separation properties of the quasitopological group $\overleftrightarrow{G}$? Is it always Tychonoff? Is the identity map $\overleftrightarrow{G} \to \text{u-}\varinjlim G_n^{\mathrm{RL}}$ a homeomorphism?

We define a topological space X to be *Tychonoff* if for each closed subset $F \subset X$ and each point $x \in X \setminus F$ there is a continuous function $f : X \to \boldsymbol{R}$ with $f(x) = 1$ and $f(F) \subset \{0\}$. It is known that each uniform (not necessarily separated) space is Tychonoff. In particular, each semiuniform group is Tychonoff.

Surprisingly, we know no (natural) example of a tower of topological groups $(G_n)_{n\in\omega}$ for which the topology of $\text{g-}\varinjlim G_n$ would be different from $\overleftrightarrow{\tau}$ or even $\overset{\rightleftarrows}{\tau}$. However, we expect counterexamples to the following problem.

PROBLEM 10.3. Is the identity map $\text{u-}\varinjlim G_n^{\mathrm{RL}} \to \text{g-}\varinjlim G_n$ a homeomorphism? What about the identity map $\text{u-}\varinjlim G_n^{\mathrm{LR}} \to \text{g-}\varinjlim G_n$?

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