



## Hyperspaces of max-plus convex subsets of powers of the real line

Lidia Bazylevych<sup>a</sup>, Dušan Repovš<sup>b,c,\*</sup>, Mykhailo Zarichnyi<sup>d,e</sup>

<sup>a</sup> National University "Lviv Polytechnica", 12 Bandery Str., 79013 Lviv, Ukraine

<sup>b</sup> Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana, 1000, Slovenia

<sup>c</sup> Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, 1000, Slovenia

<sup>d</sup> Department of Mechanics and Mathematics, Lviv National University, Universytetska Str. 1, 79000 Lviv, Ukraine

<sup>e</sup> Institute of Mathematics, University of Rzeszów, Rejtana 16 A, 35-310 Rzeszów, Poland

### ARTICLE INFO

#### Article history:

Received 24 April 2011

Available online 8 May 2012

Submitted by B. Sims

#### Keywords:

Max-plus convex set

Hyperspace

Absolute retract

Powers of the real line

### ABSTRACT

The notion of a max-plus convex subset of Euclidean space can be naturally extended to other linear spaces. The aim of this paper is to describe the topology of hyperspaces of max-plus convex subsets in Tychonov powers  $\mathbb{R}^\tau$  of the real line. We show that the corresponding spaces are absolute retracts if and only if  $\tau \leq \omega_1$ .

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### 1. Introduction

Max-plus convex sets were introduced in [1]. Max-plus convex cones also appeared in idempotent analysis, following the observation by Maslov that solutions of the Hamilton–Jacobi equation associated with a deterministic optimal control problem satisfy a “max-plus” superposition principle and therefore belong to structures similar to convex cones which are called semimodules or idempotent linear spaces [2]. In the last decade the interest in max-plus convex sets increased due to the development of the so-called “idempotent mathematics”, which is a part of mathematics where usual arithmetic operations are replaced by idempotent operations. Our paper is devoted to hyperspaces of max-plus convex subsets in Tychonov powers of the real line. The results of the first-named author cover the case of  $\mathbb{R}^n$ ,  $n \geq 2$ .

The topology of hyperspaces of compact and closed convex sets has been investigated by several authors. The classical result of Nadler et al. [3] asserts that the hyperspace of convex compact subsets of  $\mathbb{R}^n$ ,  $n \geq 2$ , is a contractible  $Q$ -manifold homeomorphic to  $Q \setminus \{*\}$  (recall that a  $Q$ -manifold is a manifold modeled on the Hilbert cube  $Q = [0, 1]^\omega$ ). Their result has found many applications in convex geometry. In particular, it enabled the proof that the hyperspace of all compact strictly convex bodies is homeomorphic to the separable Hilbert space  $\ell^2$  (see [4]). Hyperspaces of compact convex subsets of Tychonov cubes were investigated in [5].

Let  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$  and let  $\tau$  be a cardinal number. Given  $x, y \in \mathbb{R}^\tau$  and  $\lambda \in \mathbb{R}$ , we denote by  $x \oplus y$  the coordinatewise maximum of  $x$  and  $y$  and by  $\lambda \odot x$  the vector obtained from  $x$  by adding  $\lambda$  to each of its coordinates. A subset  $A$  in  $\mathbb{R}^\tau$  is said to be *tropically convex* (or *max-plus convex*) if  $\alpha \odot a \oplus \beta \odot b \in A$  for all  $a, b \in A$  and  $\alpha, \beta \in \mathbb{R}_{\max}$  with  $\alpha \oplus \beta = 0$ .

We denote the hyperspace of all nonempty max-plus convex compact subsets in  $\mathbb{R}^\tau$  by  $\text{mpcc}(\mathbb{R}^\tau)$ . Note that every max-plus convex compact subset in  $\mathbb{R}^\tau$  is a subsemilattice of  $\mathbb{R}^\tau$  with respect to the operation  $\oplus$ . In particular,  $\max A \in A$  for any max-plus convex compact subset  $A$  in  $\mathbb{R}^\tau$ .

\* Corresponding author at: Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana, 1000, Slovenia.

E-mail addresses: [izar@litech.lviv.ua](mailto:izar@litech.lviv.ua) (L. Bazylevych), [dusan.repovs@guest.arnes.si](mailto:dusan.repovs@guest.arnes.si) (D. Repovš), [topology@franko.lviv.ua](mailto:topology@franko.lviv.ua) (M. Zarichnyi).

Tychonov powers  $\mathbb{R}^\tau$ , for  $\tau > \omega$ , are the main geometric objects of the theory of noncompact nonmetrizable absolute extensors. The main result of our paper is that the hyperspace of max-plus convex subsets in the spaces  $\mathbb{R}^\tau$  is homeomorphic to  $\mathbb{R}^\tau$  if  $\tau \in \{\omega, \omega_1\}$ .

**2. Preliminaries**

The set  $\mathbb{R} \cup \{-\infty\}$  will be endowed with the metric  $\varrho$ ,  $\varrho(x, y) = |e^x - e^y|$  (conventions:  $e^{-\infty} = 0$  and  $\ln 0 = -\infty$ ). We denote the set of all nonempty compact subsets of a metric space  $(X, d)$  by  $\text{exp } X$ . The base of the Vietoris topology on  $\text{exp } X$  consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \text{exp } X \mid A \subset \bigcup_{i=1}^n U_i, A \cap U_i \neq \emptyset, \text{ for all } i = 1, \dots, n\},$$

where  $U_1, \dots, U_n$  run over the topology of  $X$ .

If  $X$  is a metric space, then one can endow  $\text{exp } X$  with the Hausdorff metric  $d_H$ :

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}$$

(hereafter,  $O_r(C)$  will denote the  $r$ -neighborhood of  $C \in \text{exp } X$ ). It is well-known that equivalent metrics on  $X$  generate equivalent Hausdorff metrics on  $\text{exp } X$ .

By ANR (resp., AR) we shall denote the class of absolute neighborhood retracts (resp., absolute retracts) for the class of metrizable spaces, i.e. the class of metrizable spaces  $X$  satisfying the following property: for every embedding  $i: X \rightarrow Y$  into a metrizable space  $Y$  there exists a retraction of a neighborhood of  $i(X)$  in  $Y$  (resp., a retraction of  $Y$ ) onto  $i(X)$ .

We say that a metric space  $X$  satisfies the strong discrete approximation property (SDAP) if for every continuous function  $\varepsilon: X \rightarrow (0, \infty)$  and every map  $f: \prod_{n=1}^\infty I^n \rightarrow X$  there exists a map  $g: \prod_{n=1}^\infty I^n \rightarrow X$  such that  $d(f(x), g(x)) < \varepsilon(x)$ ,  $x \in \prod_{n=1}^\infty I^n$ , and the family  $\{g(I^n) \mid n \in \mathbb{N}\}$  is discrete ( $d$  denotes the metric on  $X$ ). The following is a characterization theorem for  $\ell^2$ -manifolds.

**Theorem 2.1** (Toruńczyk [6]). *A complete separable nowhere locally compact ANR  $X$  is an  $\ell^2$ -manifold if and only if  $X$  satisfies the SDAP.*

Recall that a map  $f: X \rightarrow Y$  is called soft [7] if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{\psi} & Y \end{array} \tag{1}$$

such that  $A$  is a closed subset of a paracompact space  $Z$ , there exists a map  $\Phi: Z \rightarrow X$  such that  $f\Phi = \psi$  and  $\Phi|_A = \varphi$ .

A trivial  $\ell^2$ -bundle is a map  $f: X \rightarrow Y$  which is homeomorphic to the projection  $Y \times M \rightarrow Y$  onto the first factor, where  $M$  is  $\ell^2$ . A map  $f: X \rightarrow Y$  of metric spaces is said to satisfy the fiberwise discrete approximation property if for every map  $g: \prod_{n=1}^\infty I^n \rightarrow X$  and every continuous function  $\varepsilon: X \rightarrow (0, \infty)$  there is a map  $h: \prod_{n=1}^\infty I^n \rightarrow X$  such that  $d(f(x), g(x)) < \varepsilon(x)$ ,  $x \in \prod_{n=1}^\infty I^n$ , and:

- (1)  $fg = fh$ ; and
- (2) the family  $\{h(I^n) \mid i \in \mathbb{N}\}$  is discrete.

The following result was cited in [8] and was attributed to Toruńczyk and West (see [9] for the compact case).

**Theorem 2.2** (Toruńczyk–West Characterization Theorem for  $\mathbb{R}^\omega$ -Manifold Bundles). *A map  $f: X \rightarrow Y$  of complete metric ANR-spaces is a trivial  $\mathbb{R}^\omega$  if  $f$  is soft and  $f$  satisfies the fiberwise discrete approximation property (FDAP).*

The following notion was introduced in [10]: a  $c$ -structure on a topological space  $X$  is an assignment, to every nonempty finite subset  $A$  of  $X$ , of a contractible subspace  $F(A)$  of  $X$ , such that  $F(A) \subset F(A')$  whenever  $A \subset A'$ . A pair  $(X, F)$ , where  $F$  is a  $c$ -structure on  $X$ , is called a  $c$ -space. A subset  $E$  of  $X$  is called an  $F$ -set if  $F(A) \subset E$  for any finite  $A \subset E$ . A metric space  $(X, d)$  is said to be a metric l.c.-space if all the open balls are  $F$ -sets and all open  $r$ -neighborhoods of  $F$ -sets are also  $F$ -sets.

The following is a generalization of the Michael Selection Theorem for generalized convexity structures (see [11] for the proof). Recall that a multivalued map  $F: X \rightarrow Y$  of topological spaces is called lower semicontinuous if, for any open subset  $U$  of  $Y$ , the set  $\{x \in X \mid F(x) \cap U \neq \emptyset\}$  is open in  $X$ . A selection of a multivalued map  $F: X \rightarrow Y$  is a (single-valued) map  $f: X \rightarrow Y$  such that  $f(x) \in F(x)$  for every  $x \in X$ . The following was proved in [11] (see the second corollary of Theorem 2 in [11]).

**Theorem 2.3.** *Let  $(X, d, F)$  be a metric l.c.-space. Then  $X$  is an AR.*

**Theorem 2.4.** *Let  $(X, d, F)$  be a complete metric l.c.-space. Then any lower semicontinuous multivalued map  $T: Y \rightarrow X$  of a paracompact space  $Y$  whose values are nonempty closed  $F$ -sets has a continuous selection.*

### 3. Two lemmas

Recall that the countable power  $\mathbb{R}^\omega$  of the real line  $\mathbb{R}$  is homeomorphic to the pseudo-interior  $s$  of the Hilbert cube  $Q$  as well as, by the Anderson–Kadec theorem, to the separable Hilbert space  $\ell^2$ . We shall consider the following metric  $\varrho$  on  $\mathbb{R}$ :

$$\varrho(x, y) = \min\{|x - y|, 1\}.$$

We shall define a metric  $d$  on the countable power  $\mathbb{R}^\omega$  by the formula

$$d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) = \max_{i \in \mathbb{N}} \frac{\varrho(x_i, y_i)}{2^i}.$$

Note that  $d$  is a complete metric generating the Tychonov topology on  $\mathbb{R}^\omega$ .

**Lemma 3.1.** *The space  $\text{mpcc}(\mathbb{R}^\omega)$  is an absolute retract.*

**Proof.** Define a  $c$ -structure on  $\text{mpcc}(\mathbb{R}^\omega)$  as follows: given any  $A_1, \dots, A_n \in \text{mpcc}(\mathbb{R}^\omega)$ , let

$$F(\{A_1, \dots, A_n\}) = \left\{ \bigoplus_{i=1}^n \alpha_i \odot A_i \mid \alpha_1, \dots, \alpha_n \in [-\infty, 0], \bigoplus_{i=1}^n \alpha_i = 0 \right\}.$$

We are going to show that every set of the form  $F(\{A_1, \dots, A_n\})$  is contractible. Let  $A = \bigoplus_{i=1}^n A_i$ . Then  $A \in F(\{A_1, \dots, A_n\})$ . Define a map

$$H: F(\{A_1, \dots, A_n\}) \times [0, 1] \rightarrow F(\{A_1, \dots, A_n\})$$

by the formula

$$H(C, t) = C \oplus (\ln t) \odot A.$$

Note that  $H$  is well-defined,  $H(C, 0) = C \oplus \{-\infty\} = C$  and  $H(C, 1) = C \oplus 0 \odot A = A$ , for every  $C \in F(\{A_1, \dots, A_n\})$ . Thus,  $H$  contracts the set  $F(\{A_1, \dots, A_n\})$  to  $A$ .

Now let us prove that every neighborhood of a point in  $\text{mpcc}(\mathbb{R}^\omega)$  is an  $F$ -set. Let  $A \in \text{mpcc}(\mathbb{R}^\omega)$ ,  $r > 0$ , and  $d_H(A, B), d_H(A, B') < r$ . Given  $a \in A$ , find  $b \in B$  and  $b' \in B'$  such that  $d(a, b) < r$  and  $d(a, b') < r$ . Without loss of generality, we may assume that  $a = 0$ . There exist  $i, j \in \mathbb{N}$  such that

$$d(a, b) = \frac{\min\{|b_i|, 1\}}{2^i}, \quad d(a, b') = \frac{\min\{|b'_j|, 1\}}{2^j}.$$

Given  $t \in [-\infty, 0]$ , find  $k \in \mathbb{N}$  such that

$$d(a, b \oplus t \odot b') = \frac{\min\{|\max\{b_k, b'_k + t\}|, 1\}}{2^k}.$$

Without loss of generality, we may assume that  $r < 1$ . The rest of the proof splits into two cases.

Case 1.  $b_k \geq b'_k + t$ . Then

$$d(a, b \oplus t \odot b') = \frac{|b_k|}{2^k} \leq \frac{|b_i|}{2^i} < r.$$

Case 2.  $b_k \leq b'_k + t$ . Then also  $b'_k + t \leq b'_k$  and

$$d(a, b \oplus t \odot b') = \frac{|b_k + t|}{2^k} \leq \max\left\{\frac{|b_k|}{2^k}, \frac{|b'_k|}{2^k}\right\} \leq \max\left\{\frac{|b_i|}{2^i}, \frac{|b'_j|}{2^j}\right\} < r.$$

In both cases, for every  $a \in A$  there is a point  $c \in B \oplus t \odot B'$  such that  $d(a, c) < r$ . Similarly, for any  $c \in B \oplus t \odot B'$  one can find  $a \in A$  such that  $d(a, c) < r$ . This shows that  $d_H(A, B \oplus t \odot B') < r$  for every  $B, B' \in \text{mpcc}(\mathbb{R}^\omega)$  such that  $d_H(A, B) < r, d_H(A, B') < r$ .

We can demonstrate by induction that

$$d_H\left(\bigoplus_{i=1}^n \alpha_i \odot A_i, A\right) < r, \quad \text{whenever } \alpha_1, \dots, \alpha_n \in [-\infty, 0],$$

$$\bigoplus_{i=1}^n \alpha_i = 0, \quad \text{and } d_H(A_i, A) < r, \quad \text{for every } i = 1, \dots, n.$$

This shows that every  $r$ -neighborhood of a point in the space  $\text{mpcc}(\mathbb{R}^\omega)$  is an  $F$ -set.

By using a similar argument we can prove that every neighborhood of an  $F$ -set is again an  $F$ -set. It follows from the results of [11] that the space  $\text{mpcc}(\mathbb{R}^\omega)$  is an AR-space (see Theorem 2.3).  $\square$

Let  $A, B$  be nonempty sets such that  $A \subset B$ . Observe that the projection  $p = p_A^B: \mathbb{R}^B \rightarrow \mathbb{R}^A$  onto the first factor induces the map

$$\text{mpcc}(p): \text{mpcc}(\mathbb{R}^B) \rightarrow \text{mpcc}(\mathbb{R}^A)$$

given by:

$$\text{mpcc}(p)(A) = p(A), \quad A \in \text{mpcc}(\mathbb{R}^B).$$

It is easy to verify that this map is well-defined.

We may regard the construction  $\text{mpcc}$  as a covariant functor acting on the category whose objects are the powers of  $\mathbb{R}$  and where the morphisms are the projections.

**Lemma 3.2.** *Let  $p: \mathbb{R}^\omega \times \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$  be the projection onto the first factor. Then the map  $\text{mpcc}(p)$  is soft.*

**Proof.** Consider a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega) \\ \downarrow & & \downarrow \text{mpcc}(p) \\ Z & \xrightarrow{\psi} & \text{mpcc}(\mathbb{R}^\omega), \end{array} \tag{2}$$

where  $A$  is a closed subset of a paracompact space  $Z$ .

For every  $C \in \text{mpcc}(\mathbb{R}^\omega)$ , the preimage

$$\text{mpcc}(p)^{-1}(C) \subset \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega)$$

is convex with respect to the  $c$ -structure  $F$  in the space  $\text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega)$  defined as follows: given any  $A_1, \dots, A_n \in \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega)$ , let

$$F(\{A_1, \dots, A_n\}) = \left\{ \bigoplus_{i=1}^n \alpha_i \odot A_i \mid \alpha_1, \dots, \alpha_n \in [-\infty, 0], \bigoplus_{i=1}^n \alpha_i = 0 \right\}.$$

Note that this is an  $F$ -structure with respect to the Hausdorff metric  $d'_H$  on the space  $\text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega)$  generated by the metric  $d'$  on the space  $\mathbb{R}^\omega \times \mathbb{R}^\omega$  defined by the formula

$$d'((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}.$$

This can be established by repeating the corresponding arguments from the proof of Lemma 3.1.

Define a multivalued map  $\Phi: Z \rightarrow \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega)$  as follows:

$$\Phi(z) = \begin{cases} \text{mpcc}(p)^{-1}(\psi(z)), & \text{if } z \in Z \setminus A, \\ \{\varphi(z)\}, & \text{if } z \in A. \end{cases}$$

Clearly, the images of  $\Phi$  are  $F$ -sets. Since the set  $A$  is closed, we see that the map  $\Phi$  is lower semicontinuous. It follows from Theorem 2.4 that this map admits a continuous selection  $g$ . Clearly,  $g|_A = \varphi$  and  $g \text{mpcc}(p) = \psi$ . This proves the softness of  $\text{mpcc}(p)$ .  $\square$

#### 4. The main result

**Theorem 4.1.** *The hyperspace  $\text{mpcc}(\mathbb{R}^\omega)$  of compact max-plus convex subsets in the space  $\mathbb{R}^\omega$  is homeomorphic to  $\mathbb{R}^\omega$ .*

**Proof.** Since  $\mathbb{R}^\omega$  is homeomorphic to  $(\mathbb{R}^\omega)^\omega$ , one can represent the latter space as the limit of the inverse sequence

$$\mathbb{R}^\omega \leftarrow \mathbb{R}^\omega \times \mathbb{R}^\omega \leftarrow \mathbb{R}^\omega \times \mathbb{R}^\omega \times \mathbb{R}^\omega \leftarrow \dots,$$

where every arrow denotes the projection onto the first factor. Applying the functor  $\text{mpcc}$  to this sequence we obtain

$$\text{mpcc}(\mathbb{R}^\omega) \leftarrow \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega) \leftarrow \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega \times \mathbb{R}^\omega) \leftarrow \dots. \tag{3}$$

The bonding maps of the latter sequence have the following property: for every such map there exists a countable family of selections such that the family of images of these selections is discrete. Indeed, let  $C = \{c_i \mid i \in \omega\}$  be a closed countable subset of  $\mathbb{R}^\omega$ . For every  $i \in \omega$ , denote by  $s_i$  the selection of the map

$$\text{mpcc}((\mathbb{R}^\omega \times \mathbb{R}^\omega \times \dots \times \mathbb{R}^\omega) \times \mathbb{R}^\omega) \rightarrow \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega \times \dots \times \mathbb{R}^\omega)$$

defined as follows:  $s_i(A) = A \times \{c_i\}$ .

We are going to show that the limit projection of the inverse limit of (3) onto  $\text{mpcc}(\mathbb{R}^\omega)$  satisfies the FDAP. Let  $f: \varprojlim_{i \in \mathbb{N}} Q_i \rightarrow \text{mpcc}((\mathbb{R}^\omega)^\omega)$  be a map and let  $\varepsilon: \text{mpcc}((\mathbb{R}^\omega)^\omega) \rightarrow (0, \infty)$  be a function. For every  $n \in \omega$ , let

$$Y_n = \left\{ y \in Y \mid \varepsilon(f(y)) \geq \frac{1}{2^n} \right\}.$$

Note that

$$Y_0 \subset \text{Int}(Y_1) \subset Y_1 \subset \text{Int}(Y_2) \subset Y_2 \cdots$$

Define, for every  $l = 0, 2, 4, \dots$ , a map  $g_l: Y_{l-1} \cup Y_l \cup Y_{l+1} \rightarrow \text{mpcc}((\mathbb{R}^\omega)^\omega)$  by the formula

$$g_l(y) = \text{mpcc}(\text{pr}_{l+1})(f(y)) \times \{c_i\} \times \{c_i\} \times \cdots,$$

whenever  $y \in Q_i$ . Now, for every  $l = 1, 3, 5, \dots$ , let  $\varphi_l: Y_{l-1} \cup Y_l \cup Y_{l+1} \rightarrow [0, 1]$  be a function such that  $\varphi_l|_{Y_{l-1}} \equiv 0$ ,  $\varphi_l|_{Y_{l+1}} \equiv 1$ .

Define a map  $g: \varprojlim_{i \in \mathbb{N}} Q_i \rightarrow \text{mpcc}((\mathbb{R}^\omega)^\omega)$  by the following condition. Let  $y \in Y_l$ , where  $l = 0, 2, 4, \dots$ . Then define  $g(y) = g_l(y)$ . If  $y \in Y_l \cap Q_i$ , where  $l = 1, 3, 5, \dots$ , then define

$$g(y) = \{(a_1, \dots, a_l, \varphi_l(y)a_{l+1} + (1 - \varphi_l(y))c_i, c_i, c_i, \dots) \mid (a_k)_{k=1}^\infty \in f(y)\}.$$

It is easy to see that the map  $g$  is well-defined,  $\text{mpcc}(\text{pr}_1)f = \text{mpcc}(\text{pr}_1)g$ , and that  $d(f(x), g(x)) < \varepsilon(x)$ , for every  $x \in \varprojlim_{i \in \mathbb{N}} Q_i$ .

We are going to prove that the map  $g$  is a closed embedding. Suppose the contrary. Then there exists a sequence  $(y_{k_i})_{i=1}^\infty$ , where  $y_{k_i} \in Q_{k_i}$  for every  $i$  (here we assume that  $k_1 < k_2 < k_3 < \dots$ ), such that  $\lim_{i \rightarrow \infty} g(y_{k_i}) = A$ , for some  $A \in \text{mpcc}((\mathbb{R}^\omega)^\omega)$ . Without loss of generality, one may assume that  $k_i = i$ , for all  $i$ .

Since  $\varepsilon(A) > 0$ , one may assume that  $\varepsilon(g(y_i)) > 2^{-n}$  for some  $n < \omega$ . Denote by  $\pi_k: (\mathbb{R}^\omega)^\omega \rightarrow \mathbb{R}^\omega$  the projection onto the  $k$ th factor. Then from the construction of the map  $g$  it follows that  $\text{mpcc}(\pi_{n+1})(g(y_i)) = \{c_i\}$ . Since the set  $C = \{c_i \mid i \in \omega\}$  is closed in  $\mathbb{R}^\omega$ , we obtain a contradiction.

It now follows from Theorem 2.2 that the limit projection of the inverse limit of the inverse sequence (3) onto  $\text{mpcc}(\mathbb{R}^\omega)$  is a trivial  $\ell^2$ -bundle. Since the space  $\text{mpcc}(\mathbb{R}^\omega)$  is an absolute retract, we conclude that

$$\text{mpcc}(\mathbb{R}^\omega) \simeq \text{mpcc}((\mathbb{R}^\omega)^\omega) \simeq \text{mpcc}(\mathbb{R}^\omega) \times \ell^2 \simeq \ell^2,$$

which proves the theorem.  $\square$

**Remark 4.2.** As a by-product of the proof we see that the map  $\text{mpcc}(p_1): \text{mpcc}(\mathbb{R}^\omega \times \mathbb{R}^\omega) \rightarrow \text{mpcc}(\mathbb{R}^\omega)$  is a trivial  $\ell^2$ -bundle (here  $p_1$  denotes the projection onto the first factor).

The following result is an analogue of a theorem of the first-named author [12], proved for the open sets in  $\mathbb{R}^n$ ,  $n \geq 2$ .

**Theorem 4.3.** *Let  $X$  be an open subset in the space  $\mathbb{R}^\omega$ . Then the hyperspace of max-plus convex subsets in  $X$  is homeomorphic to  $X$ .*

**Proof.** The set  $X$  is an  $\mathbb{R}^\omega$ -manifold, being an open subset of  $\text{mpcc}(\mathbb{R}^\omega)$ . We identify the set  $X$  with the set of all singletons in  $X$ . The map  $\max: \text{mpcc}(X) \rightarrow X$  is therefore a retraction. Denote the homotopy  $H: \text{mpcc}(X) \times [0, 1] \rightarrow \text{mpcc}(X)$  by the formula

$$H(A, t) = \{a \oplus \ln t \max A \mid a \in A\}, \quad A \in \text{mpcc}(X), \quad t \in [0, 1]$$

(convention:  $\ln 0 = -\infty$ ).

Therefore, the space  $X$  is a deformation retract of the space  $\text{mpcc}(X)$ , whence we conclude that the spaces  $X$  and  $\text{mpcc}(X)$  are homotopically equivalent. The classification theorem for  $\mathbb{R}^\omega$ -manifolds implies that the spaces  $X$  and  $\text{mpcc}(X)$  are homeomorphic.  $\square$

**Theorem 4.4.** *The hyperspace  $\text{mpcc}(\mathbb{R}^{\omega_1})$  is homeomorphic to  $\mathbb{R}^{\omega_1}$ .*

**Proof.** We represent  $\mathbb{R}^{\omega_1}$  as the limit of the inverse system  $\mathcal{S} = \{(\mathbb{R}^\omega)^\alpha, p_{\alpha\beta}; \omega_1\}$ , where, for  $\alpha > \beta$ , the map  $p_{\alpha\beta}: (\mathbb{R}^\omega)^\alpha \rightarrow (\mathbb{R}^\omega)^\beta$  is the projection map. Then, recall that every projection map  $p_{\alpha\beta}$  induces the map  $\text{mpcc}(p_{\alpha\beta}): \text{mpcc}((\mathbb{R}^\omega)^\alpha) \rightarrow \text{mpcc}((\mathbb{R}^\omega)^\beta)$  and therefore we obtain the inverse system

$$\text{mpcc}(\mathcal{S}) = \{\text{mpcc}((\mathbb{R}^\omega)^\alpha), \text{mpcc}(p_{\alpha\beta}); \omega_1\}.$$

Since by Remark 4.2 every bonding map  $\text{mpcc}(p_{\alpha\beta})$  is homeomorphic to the projection  $p: \mathbb{R}^\omega \times \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ , we conclude that

$$\text{mpcc}(\mathbb{R}^{\omega_1}) = \text{mpcc}(\varprojlim(\mathcal{S})) = \varprojlim(\text{mpcc}(\mathcal{S})) \simeq \mathbb{R}^{\omega_1}$$

(the second equality is simply the continuity of the functor  $\text{mpcc}$ ; see [13] for details.)  $\square$

In the sequel, we shall speak of the theory of noncompact nonmetrizable absolute extensors in the sense of [8]. They are defined as retracts of functionally open subspaces of powers of the real line. Recall that a set  $U$  in a topological space  $X$  is called *functionally open* if  $U = f^{-1}((0, 1])$  for some continuous function  $f : X \rightarrow [0, 1]$ .

**Theorem 4.5.** *Let  $M$  be a functionally open subset of  $\mathbb{R}^{\omega_1}$ . Then  $\text{mpcc}(M)$  is homeomorphic to  $M$ .*

**Proof.** Note that the set  $\text{mpcc } M$  is also functionally open. Indeed, let  $f : \mathbb{R}^{\omega_1} \rightarrow [0, 1]$  be a continuous function such that  $M = f^{-1}((0, 1])$ . Define the function  $\tilde{f} : \mathbb{R}^{\omega_1} \rightarrow [0, 1]$  by the formula  $\tilde{f}(A) = \inf A$ . Then, clearly,  $\tilde{f}^{-1}((0, 1]) = \text{mpcc } M$ .

There exists a countable subset  $S \subset \omega_1$  and a function  $g : \mathbb{R}^S \rightarrow [0, 1]$  such that  $f = \text{gpr}_S$ . Therefore,  $M = U \times \mathbb{R}^{\omega_1 \setminus S}$ . Without loss of generality, one may conclude that  $S = \omega \subset \omega_1$ . We conclude that

$$M = \varprojlim \{U \times \mathbb{R}^{\alpha \setminus \omega}, p_{\alpha \setminus \omega}^{\beta \setminus \omega}; \omega < \alpha < \beta < \omega_1\}$$

and therefore

$$\text{mpcc}(M) = \varprojlim \{\text{mpcc}(U \times \mathbb{R}^{\alpha \setminus \omega}), \text{mpcc}(p_{\alpha \setminus \omega}^{\beta \setminus \omega}); \omega < \alpha < \beta < \omega_1\}.$$

Since by Theorem 4.3, the space  $\text{mpcc}(U)$  is homeomorphic to  $U$  and every projection map in the latter inverse system is soft, we conclude that

$$\text{mpcc}(M) \simeq \text{mpcc}(U) \times \mathbb{R}^{\omega_1} \simeq U \times \mathbb{R}^{\omega_1} \simeq M. \quad \square$$

**Theorem 4.6.** *The hyperspace  $\text{mpcc}(\mathbb{R}^\tau)$  is not an absolute retract, for any  $\tau > \omega_1$ .*

**Proof.** First, note that it suffices to consider the case  $\tau = \omega_2$ . Now, recall that  $\text{mpcc}$  is a functor acting on the category whose objects are spaces  $\mathbb{R}^\tau$  and the morphisms are the projections. Assuming that  $\text{mpcc}(\mathbb{R}^{\omega_2})$  is an absolute retract we conclude, by Chigogidze’s characterization theorem [14], that  $\text{mpcc}(\mathbb{R}^{\omega_2})$  is homeomorphic to  $\mathbb{R}^{\omega_2}$ .

By general results concerning the functors in the category of Tychonov spaces [8,7], we obtain that any homeomorphism of  $\mathbb{R}^{\omega_2}$  and  $\text{mpcc}(\mathbb{R}^{\omega_2})$  implies the isomorphism of the square diagram

$$\mathcal{D} = \begin{array}{ccc} (\mathbb{R}^\omega)^3 & \xrightarrow{\text{pr}_{12}} & (\mathbb{R}^\omega)^2 \\ \text{pr}_{13} \downarrow & & \downarrow \text{pr}_1 \\ (\mathbb{R}^\omega)^2 & \xrightarrow{\text{pr}_1} & \mathbb{R}^\omega \end{array},$$

where  $\text{pr}_{ij}, \text{pr}_k$  denote the projections onto the corresponding factors, and  $\text{mpcc}(\mathcal{D})$ .

We are going to show that the diagram  $\text{mpcc}(\mathcal{D})$  is not a pullback diagram. Let

$$A = \{0\} \subset \mathbb{R}^\omega, \quad B = C = \{0\} \times \{(x_i) \mid x_0 \in [0, 1], x_i = 0, \text{ if } i > 0\} \subset (\mathbb{R}^\omega)^2.$$

Let also

$$D_1 = \{0\} \times \{(x_i), (y_i) \mid x_0 = y_0 \text{ and } x_i = y_i = 0, \text{ if } i > 0\} \subset (\mathbb{R}^\omega)^3;$$

then

$$\text{mpcc}(\text{pr}_{12})(D) = \text{mpcc}(\text{pr}_{12})(D_1) = B, \quad \text{mpcc}(\text{pr}_{13})(D) = \text{mpcc}(\text{pr}_{13})(D_1) = C.$$

Thus  $\text{mpcc}(\mathcal{D})$  is not a pullback diagram and this completes the proof.  $\square$

### 5. Epilogue

The following question is related to Theorem 4.3.

**Question 5.1.** Let  $U$  be an open subset of  $\mathbb{R}^{\omega_1}$  which is an  $\mathbb{R}^{\omega_1}$ -manifold (see [14] for the background of the theory of  $\mathbb{R}^{\omega_1}$ -manifolds). Is  $\text{mpcc}(U)$  then homeomorphic to  $U$ ?

The following notion was introduced in [15] and investigated in [16,17]. A subset  $B$  of  $\mathbb{R}_+^n$  is said to be  $\mathbb{B}$ -convex if for all  $x, y \in B$  and all  $t \in [0, 1]$  one has  $\max(tx, y) \in B$ . For the hyperspace  $\mathbb{B}\text{-cc}(\mathbb{R}^n)$ ,  $n \geq 2$ , of compact  $\mathbb{B}$ -convex subsets of  $\mathbb{R}_+^n$  one can prove analogues of the results in [13].

One can extend this notion over an arbitrary vector lattice. Let  $\ell_+^2$  denote the positive cone of the separable Hilbert space  $\ell^2$ . We say that a subset  $B$  of  $\ell_+^2$  is  $\mathbb{B}$ -convex if for all  $x, y \in B$  and all  $t \in [0, 1]$  one has  $\max(tx, y) \in B$ . We conjecture that the hyperspace of compact  $\mathbb{B}$ -convex subsets in  $\ell_+^2$  is homeomorphic to  $\ell^2$ . An analogous question can be formulated for the nonseparable case.

**Question 5.2.** Let  $\ell^2(A)_+$  denote the positive cone in a nonseparable Hilbert space  $\ell^2(A)$ . Is the hyperspace  $\mathbb{B}\text{-cc}(\ell^2(A)_+)$  homeomorphic to  $\ell^2(A)$ ?

## Acknowledgments

This research was supported by the Slovenian Research Agency grants P1-0292-0101, J1-2057-0101, and J1-4144-0101. We thank the referee for comments and suggestions.

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