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On the length of chains of proper subgroups covering a topological group

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Abstract We prove that if an ultrafilter $\mathcal{L}$ is not coherent to a Q-point, then each analytic non-σ-bounded topological group G admits an increasing chain $\langle G_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ of its proper subgroups such that: (i) $\bigcup_\alpha G_\alpha = G$; and (ii) For every σ-bounded subgroup H of G there exists α such that $H \subset G_\alpha$. In case of the group $\mathrm{Sym}(\omega)$ of

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all permutations of ω with the topology inherited from ω^ω this improves upon earlier results of S. Thomas.

Keywords Q-points · P_κ-point · σ-bounded group · ω-bounded group · Menger property · $[\mathcal{F}]$-Menger property

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1 Introduction

A theorem of Macpherson and Neumann [13] states that if the group $\mathrm{Sym}(\omega)$ can be written as a union of an increasing chain $\langle G_i : i < \lambda \rangle$ of proper subgroups G_i, then $\lambda > \omega$. Throughout this paper the minimal λ with this property will be denoted by $\mathrm{cf}(\mathrm{Sym}(\omega))$. For every increasing function $f \in \omega^\omega$ we denote by S_f the subgroup of $\mathrm{Sym}(\omega)$ generated by $\{\pi \in \mathrm{Sym}(\omega) : \pi, \pi^{-1} \leq^* f\}$, where $x \leq^* y$ means that $x(n) \leq y(n)$ for all but finitely many $n \in \omega$. If we additionally require that for every $f \in \omega^\omega$ there exists $i \in \lambda$ such that $S_f \subset G_i$, then the minimal length of such a chain will be denoted by $\mathrm{cf}^*(\mathrm{Sym}(\omega))$. It is clear that $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \geq \max\{\mathrm{cf}(\mathrm{Sym}(\omega)), \mathfrak{b}\}$. The consistency of $\mathrm{cf}^*(\mathrm{Sym}(\omega)) > \mathrm{cf}(\mathrm{Sym}(\omega))$ and the inequality $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \leq \mathrm{cf}(\mathfrak{d})$ were established in [18, Proposition 2.5]. The initial aim of this paper was to sharpen the latter upper bound on $\mathrm{cf}^*(\mathrm{Sym}(\omega))$. This led us to consider increasing chains of proper submonoids of topological monoids.

We recall that a *semigroup* is a set with a binary associative operation $\cdot : X \times X \to X$. A semigroup with a two-sided unit 1 is called a *monoid*. It is clear that each group is a monoid. By a *topological monoid* we understand a monoid X with a topology τ making the binary operation $\cdot : X \times X \to X$ of X continuous.

Definition 1.1 Let X be a topological monoid (resp. group). The minimal length of an increasing chain $\langle X_i : i < \lambda \rangle$ of proper submonoids (resp. subgroups) X_i of X such that $X = \bigcup_{i<\lambda} X_i$ and for every σ-bounded subset H of X there exists $i \in \lambda$ such that $H \subset X_i$ will be denoted by $\mathrm{cf}^*_m(X)$ (resp. $\mathrm{cf}^*_g(X)$).

We recall that a subset B of a topological monoid X is said to be *totally bounded,* if for every open neighborhood U of the identity 1 of X there exists a finite subset F of X such that $X \subset FU \cap UF$.

A subset B is said to be *σ-bounded,* if it can be written as a countable union of totally bounded subsets. A direct verification shows that $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ as defined in [18] and $\mathrm{cf}^*_g(\mathrm{Sym}(\omega))$ in the sense of our Definition 1.1 coincide.

It is clear that $\mathrm{cf}^*_m(X) \leq \mathrm{cf}^*_g(X)$ for every topological group X. We do not know whether these cardinals can be different. Probably the most interesting case is the group $\mathrm{Sym}(\omega)$.

Let R be a relation on ω and $x, y \in \omega^\omega$. We denote by $[x\,R\,y]$ the set $\{n \in \omega : x(n)\,R\,y(n)\}$. For an ultrafilter $\mathcal{F}$ the notation $x \leq_\mathcal{F} y$ means $[x \leq y] \in \mathcal{F}$. Let $\mathfrak{b}(\mathcal{F})$ be the cofinality of the linearly ordered set $(\omega^\omega, \leq_\mathcal{F})$.

Following [2] we define a point $x \in X$ of a topological monoid X to be *left balanced* (resp. *right balanced*) if for every neighborhood $U \subset X$ of the unit 1 of X there is a neighborhood $V \subset X$ of 1 such that $Vx \subset xU$ (resp. $xV \subset Ux$). Observe that x is left balanced if the left shift $l_x : X \to X$, $l_x : y \mapsto xy$, is open at 1. Let B_L and B_R denote respectively the sets of all left and right balanced points of the monoid X. A topological monoid X is defined to be *left balanced* (resp. *right balanced*) if $X = B_L \cdot U$ (resp. $X = U \cdot B_R$) for every neighborhood $U \subset X$ of the unit 1 in X. If a topological monoid X is both left and right balanced, then we say that X is *balanced*.

We define a topological monoid X to be a Menger monoid,[1] if for every sequence $\langle U_n : n \in \omega\rangle$ of open neighborhoods of 1 there exists a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of X such that $X = \bigcup_{n\in\omega} F_n U_n \cap U_n F_n$. A topological monoid X is said to be *ω-bounded*, if for every neighborhood U of 1 there exists a countable $C \subset X$ such that $X = C \cdot U$.

The following two theorems are the principal results of this paper.

Theorem 1.2 *Let X be a first countable ω-bounded balanced topological monoid such that one of its finite powers is not a Menger monoid. Then* $\mathrm{cf}^*_m(X) \le \mathfrak{b}(\mathcal{L})$ *for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

Theorem 1.3 *Let G be an ω-bounded topological group such that one of its finite powers is not a Menger monoid. Then* $\mathrm{cf}^*_g(G) \le \mathfrak{b}(\mathcal{L})$ *for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

Applying [2, Proposition 7.5] we conclude that the Baire space ω^ω with the operation of composition is a balanced topological monoid, and σ-bounded subsets of this topological monoid are exactly those which are contained in the σ-compact subsets of ω^ω. It is easy to see that ω^ω is not a Menger monoid. Thus we get the following

Corollary 1.4 *Let $\mathcal{L}$ be an ultrafilter coherent to no Q-point. Then ω^ω can be written as the union of an increasing chain of its proper subsets of length* $\le \mathfrak{b}(\mathcal{L})$*, each of which is closed under composition, and such that every σ-compact subset of ω^ω is contained in one of the elements of this chain.*

A metrizable space X is said to be *analytic*, if it is a continuous image of ω^ω. A topological group G is called *analytic* if such is the underlying topological space. Theorem 1.3 implies the following:

Corollary 1.5 *Let G be an analytic group which is not σ-bounded. Then* $\mathrm{cf}^*_g(G) \le \mathfrak{b}(\mathcal{L})$ *for every ultrafilter $\mathcal{L}$ which is not coherent to any Q-point.*

$\mathrm{Sym}(\omega)$ is easily seen to be a G_δ-subset of ω^ω and the composition as well as the inversion are continuous with respect to the topology inherited from ω^ω. Therefore $\mathrm{Sym}(\omega)$ with this topology is a Polish topological group. A direct verification also shows that it is not σ-bounded.

Corollary 1.6 $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \le \mathfrak{b}(\mathcal{L})$ *for every ultrafilter $\mathcal{L}$ which is not coherent to a Q-point.*

[1] In terms of [2] this means that $(X, \mu_L \wedge \mu_R)$ is a Menger monoid.

Combined with the following consequence of [12, Theorem 2.8], Corollary 1.6 yields the upper bound for $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ obtained earlier in [18].

Proposition 1.7 *There exists an ultrafilter $\mathcal{L}$ which is not coherent to any Q-point and such that* $\mathfrak{b}(\mathcal{L}) = \mathrm{cf}(\mathfrak{d})$.

We recall from [5] that ultrafilters $\mathcal{F}$ and $\mathcal{U}$ on ω are said to be *nearly coherent*, if there exists an increasing sequence $\langle k_n : n \in \omega\rangle$ of natural numbers such that $\bigcup_{n\in I}[k_n, k_{n+1}) \in \mathcal{F}$ if and only if $\bigcup_{n\in I}[k_n, k_{n+1}) \in \mathcal{U}$ for every subset I of ω. In what follows we shall drop "near" and simply say that two ultrafilters are coherent. In other words, $\mathcal{F}$ and $\mathcal{U}$ are coherent if and only if $\phi(\mathcal{F}) = \phi(\mathcal{U})$ for some increasing surjection $\phi : \omega \to \omega$. The coherence relation is an equivalence relation. NCF is the statement that all ultrafilters are coherent. Its consistence was established in [7].

An ultrafilter $\mathcal{L}$ is called:

- *a (pseudo-) P_κ-point*, where κ is a cardinal, if for every $\mathcal{L}' \in [\mathcal{L}]^{<\kappa}$ there exists $L \in \mathcal{L}$ (resp. $L \in [\omega]^\omega$) such that $L \subset^* L'$ for all $L' \in \mathcal{L}'$. P_{ω_1}-points are also called P-points;
- *a simple P_κ-point*, if there exists a sequence $\langle L_\alpha : \alpha < \kappa\rangle$ of infinite subsets of ω such that $L_\alpha \subset^* L_\beta$ for all $\kappa > \alpha > \beta$ and $\mathcal{L} = \{X \subset \omega : L_\alpha \subset X$ for some $\alpha < \kappa\}$;
- *a Q-point*, if for every increasing surjection $\phi : \omega \to \omega$ there exists $L \in \mathcal{L}$ such that $\phi \restriction L$ is injective;
- *a Ramsey ultrafilter*, if it is simultaneously both a P- and a Q-point.

Corollary 1.6 implies the following statements.

Corollary 1.8 *Suppose that there exists a pseudo-$P_{\mathfrak{b}^+}$-point. Then* $\mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{b}$.

Corollary 1.9 *Suppose that* $\mathfrak{u} < \mathrm{cf}^*(\mathrm{Sym}(\omega))$. *Every two ultrafilters that are not coherent to Q-points are coherent. In particular, if there is no Q-point, then NCF holds.*

Corollary 1.8 can be compared to the following theorem: If $\lambda < \kappa$ are regular uncountable cardinals such that there exists a simple P_λ-point $\mathcal{U}$ and a P_κ-point $\mathcal{F}$, then $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \le \lambda$ (cf. [18, Theorem 3.4]). The assumption of this theorem (whose consistency was conjectured in [7]) clearly implies that $\mathfrak{u} < \mathfrak{s}$ and $\mathcal{U}$ is not coherent to $\mathcal{F}$, and hence there are exactly two coherence classes of ultrafilters (cf. [6, Corollary 13]). The question whether there can be exactly n coherence classes of ultrafilters for $1 < n < \omega$ remains open.

On the other hand, given any ground model of GCH and a regular cardinal ν in it, the forcing from [8] with $\delta = \omega_1$ and $\nu = \kappa$ (δ and ν are the two parameters there) yields a model of "there exists a simple P_κ-point $\mathcal{U}$ and $\mathfrak{b} = \omega_1 \le 2^\omega = \kappa$". Combined with Theorem 1.3 this gives the consistency of the statement "there exists a simple P_κ-point $\mathcal{U}$ and $\omega_1 = \mathfrak{b} = \mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{b}(\mathcal{U}) < \kappa$".

We shall denote the set of all unbounded nondecreasing elements of ω^ω by $\omega^{\uparrow\omega}$. We call a set $F \subset \omega^{\uparrow\omega}$ *finitely dominating*, if for every $x \in \omega^\omega$ there exists a finite subset

$\{f_0, \ldots, f_n\}$ of F such that $x \leq^* \max\{f_0, \ldots, f_n\}$. Following [14] we denote the minimal size of a family of non-finitely dominating sets covering $\omega^{\uparrow\omega}$ by $\mathrm{cov}(\mathfrak{D}_{fin})$.

As the next theorem shows, NCF implies that $\mathrm{cf}^*(\mathrm{Sym}(\omega))$ is maximal possible.

Theorem 1.10 $\mathrm{cf}^*(\mathrm{Sym}(\omega)) \geq \mathrm{cov}(\mathfrak{D}_{fin})$. *Moreover, NCF implies that* $\mathrm{cf}^*(\mathrm{Sym}(\omega)) = \mathfrak{d}$.

Shelah and Tsaban [17] proved that $\max\{\mathfrak{b}, \mathfrak{g}\} \leq \mathrm{cov}(\mathfrak{D}_{fin})$, and the strict inequality is consistent (cf. [14]). Thus Theorem 1.10 improves the lower bound in $\mathfrak{g} \leq \mathrm{cf}^*(\mathrm{Sym}(\omega))$ [18, Theorem 2.6]. Combining Corollary 1.9 and the fact that there are no Q-points under $\mathfrak{u} < \mathfrak{s}$ (cf. [3, Theorems 13.6.2, 13.8.1]),[2] we get the following:

Corollary 1.11 *If* $\mathfrak{u} < \min\{\mathfrak{s}, \mathrm{cf}^*(\mathrm{Sym}(\omega))\}$, *then NCF holds.*

We do not know whether the inequality $\mathfrak{u} < \mathrm{cf}^*(\mathrm{Sym}(\omega))$ (or even $\mathfrak{u} < \mathrm{cf}(\mathrm{Sym}(\omega))$) implies NCF. This would be true if $\mathrm{cf}(\mathrm{Sym}(\omega)) \leq \mathfrak{mcf} = \min\{\mathfrak{b}(\mathcal{F}) : \mathcal{F}$ is an ultrafilter$\}$ (in particular, if $\mathfrak{mcf}$ is attained at some ultrafilter not coherent to a Q-point). It would also be interesting to establish whether NCF implies $\mathrm{cf}(\mathrm{Sym}(\omega)) = \mathfrak{d}$.

This work is a continuation of our previous paper [2]. We refer the reader to [19] for the definitions and basic properties of small cardinals which are used but not defined in this paper. All filters are assumed to be non-principal.

2 Proofs

The main technical tool for the proofs of Theorems 1.2 and 1.3 was developed in [2]. This will allow us to prove some stronger technical statements in this section, namely Propositions 2.5 and 2.6. In order to formulate them we need to recall some definitions.

Let $\mathcal{F}$ be a filter. Following [4] (our definition of an $[\mathcal{F}]$-cover differs slightly from the one given in [2,4], however, by [3, 5.5.2, 5.5.3] the two versions are equivalent), we define an indexed cover $\langle B_n : n \in \omega\rangle$ of a set X to be an $[\mathcal{F}]$-*cover* if there is an increasing surjection $\phi : \omega \to \omega$ such that $\phi(\{n \in \omega : x \in B_n\}) \in \mathcal{F}$ for every $x \in X$.

A subset X of a topological monoid M is defined to be $[\mathcal{F}]$-*Menger* if for every sequence $\langle U_n : n \in \omega\rangle$ of neighborhoods of 1 in M there is a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of M such that $\langle U_n \cdot F_n \cap F_n \cdot U_n : n \in \omega\rangle$ is an $[\mathcal{F}]$-cover of X. The latter happens if and only if

$$X \subset \bigcup_{F \in \mathcal{F}} \bigcap_{n \in \phi(F)} U_n \cdot F_n \cap F_n \cdot U_n$$

for some monotone surjection $\phi : \omega \to \omega$.

Definition 2.1 For a topological monoid (group) X and a free filter $\mathcal{F}$ on ω by $\mathrm{cf}^{\mathcal{F}}_m(X)$ (resp. $\mathrm{cf}^{\mathcal{F}}_g(X)$) we denote the minimal length of an increasing chain $\langle X_i : i < \lambda\rangle$

[2] A complete proof of this fact is included in the Arxiv-version of this paper, see http://arxiv.org/pdf/1011.1031.

of proper submonoids (subgroups) X_i of X such that $X = \bigcup_{i<\lambda} X_i$ and for every $[\mathcal{F}]$-Menger subset H of X there exists $i \in \lambda$ such that $H \subset X_i$.

If no such chain exists, then we say that $\mathrm{cf}_m^{\mathcal{F}}(X)$ (resp. $\mathrm{cf}_g^{\mathcal{F}}(X)$) is undefined.

It is easy to check that $\mathrm{cf}_m^*(X)$ (resp. $\mathrm{cf}_g^*(X)$) is $\mathrm{cf}_m^{\mathfrak{Fr}}(X)$ (resp. $\mathrm{cf}_g^{\mathfrak{Fr}}(X)$), where $\mathfrak{Fr}$ denotes the Fréchet filter consisting of all cofinite subsets of ω.

Let $\mathcal{F}$ be an ultrafilter. A sequence $\langle b_\alpha : \alpha < \mathfrak{b}(\mathcal{F})\rangle$ of increasing elements of ω^ω is called a $\mathfrak{b}(\mathcal{F})$-*scale*, if it is cofinal with respect to $\leq_{\mathcal{F}}$ and $b_\alpha \leq_{\mathcal{F}} b_\beta$ for all $\alpha \leq \beta < \mathfrak{b}(\mathcal{F})$.

Let us denote the family of all monotone surjections from ω to ω by $\mathcal{S}$. Following [3, §10.1] (see also [9]) we denote for an ultrafilter $\mathcal{F}$ by $\mathfrak{q}(\mathcal{F})$ the minimal size of a subfamily Φ of $\mathcal{S}$ such that for every $\psi \in \mathcal{S}$ there exists $\phi \in \Phi$ such that $[\phi \leq \psi] \in \mathcal{F}$. It is clear that there exists a sequence $\langle \phi_\alpha : \alpha < \mathfrak{q}(\mathcal{F})\rangle \in \mathcal{S}^{\mathfrak{q}(\mathcal{F})}$ such that $[\phi_\beta < \phi_\alpha] \in \mathcal{F}$ for all $\beta > \alpha$ and for every $\psi \in \mathcal{S}$ there exists α with the property $[\phi_\alpha < \psi] \in \mathcal{F}$. Such a family will be called a $\mathfrak{q}(\mathcal{F})$-*scale*.

Cardinals $\mathfrak{b}(\mathcal{F})$ and $\mathfrak{q}(\mathcal{F})$ are the cofinality and the coinitiality of the linearly ordered set $(\omega^{\uparrow\omega}, \leq_{\mathcal{F}})$, which in a certain sense makes them dual.

If an ultrafilter $\mathcal{F}$ is not coherent to any Q-point then $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$, for a proof see [10,12] or [3, 10.2.5]. On the other hand, there can be ultrafilters $\mathcal{F}$ with $\mathfrak{b}(\mathcal{F}) \neq \mathfrak{q}(\mathcal{F})$, see [9]. As we shall see later, this means that $\mathrm{cf}_g^{\mathcal{F}}(X)$ and $\mathrm{cf}_m^{\mathcal{F}}(X)$ are not always well-defined.

Theorem 2.2 *Let $\mathcal{F}$ be an ultrafilter and X a first countable ω-bounded balanced topological monoid (resp. first countable topological group) and suppose that one of its finite powers is not a Menger monoid.*

(1) *If the cardinal $\mathrm{cf}_m^{\mathcal{F}}(X)$ (resp. $\mathrm{cf}_g^{\mathcal{F}}(X)$) exists, then it is equal to $\mathfrak{b}(\mathcal{F})$ and $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is not coherent to any Q-point, then the cardinal $\mathrm{cf}_m^{\mathcal{F}}(X)$ (resp. $\mathrm{cf}_g^{\mathcal{F}}(X)$) exists and hence it is equal to $\mathfrak{b}(\mathcal{F}) = \mathfrak{q}(\mathcal{F})$.*
(3) *For the group $X = \mathrm{Auth}(\mathbb{R}_+)$ of the homeomorphisms of the half-line the cardinal $\mathrm{cf}_m^{\mathcal{F}}(X)$ exists if and only if $\mathrm{cf}_g^{\mathcal{F}}(X)$ exists if and only if $\mathcal{F}$ is not coherent to a Q-point.*

We postpone the proof of Theorem 2.2 for the moment. It is clear that for a topological group X the existence of $\mathrm{cf}_g^{\mathcal{F}}(X)$ implies the existence of $\mathrm{cf}_m^{\mathcal{F}}(X)$, and in this case $\mathrm{cf}_m^{\mathcal{F}}(X) \leq \mathrm{cf}_g^{\mathcal{F}}(X)$.

Question 2.3 Is the existence of $\mathrm{cf}_g^{\mathcal{F}}(X)$ equivalent to the existence of $\mathrm{cf}_m^{\mathcal{F}}(X)$ (at least for the group $\mathrm{Sym}(\omega)$)? Are these cardinals always equal (if they exist)?

The following result was established in [2].

Lemma 2.4 *A topological group (resp. balanced topological monoid) H is $[\mathcal{L}]$-Menger for some ultrafilter $\mathcal{L}$ coherent to no Q-point if and only if H is algebraically generated by an $[\mathcal{L}]$-Menger subspace $X \subset H$.*

The condition in Lemma 2.4 that $\mathcal{L}$ is not coherent to any Q-point is essential by [2, Theorem 6.4]. However, we do not know whether it can be omitted from Theorem 1.2, Theorem 1.3 or Corollary 1.6.

Theorem 1.2 is a special case of the following result:

Proposition 2.5 *Let X be a first countable ω-bounded balanced topological monoid such that one of its finite powers is not a Menger monoid, and let $\mathcal{F}$ be a filter on ω. If there exists an ultrafilter $\mathcal{L} \supset \mathcal{F}$ that is not coherent to any Q-point, then $\mathrm{cf}_m^{\mathcal{F}}(X)$ is well-defined and is less than or equal to $\mathfrak{b}(\mathcal{L})$.*

Proof Let $\mathcal{L} \supset \mathcal{F}$ be an ultrafilter that is not coherent to any Q-point, $\langle b_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ be a $\mathfrak{b}(\mathcal{L})$-scale, and $\langle \phi_\alpha : \alpha < \mathfrak{q}(\mathcal{L}) = \mathfrak{b}(\mathcal{L})\rangle$ be a $\mathfrak{q}(\mathcal{L})$-scale. Assume that X^k is not a Menger monoid for some $k \in \omega$. Let $\{U_n : n \in \omega\}$ be a local base at the neutral element 1 of X. Without loss of generality, we may assume that $U_{n+1}^3 \subset U_n$ for all $n \in \omega$. Applying [2, Proposition 7.1], we can additionally assume that there exists a sequence $\langle C_n : n \in \omega\rangle$ of countable subsets of X such that $U_n \cdot C_n = C_n \cdot U_n = X$ for all n, and for every $F \in [X]^{<\omega}$ there exists $F' \in [C_n]^{<\omega}$ such that $FU_{n+1} \cap U_{n+1}F \subset F'U_n \cap U_n F'$. Fix an enumeration $\{c_{n,m} : m \in \omega\}$ of C_n. For a pair $(\phi, b) \in \mathcal{S} \times \omega^\omega$ we set

$$
\begin{aligned}
Y_{\phi,b} = \bigcup_{L\in\mathcal{L}} \bigcap_{n\in L} & U_{\phi(n)} \cdot \{c_{k,m} : \phi(n) \le k \le n,\ m \le b(n)\} \\
& \cap \{c_{k,m} : \phi(n) \le k \le n,\ m \le b(n)\} \cdot U_{\phi(n)}
\end{aligned}
$$

and denote by X_α the submonoid of X generated by $Y_{\phi_\alpha, b_\alpha}$. A direct verification shows that $Y_{\phi,b}$ is an $[\mathcal{L}]$-Menger subset of X for arbitrary pair $(\phi, b) \in \mathcal{S} \times \omega^\omega$ (cf. e.g., the proof of [2, Lemma 3.2]), and hence by Lemma 2.4 X_α is an $[\mathcal{L}]$-Menger submonoid of X. Thus $\langle X_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ is an increasing sequence of $[\mathcal{L}]$-Menger submonoids of X. Since X^k is not a Menger monoid and the $[\mathcal{L}]$-Menger property is preserved by finite powers [2, Corollary 3.5], each X_α is a proper submonoid of X.

It suffices to show that each $[\mathcal{F}]$-Menger submonoid H of X is contained in some X_α. Given such H let us find an increasing $f \in \omega^\omega$ and $\phi \in \mathcal{S}$ such that

$$
H \subset \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} U_{\phi(n)} \cdot \{c_{\phi(n),m} : m \le f(n)\} \cap \{c_{\phi(n),m} : m \le f(n)\} \cdot U_{\phi(n)}.
$$

(Such f and ϕ can be easily constructed by the definition of the $[\mathcal{F}]$-Menger property.)

Choose α such that $f \le_{\mathcal{L}} b_\alpha$ and $\phi_\alpha \le_{\mathcal{L}} \phi$. We claim that $H \subset X_\alpha$. Indeed, let us fix $h \in H$ and pick $F_0 \in \mathcal{F}$ such that

$$
h \in \bigcap_{n\in F_0} U_{\phi(n)} \cdot \{c_{\phi(n),m} : m \le f(n)\} \cap \{c_{\phi(n),m} : m \le f(n)\} \cdot U_{\phi(n)}.
$$

Set $A = [\phi_\alpha \le \phi]$, $B = [f \le b_\alpha]$, and observe that $A, B \in \mathcal{L}$. Then

$$
h \in \bigcap_{n\in F_0} U_{\phi(n)} \cdot \big\{c_{\phi(n),m} : m \le f(n)\big\} \cap \big\{c_{\phi(n),m} : m \le f(n)\big\} \cdot U_{\phi(n)}
$$

$$\subset \bigcap_{n\in F_0\cap A} U_{\phi_\alpha(n)} \cdot \left\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le f(n)\right\}$$
$$\cap \left\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le f(n)\right\} \cdot U_{\phi_\alpha(n)}$$
$$\subset \bigcap_{n\in F_0\cap A\cap B} U_{\phi_\alpha(n)} \cdot \left\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le b_\alpha(n)\right\}$$
$$\cap \left\{c_{k,m} : \phi_\alpha(n) \le k \le n, m \le b_\alpha(n)\right\} \cdot U_{\phi_\alpha(n)} \subset X_\alpha,$$

which completes our proof. □

Theorem 1.3 is a consequence of the following:

Proposition 2.6 *Let G be an ω-bounded topological group such that one of its finite powers is not a Menger monoid and let $\mathcal{F}$ be a filter on ω. If there exists an ultrafilter $\mathcal{L} \supset \mathcal{F}$ that is not coherent to any Q-point, then $\mathrm{cf}_g^{\mathcal{F}}(G)$ is well-defined and is less than or equal to $\mathfrak{b}(\mathcal{L})$.*

Proof By a result of Guran [11], G is topologically isomorphic to a subgroup of a product $\prod_{i\in I} Q_i$, where each Q_i is a second countable group. There exists $J \in [I]^\omega$ with the property that one of the finite powers of $H := \mathrm{pr}_J(G)$ is not a Menger monoid. Indeed, let $k \in \omega$ be such that G^k is not a Menger monoid. There exists a sequence $\langle U_n : n \in \omega\rangle$ of open neighbourhoods of the neutral element of G such that $G^k \ne \bigcup_{n\in\omega} F_n U_n^k \cap U_n^k F_n$ for any sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of G^k. Shrinking U_n, if necessary, we may additionally assume that $U_n = \prod_{i\in J_n} W_{i,n} \times \prod_{i\in I\setminus J_n} Q_i$, where J_n is a finite subset of I and $W_{i,n}$ is an open neighbourhood of the neutral element of Q_i. Set $J = \bigcup_{n\in\omega} J_n$, $H = \mathrm{pr}_J(G)$, and $V_n = \prod_{i\in J_n} W_{i,n} \times \prod_{i\in J\setminus J_n} Q_i$. It follows from the above that $H^k \ne \bigcup_{n\in\omega} K_n V_n^k \cap V_n^k K_n$ for any sequence $\langle K_n : n \in \omega\rangle$ of finite subsets of H^k, which means that H^k is not a Menger monoid.

By applying the same argument as in the proof of Proposition 2.5 to the (first countable) group H, we conclude that there exists an appropriate increasing chain $\langle H_\alpha : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ of proper subgroups of H such that $H = \bigcup_\alpha H_\alpha$. Now $\langle \mathrm{pr}_J^{-1}(H_\alpha) : \alpha < \mathfrak{b}(\mathcal{L})\rangle$ is a witness for $\mathrm{cf}_g^{\mathcal{F}}(G) \le \mathfrak{b}(\mathcal{L})$, which completes our proof. □

Proof of Theorem 2.2 (1) Suppose that $\kappa := \mathrm{cf}_m^{\mathcal{F}}(X)$ exists and $\kappa < \mathfrak{q}(\mathcal{F})$. All other cases ($\kappa > \mathfrak{q}(\mathcal{F})$, $\kappa < \mathfrak{b}(\mathcal{F})$, $\kappa > \mathfrak{b}(\mathcal{F})$, or X is a topological group, $\mathrm{cf}_g^{\mathcal{F}}(X)$ exists and $\mathrm{cf}_g^{\mathcal{F}}(X) < \mathfrak{q}(\mathcal{F})$, $\mathrm{cf}_g^{\mathcal{F}}(X) > \mathfrak{q}(\mathcal{F})$, $\mathrm{cf}_g^{\mathcal{F}}(X) < \mathfrak{b}(\mathcal{F})$, or $\mathrm{cf}_g^{\mathcal{F}}(X) > \mathfrak{b}(\mathcal{F})$) are analogous.

We use the notations from the proof of Proposition 2.5. For every $\alpha < \mathfrak{q}(\mathcal{F})$ let

$$Z_\alpha = \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} U_{\phi_\alpha(n)} \cdot \{c_{\phi_\alpha(n),m} : m \le n\} \cap \{c_{\phi_\alpha(n),m} : m \le n\} \cdot U_{\phi_\alpha(n)}$$

and observe that $\langle Z_\alpha : \alpha < \mathfrak{q}(\mathcal{F})\rangle$ is an increasing sequence of $[\mathcal{F}]$-Menger subspaces of X covering X. Let $\langle X_\xi : \xi < \kappa\rangle$ be a sequence of proper submonoids of X witnessing for $\mathrm{cf}_m^{\mathcal{F}}(X) = \kappa$. Since $\mathfrak{q}(\mathcal{F})$ is regular and for every $\alpha < \mathfrak{q}(\mathcal{F})$ there exists $\xi < \kappa$ with $Z_\alpha \subset X_\xi$, we conclude that there exists ξ

such that $X_\xi \supset Z_\alpha$ for cofinally many $\alpha \in \mathfrak{q}(\mathcal{F})$, which means $X_\xi = X$ and thus contradicts the assumption that X_ξ is a proper submonoid of X.

(2) The existence of $\mathrm{cf}_m^{\mathcal{F}}(X)$ (resp. $\mathrm{cf}_g^{\mathcal{F}}(X)$) follows from Proposition 2.5 (resp. Proposition 2.6.) The rest is a consequence of the previous item.

(3) This item follows directly from [2, Theorem 6.4].

□

A sequence $\langle U_n : n \in \omega\rangle$ is called an *ω-cover* of a set X if for every finite $F \subset X$ there exists $n \in \omega$ such that $F \subset U_n$. If, moreover, there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ of integers such that for every finite $F \subset X$ and for all but finitely many $k \in \omega$ there exists $n \in [n_k, n_{k+1})$ such that $F \subset U_n$, then the cover $\langle U_n : n \in \omega\rangle$ is called *ω-groupable*.

Proof of Corollary 1.5 In light of Theorem 1.3 it is enough to verify the following: □

Claim 2.7 *If all finite powers of an analytic topological group G are Menger monoids, then G is σ-bounded.*[3]

Proof Suppose that all finite powers of G are Menger monoids. By applying [21, Lemma 17] and [2, Proposition 3.1, Lemma 3.2], we can conclude that G is $[\mathcal{U}]$-Menger for some ultrafilter $\mathcal{U}$. Given a decreasing base $\langle U_n : n \in \omega\rangle$ at the identity of G we can find a sequence $\langle F_n : n \in \omega\rangle$ of finite subsets of G such that $\langle B_n = F_n U_n \cap U_n F_n : n \in \omega\rangle$ is an $[\mathcal{U}]$-cover of G. For every $g \in G$ denote the set $\{n \in \omega : g \in B_n\}$ by $\mathcal{N}_g$.

It follows that there exists an increasing number sequence $\langle n_k : k \in \omega\rangle$ such that $\bigcup_{\mathcal{N}_g \cap [n_k, n_{k+1}) \neq \emptyset}[n_k, n_{k+1}) \in \mathcal{U}$ for all $g \in G$ (if ϕ is a finite-to-one surjection witnessing for $\langle B_n : n \in \omega\rangle$ being an $[\mathcal{U}]$-cover, then the sequence $\langle \min \phi^{-1}(k)\rangle_{k\in\omega}$ is as required.) Let F'_k be a finite subset of G such that $D_k := U_k F'_k \cap F'_k U_k \supset \bigcup_{n\in[n_k,n_{k+1})} B_n$.$\langle D_k : k \in \omega\rangle$ is clearly an ω-cover of G. Applying [16, Theorem 4.5] (see also [20, Theorem 7]), we conclude that $\langle D_k : k \in \omega\rangle$ is ω-groupable.

Let $\langle k_m : m \in \omega\rangle$ be an increasing number sequence witnessing for this. Set $Y_m = \bigcap_{l\geq m} \bigcup_{k\in[k_m,k_{m+1})} D_k$. A direct verification shows that each Y_m is totally bounded and $G = \bigcup_{m\in\omega} Y_m$. □

Proof of Corollary 1.8 Suppose that $\mathcal{U}$ is a pseudo-$P_{\mathfrak{b}^+}$-point. Since $\phi(\mathcal{U})$ is clearly a pseudo-$P_{\mathfrak{b}^+}$-point for every finite-to-one ϕ, $\mathcal{U}$ is not coherent to a Q-point by [3, Theorem 13.8.1]. Therefore $\mathrm{cf}^*(\mathrm{Sym}(\kappa)) \leq \mathfrak{b}(\mathcal{U})$. It suffices to apply the following result of Nyikos [15] (see [6, Proposition 5] or [3, Theorem 13.2.1, Corollary 10.3.2] for its proof): *If $\mathcal{L}$ is pseudo-$P_{\mathfrak{b}^+}$-point, then $\mathfrak{b}(\mathcal{L}) = \mathfrak{b}$.* □

Proof of Corollary 1.9 Let $\mathcal{U}$ be an ultrafilter generated by $\mathfrak{u}$ many subsets of ω. It is well-known that $\mathfrak{b}(\mathcal{U}) = \mathfrak{d}$ and $\mathcal{U}$ is coherent to any ultrafilter $\mathcal{F}$ such that $\mathfrak{b}(\mathcal{F}) > \mathfrak{u}$, see [3, Theorem 10.3.1] or [6, Theorem 12]. It suffices to apply Corollary 1.5 and the transitivity of the coherence relation. □

[3] This fact can be thought of as the analogue for topological groups of the following result proven in [1]: if for every sequence $\langle u_n : n \in \omega\rangle$ of open covers of an analytic space X there exists a sequence $\langle v_n : n \in \omega\rangle$ such that $v_n \in [u_n]^{<\omega}$ and $X = \bigcup_{n\in\omega} \cup v_n$, then X is σ-compact.

Lemma 2.8 *If $F \subset \omega^\omega$ is a finitely dominating family of strictly increasing functions, then $\bigcup_{f\in F} S_f$ generates* $\mathrm{Sym}(\omega)$.

Proof Let $H = \langle \bigcup_{f\in F} S_f \rangle$ and $\pi \in \mathrm{Sym}(\omega)$ be such that all its orbits are finite, i.e. for every $n \in \omega$ the set $\{\pi^k(n) : k \in \omega\}$ is finite, where $\pi^1 = \pi$ and $\pi^{k+1} = \pi \circ \pi^k$. Let $\mathcal{A} = \{a_i : i \in \omega\}$ be the enumeration of orbits of π such that $\min a_i < \min a_{i+1}$ for all i. The following claim is obvious. □

Claim 2.9 *There exist two increasing sequences $\langle n_i^0 : i \in \omega\rangle$ and $\langle n_i^1 : i \in \omega\rangle$ of natural numbers such that for every $a \in \mathcal{A}$ there exists a pair $\langle i, j\rangle \in \omega \times 2$ such that $a \subset [n_i^j, n_{i+1}^j)$.*

Let $h \in \omega^\omega$ be an increasing function such that $h(n_i^j) \geq \max\{\pi(m), \pi^{-1}(m) : m \in [n_i^j, n_{i+1}^j)\}$ for all i and j, and F_0 be a finite subset of F such that $h \leq^* \max F_0$. Fix any $a \in \mathcal{A}$ and find $\langle i, j\rangle \in \omega \times 2$ such that $a \subset [n_i^j, n_{i+1}^j)$. Let $f \in F_0$ be such that $f(n_i^j) > h(n_i^j)$. By the definition of h the above implies $\pi(m), \pi^{-1}(m) \leq h(n_i^j) \leq f(n_i^j) \leq f(m)$ for every $m \in a$. Therefore for every $a \in \mathcal{A}$ there exists $f_a \in F_0$ such that $\pi(m), \pi^{-1}(m) < f_a(m)$ for all $m \in a$. Set $\pi_f = \pi \restriction \bigcup\{a \in \mathcal{A} : f_a = f\}$ and note that $\pi_f \in S_f$ and $\pi = \circ_{f\in F_0} \pi_f$ (the latter composition obviously does not depend on the order in which we take π_f's). Hence $\pi \in H$.

$\mathrm{Sym}(\omega)$ is easily seen to be a G_δ-subset of ω^ω. Therefore $\mathrm{Sym}(\omega)$ with the topology τ inherited from ω^ω is a Polish topological group. It is also easy to check that the set E of all permutations of ω with finite orbits is a dense G_δ of $(\mathrm{Sym}(\omega), \tau)$, and hence $E \circ E \supset \mathrm{Sym}(\omega)$ by the Baire Category Theorem. It suffices to note that $E \circ E \subset H$.

Proof of Theorem 1.10 The first statement is a direct consequence of Lemma 2.8: Suppose that $\kappa = \mathrm{cf}^*(\mathrm{Sym}(\omega)) < \mathrm{cov}(\mathfrak{D}_{fin})$ and $\langle G_\alpha : \alpha < \kappa\rangle$ is an increasing sequence of proper subgroups of $\mathrm{Sym}(\omega)$ witnessing for that. Set $B_\alpha = \{f \in \omega^{\uparrow\omega} : S_f \subset G_\alpha\}$. By the definition of $\mathrm{cf}^*(\mathrm{Sym}(\omega))$, $\bigcup_{\alpha<\kappa} B_\alpha = \omega^{\uparrow\omega}$. Since $\kappa < \mathrm{cov}(\mathfrak{D}_{fin})$, there exists $\alpha < \kappa$ such that B_α is finitely dominating, which by Lemma 2.8 implies that $G_\alpha = \mathrm{Sym}(\omega)$ and hence contradicts the properness of G_α.

The second one follows from the fact that NCF implies that $\mathrm{cov}(\mathfrak{D}_{fin}) = \mathfrak{d}$. Indeed, suppose that NCF holds. Then $\mathfrak{b}(\mathcal{F}) = \mathfrak{d}$ for all ultrafilters $\mathcal{F}$, see e.g. [5, Theorem 16] or [3, 12.3.1]. In addition, every not finitely dominating subset of $\omega^{\uparrow\omega}$ is $\leq_{\mathcal{F}}$-bounded for every ultrafilter $\mathcal{F}$. □

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