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Hyperspace of convex compacta of nonmetrizable compact convex subspaces of locally convex spaces ☆

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Abstract

Our main result states that the hyperspace of convex compact subsets of a compact convex subset X in a locally convex space is an absolute retract if and only if X is an absolute retract of weight $\leqslant \omega_1$. It is also proved that the hyperspace of convex compact subsets of the Tychonov cube I^{ω_1} is homeomorphic to I^{ω_1}. An analogous result is also proved for the cone over I^{ω_1}. Our proofs are based on analysis of maps of hyperspaces of compact convex subsets, in particular, selection theorems for such maps are proved.
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1. Introduction

For any uncountable cardinal number τ, the Tychonov and the Cantor cubes (denoted by I^τ and D^τ, respectively), belong to the class of main geometric objects in the topology of non-metrizable compact Hausdorff spaces. The spaces I^τ (we denote by I the segment $[0, 1]$) and D^τ were first characterized by Shchepin [13]. In particular, the Tychonov cubes are characterized as the homogeneous-by-character nonmetrizable compact Hausdorff absolute retracts [12]. This characterization was later applied to the study of topology of the *functor-powers*, i.e. spaces of the form $F(K^\tau)$, where K is a compact metrizable space and F is a covariant functor in the category of compact Hausdorff spaces. In particular, it was proved that, for an uncountable τ, the space $P(I^\tau)$, where P denotes the probability measure functor, is homeomorphic to I^τ if and only if $\tau = \omega_1$. For the hyperspace functor exp it is known that $\exp(D^\tau)$ is homeomorphic to D^τ if and only if $\tau = \omega_1$ and $\exp(I^\tau)$ is not an absolute retract whenever $\tau > \omega$.

In this paper we consider the hyperspaces $cc(X)$ of nonempty compact convex subsets in X, for compact convex subsets in locally convex spaces X. For metrizable X, this object was investigated by different authors (see, e.g.,

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[7,9]). In particular, it was proved in [9] that the hyperspace of convex compact subsets of the Hilbert cube $Q = I^\omega$ is homeomorphic to I^ω.

The aim of this paper is to consider the nonmetrizable compact convex subsets in locally convex spaces. One of our main results is Theorem 4.1, which characterizes the compact convex spaces X by $cc(X)$ being an absolute retract. We also show that the space $cc(X)$ is homeomorphic to I^{ω_1} (resp., the cone over I^{ω_1}) if and only if X is homeomorphic to I^{ω_1} (resp., the cone over I^{ω_1}).

These results are in the spirit of the corresponding results concerning the functor-powers of compact metric spaces (see [13]). The proofs are based on the spectral analysis of nonmetrizable compact Hausdorff spaces, in particular on the Shchepin Spectral Theorem [13] as well as on analysis of the selection type properties of the maps of the hyperspaces of compact convex subsets.

The construction cc determines a functor acting on the category **Conv** of compact convex subsets of locally convex spaces. The results of this paper demonstrate that the functor cc is closer to the functor P of probability measures than to the hyperspace functor exp.

2. Preliminaries

All topological spaces are assumed to be Tychonov, all maps are continuous. By $\bar{A}$ we denote the closure of a subset A of a topological space. Let X be any space.

The *hyperspace* $\exp X$ of X is the space of all nonempty compact subsets in X endowed with the *Vietoris* topology. A base of this topology is formed by the sets of the form

$$\langle U_1, \ldots, U_n \rangle = \{A \in \exp X \mid A \subset U_1 \cup \cdots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for every } i\},$$

where $U_1, \ldots, U_n$ run through the topology of X, $n \in \mathbb{N}$. For a metric space (X, ρ), the Vietoris topology on $\exp(X)$ is induced by the Hausdorff metric ρ_H:

$$\rho_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), \ B \subset O_\varepsilon(A)\}.$$

The hyperspace construction determines a functor in the category **Comp** of compact Hausdorff spaces and continuous maps. Given a map $f : X \to Y$ in **Comp**, we define $\exp(f) : \exp(X) \to \exp(Y)$ by $\exp(f)(A) = f(A)$, $A \in \exp(X)$.

Let **Conv** denote the category of compact convex subsets in locally convex spaces and affine continuous maps. If X is an object of **Conv** we define

$$cc(X) = \{A \in \exp(X) \mid A \text{ is convex}\} \subset \exp(X).$$

If $f : X \to Y$ is a map in **Conv**, then the map $cc(f) : cc(X) \to cc(Y)$ is defined as the restriction of $\exp(f)$ on $cc(X)$.

In the sequel, for a nonempty compact subset X in a locally convex space Y, we denote the closed convex hull map by $h : \exp X \to cc(Y)$. Let X be a subset of a metrizable locally convex space M. In the sequel, we identify any point $x \in X$ with the singleton $\{x\} \in cc(X)$.

Recall that the *Minkowski operation* in $cc(X)$ is defined as follows:

$$\lambda_1 A_1 + \lambda_2 A_2 = \{\lambda_1 x_1 + \lambda_2 x_2 \mid x_1 \in A_1, \ x_2 \in A_2\},$$

$\lambda_1, \lambda_2 \in \mathbb{R}$, $A_1, A_2 \in cc(X)$.

Lemma 2.1. *Let X be a compact convex subset in a locally convex space. There exists an embedding α of the space $cc(X)$ into a locally convex space L satisfying the condition*

$$\alpha(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \alpha(A_1) + \lambda_2 \alpha(A_2) \tag{2.1}$$

for every $\lambda_1, \lambda_2 \in \mathbb{R}$, $A_1, A_2 \in cc(X)$.

Proof. Let X be a compact convex subset in a metrizable locally convex space M. Following [11], consider the equivalence relation $\sim$ on $cc(M) \times cc(M)$ defined by the condition: $(A, B) \sim (C, D)$ if and only if $A + D = B + C$. Denote by L the set of equivalence classes of $\sim$ (in the sequel, we denote by $[A, B]$ the equivalence class that contains

(A, B)). It is well known that L is a linear space with respect to the naturally defined operations. Let U be a convex neighborhood of zero in M and define

$$U^* = \big\{[A, B] \in L \mid A \subset B + U, \ B \subset A + U\big\}.$$

The sets U^* form a base at zero in L. The map $\alpha : \mathrm{cc}(X) \to L$ defined by the formula $\alpha(A) = [A, \{0\}]$ is the required embedding. $\square$

3. Functor cc and soft maps

A map $f : X \to Y$ is *soft* (see [13]) if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[\ \varphi\ ]{} & Y \end{array}$$

where $i : A \to Z$ is a closed embedding into a paracompact space Z, there exists a map $\Phi : Z \to X$ such that $\Phi | A = \psi$ and $f\Phi = \varphi$.

In other words, a map is soft if it satisfies the parameterized selection extension property.

The following proposition is close to the Michael selection theorem for convex-valued maps [8].

Proposition 3.1. *Let $f : X \to Y$ be an affine open map of compact convex metrizable subsets of locally convex spaces. Then the map $\mathrm{cc}(f) : \mathrm{cc}(X) \to \mathrm{cc}(Y)$ is soft.*

Proof. We first prove that the map $\mathrm{cc}(f)$ is open. It is well known that the map $\exp(f)$ is open. Since the diagram

$$\begin{array}{ccc} (\exp(f))^{-1}(\mathrm{cc}(Y)) & \xrightarrow{\ \ \ h\ \ \ } & \mathrm{cc}(X) \\ & \searrow{\scriptstyle \exp(f)|(\exp(f))^{-1}(\mathrm{cc}(Y))} \quad \swarrow{\scriptstyle \mathrm{cc}(f)} & \\ & \mathrm{cc}(Y) & \end{array}$$

is commutative and the closed convex hull map h is a retraction of $(\exp(f))^{-1}(\mathrm{cc}(Y))$ onto $\mathrm{cc}(X)$, we see that the map $\mathrm{cc}(f)$ is also open.

There exists an embedding $\alpha : \mathrm{cc}(X) \to L$ satisfying condition (2.1). Choose a countable family of functionals $\{\varphi_1, \varphi_2, \ldots\} \subset L^*$ such that this family separates the points and $\varphi_i(\alpha(\mathrm{cc}(X))) \subset [-1/i, 1/i]$. Then the map $\varphi = (\varphi_1, \varphi_2, \ldots)$, defined on $\alpha(\mathrm{cc}(X))$, embeds $\alpha(\mathrm{cc}(X))$ into the Hilbert space ℓ^2. Denote by

$$\xi : \varphi\big(\alpha\big(\mathrm{cc}(X)\big)\big) \times \mathrm{cc}\big(\varphi\big(\alpha(\mathrm{cc}(X))\big)\big) \to \varphi\big(\alpha\big(\mathrm{cc}(X)\big)\big)$$

the nearest point map: $y = \xi(x, A)$ if and only if $\|z - x\| > \|y - x\|$, for every $z \in A \setminus \{y\}$ (here $\|\cdot\|$ denotes the norm in ℓ^2).

Suppose a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ p\ } & \mathrm{cc}(X) \\ \cap\downarrow & & \downarrow{\scriptstyle \mathrm{cc}(f)} \\ Z & \xrightarrow[\ q\ ]{} & \mathrm{cc}(Y) \end{array}$$

is given, where A is a closed subset of a paracompact space Z.

Since $\mathrm{cc}(X)$ is an absolute retract, there exists a map $r : Z \to \mathrm{cc}(X)$ such that $r|A = p$. Note that for every $B \in \mathrm{cc}(Y)$, the set $\varphi(\alpha(\mathrm{cc}(f)^{-1}(B)))$ is a convex closed subset of $\varphi(\alpha(\mathrm{cc}(X)))$, i.e. an element of the space $\mathrm{cc}(\varphi(\alpha(\mathrm{cc}(X))))$. Since the map $\mathrm{cc}(f)$ is open, the map

$$\delta : \mathrm{cc}(Y) \to \mathrm{cc}\big(\varphi\big(\alpha(\mathrm{cc}(X))\big)\big), \qquad \delta(B) = \varphi\big(\alpha\big(\mathrm{cc}(f)^{-1}(B)\big)\big),$$

is continuous.

Define the map $R : Z \to \mathrm{cc}(X)$ by the formula

$$R(z) = \alpha^{-1}\big(\varphi^{-1}\big(\xi(\varphi(\alpha(r(z))), \delta(q(z))))\big)\big), \quad z \in Z.$$

It is easy to see that R is continuous, $R|A = p$, and $\mathrm{cc}(f)R = q$. □

A point p of a set X in a locally convex space E is called an *exposed point* of X if there exists a continuous linear functional f on E such that $f(x) > f(p)$, for each $x \in X \setminus \{p\}$.

Lemma 3.2. *Let $f : X \to Y$ be an open affine continuous map of compact convex subsets in locally convex spaces such that $|f^{-1}(y)| > 1$ for every $y \in Y$. Then $|\mathrm{cc}(f)^{-1}(B)| > 1$, for every $B \in \mathrm{cc}(Y)$.*

Proof. As in the proof of Proposition 3.1, one may assume that X is affinely embedded in the Hilbert space ℓ^2. Let $B \in \mathrm{cc}(Y)$ and $A \in \mathrm{cc}(f)^{-1}(B)$. If $A \neq f^{-1}(B)$, then we define A' as the closure of the convex hull of $A \cup \{x\}$, where $x \in f^{-1}(B) \setminus A$. Then $A' \neq A$ and $A' \in \mathrm{cc}(f)^{-1}(B)$.

If $A = f^{-1}(B)$, then it is well-known (see e.g., [1]) that there exists an exposed point, x of A. Since f is open, there exists a neighborhood U of x such that $f(A \setminus U) = B$. In this case we define A' as the closure of the convex hull of $A \setminus U$. Note that $A' \in \mathrm{cc}(f)^{-1}(B)$. That $A \neq A'$ easily follows from the fact that x is an exposed point. □

Lemma 3.3. *Suppose that $f : X \to Y$ is a continuous affine map of compact convex subsets of locally convex spaces. If the map $\mathrm{cc}(f)$ is open then so is the map f.*

Proof. Suppose to the contrary, that f is not open. Then there exists $x \in X$ and a net $(y_\alpha)_{\alpha \in A}$ in Y converging to $y = f(x)$, such that there is no net $(x_\alpha)_{\alpha \in A}$ in X converging to x with $x_\alpha \in f^{-1}(y_\alpha)$, for every $\alpha \in A$.

Assuming that the map $\mathrm{cc}(f)$ is open, we obtain that there exists a net $(C_\alpha)_{\alpha \in A}$ in $\mathrm{cc}(X)$ converging to $\{x\}$ and such that $\mathrm{cc}(f)(C_\alpha) = \{y_\alpha\}$, for every $\alpha \in A$. Then, obviously, the net $(c_\alpha)_{\alpha \in A}$ converges to x, for every choice $c_\alpha \in C_\alpha$, $\alpha \in A$. This gives a contradiction. □

A commutative diagram

$$\mathcal{D} = \quad \begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ Z & \xrightarrow{\ v\ } & T \end{array} \tag{3.1}$$

is called *soft* if its *characteristic map*

$$\chi_\mathcal{D} = (f, g) : X \to Y \times_T Z = \big\{(y, z) \in Y \times Z \mid u(y) = v(z)\big\}$$

is soft.

Lemma 3.4. *Suppose that a commutative diagram $\mathcal{D}$ (see formula (3.1)) in the category **Conv** consists of metrizable spaces. If the diagram $\mathrm{cc}(\mathcal{D})$ is soft, then so is the diagram $\mathcal{D}$.*

Proof. First we show that the diagram $\mathcal{D}$ is open if such is $\mathrm{cc}(\mathcal{D})$. Let $(y_i, z_i)_{i=1}^\infty$ be a sequence in $Y \times_T Z$ converging to a point (y, z) and let $x \in X$ be such that $\chi_\mathcal{D}(x) = (y, z)$. Since $\mathrm{cc}(\mathcal{D})$ is soft (and therefore open), there exists a sequence $(A_i)_{i=1}^\infty$ in $\mathrm{cc}(X)$ such that $(f(A_i), g(A_i)) = (\{y_i\}, \{z_i\})$, for every i, and $(A_i)_{i=1}^\infty$ converges to $\{x\}$ in $\mathrm{cc}(X)$. Choose arbitrary $x_i \in A_i$, then $(f(x_i), g(x_i)) = (y_i, z_i)$, for every i, and $(x_i)_{i=1}^\infty$ converges to x in X. This shows that the map $\chi_\mathcal{D}$ is open.

Now the map $\chi_\mathcal{D}$, being an open affine map of convex compact metrizable subspaces of locally convex spaces, is soft. This follows from the Michael Selection Theorem [8] (see e.g., [13]). □

4. Hyperspaces cc(X) homeomorphic to Tychonov cubes

We are going to recall some definitions and results related to the Shchepin Spectral Theorem (see [13] for details). In what follows, an *inverse system* $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ satisfies the following conditions:

(1) X_α are compact Hausdorff spaces;
(2) $p_{\alpha\beta}$ are surjective;
(3) the partially ordered set $\mathcal{A}$ (by $\leqslant$) is directed, i.e., for every $\alpha, \beta \in \mathcal{A}$ there exists $\gamma \in \mathcal{A}$ with $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$.

An inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ is called *open* if all the maps $p_{\alpha\beta}$ are open. An inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ is called *continuous* if for every $\alpha \in \mathcal{A}$ we have $X_\alpha = \varprojlim\{X_{\alpha'}, p_{\alpha'\beta'}; \alpha', \beta' < \alpha\}$.

By w(X) we denote the *weight* of a space X. An inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ is called a τ-*system*, τ being a cardinal number, if the following holds:

(1) the directed set $\mathcal{A}$ is τ-complete, i.e. every chain of cardinality $\leqslant \tau$ in $\mathcal{A}$ has the least upper bound;
(2) $\mathcal{S}$ is continuous;
(3) w$(X_\alpha) \leqslant \tau$, for every $\alpha \in \mathcal{A}$.

If $\tau = \omega$, we use the terms σ-*complete* and σ-*system*.

For every A, we denote the family of all countable subsets of A ordered by inclusion by $\mathcal{P}_\omega(A)$.

A standard way to represent a compact Hausdorff space X as a limit of a σ-system is to embed it into a Tychonov cube I^τ, for some τ. For any countable $A \subset \tau$, let $X_A = p_A(X)$, where $p_A : I^\tau \to I^A$ denotes the projection. In this way we obtain an inverse system $\mathcal{S} = \{X_A, p_{AB}; \mathcal{P}_\omega(\tau)\}$, where, for $A \supset B$, $p_{AB} : X_A \to X_B$ denotes the (unique) map with the property $p_B | X = p_{AB}(p_A | X)$. The resulting inverse system $\mathcal{S}$ is a σ-system and $X = \varprojlim \mathcal{S}$.

If X is a compact convex subset of a locally convex space, we can affinely embed X into I^τ, for some τ. The above construction gives us an inverse σ-system $\mathcal{S}$ in the category **Conv** such that $X = \varprojlim \mathcal{S}$.

In the sequel, we will use the well-known fact that the functor cc is continuous in the sense that it commutes with the limits of inverse systems.

A compact Hausdorff space X is *openly generated* if X is the limit of an inverse σ-system with open short projections. The *absolute retracts* (ARs) are considered in the class of compact Hausdorff spaces.

Theorem 4.1. *Let X be a convex compact subset of a locally convex space. Then the space* cc(X) *is an absolute retract if and only if X is openly generated and of weight $\leqslant \omega_1$.*

Proof. If X is openly generated and of weight $\leqslant \omega_1$, then X is homeomorphic to $\varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ is an inverse system consisting of convex compact subsets of metrizable locally convex spaces and open maps. Then cc(X) is homeomorphic to $\varprojlim$cc$(\mathcal{S})$. Since the spaces cc(X_α) are ARs and the maps cc$(p_{\alpha\beta})$ are soft (see Proposition 3.1), the space cc(X) is an AR.

Suppose now that cc(X) is an AR of weight $\geqslant \omega_2$. It easily follows from standard results of Shchepin's theory that there exists a compact convex space $\tilde{X}$ of weight ω_2 such that cc$(\tilde{X})$ is an AR (see [13] and also [4], where the case of locally convex spaces is considered). We may assume that cc$(\tilde{X}) = \varprojlim$cc$(\tilde{\mathcal{S}})$, where $\tilde{\mathcal{S}} = \{\tilde{X}_\alpha, \tilde{p}_{\alpha\beta}; \omega_2\}$ is an inverse system such that for every $\alpha < \omega_2$ the space cc$(\tilde{X}_\alpha)$ is an AR and for every $\alpha, \beta, \beta \leqslant \alpha < \omega_2$, the map cc$(\tilde{p}_{\alpha\beta})$ is soft. In its turn, every $\tilde{X}_\alpha$ can be represented as $\varprojlim \tilde{\mathcal{S}}_\alpha$, where $\tilde{\mathcal{S}}_\alpha = \{\tilde{X}_{\alpha\gamma}, \tilde{q}_{\gamma\delta}^\alpha; \omega_1\}$ is an inverse system in **Conv** and it follows from the results of Chigogidze [4] that for every α, β, where $\beta \leqslant \alpha < \omega_2$, the map $\tilde{p}_{\alpha\beta}$ is the limit of a morphism $(\tilde{p}_{\alpha\beta}^\gamma)_{\gamma < \omega_1} : \tilde{\mathcal{S}}_\alpha \to \tilde{\mathcal{S}}_\beta$ such that the maps cc$(\tilde{p}_{\alpha\beta}^\gamma)$ are soft and for every $\gamma \geqslant \delta$, $\gamma, \delta < \omega_1$, the diagram

$$
\begin{array}{ccc}
\mathrm{cc}(\tilde{X}_{\alpha\gamma}) & \xrightarrow{\mathrm{cc}(\tilde{p}_{\alpha\beta}^\gamma)} & \mathrm{cc}(\tilde{X}_{\beta\gamma}) \\
{\scriptstyle \mathrm{cc}(q_{\gamma\delta}^\alpha)} \downarrow & & \downarrow {\scriptstyle \mathrm{cc}(q_{\gamma\delta}^\beta)} \\
\mathrm{cc}(\tilde{X}_{\alpha\delta}) & \xrightarrow[\mathrm{cc}(\tilde{p}_{\alpha\beta}^\delta)]{} & \mathrm{cc}(\tilde{X}_{\beta\delta})
\end{array}
$$

is soft. Since all the spaces in the above diagram are metrizable, by Lemma 3.4, the diagram

$$
\begin{array}{ccc}
\tilde{X}_{\alpha\gamma} & \xrightarrow{\ \tilde{p}^{\gamma}_{\alpha\beta}\ } & \tilde{X}_{\beta\gamma} \\
{\scriptstyle q^{\alpha}_{\gamma\delta}}\Big\downarrow & & \Big\downarrow{\scriptstyle q^{\beta}_{\gamma\delta}} \\
\tilde{X}_{\alpha\delta} & \xrightarrow[\ \tilde{p}^{\delta}_{\alpha\beta}\ ]{} & \tilde{X}_{\beta\delta}
\end{array}
$$

is also soft. As the limits of soft morphisms, the maps $\tilde{p}_{\alpha\beta}$ are soft and we conclude that the space $\tilde{X}$ is an absolute retract.

Since the space $\tilde{X}$ is an AR, it contains a copy of the Tychonov cube I^{ω_2}. It follows from the Shchepin Spectral Theorem that, without loss of generality, one may assume that every $\tilde{X}_\alpha$ contains the space $(I^{\omega_1})^\alpha$ and for every α, β, where $\beta \leqslant \alpha < \omega_2$, the map $\tilde{p}_{\alpha\beta}|(I^{\omega_1})^\alpha$ is the projection map of $(I^{\omega_1})^\alpha$ onto $(I^{\omega_1})^\beta$.

Denote by D the Aleksandrov supersequence of weight ω_1, i.e. the one-point compactification of a discrete space of cardinality ω_1.

Claim. *There exists $\alpha < \omega_2$ such that the subspace $(I^{\omega_1})^\alpha \subset \tilde{X}_\alpha$ contains an affinely independent copy of the space D.*

Proof. Represent D as $\{d_\gamma \mid \gamma \leqslant \omega_1\}$, where d_{ω_1} denotes the unique non-isolated point of D. For $\gamma < \omega_1$, let $r_\gamma : D \to \{d_\delta \mid \delta \leqslant \gamma\} \cup \{d_{\omega_1}\}$ denote the retraction that sends $\{d_\delta \mid \gamma < \delta < \omega_1\}$ into d_{ω_1}.

Define by transfinite induction maps $f_\gamma : D \to (I^{\omega_1})^{\alpha_\gamma} \subset \tilde{X}_{\alpha_\gamma}$, where $\gamma < \omega_1$ and $\alpha_\gamma < \omega_2$, so that $\alpha_\gamma \leqslant \alpha_{\gamma'}$ and $\tilde{p}_{\alpha_{\gamma'}\alpha_\gamma} f_{\gamma'} = f_\gamma$ for every $\gamma \leqslant \gamma'$.

Let $f_0 : D \to (I^{\omega_1})^{\alpha_0} \subset \tilde{X}_{\alpha_0}$ be an arbitrary constant map, for some $\alpha_0 < \omega_2$. Suppose that, for some $\delta < \omega_1$, maps f_γ are already defined for every $\gamma < \delta$ so that $f_\gamma = i_\gamma r_\gamma$ for some embedding $i_\gamma : r_\gamma(D) \to \tilde{X}_{\alpha_\gamma}$. If δ is a limit ordinal, let $\alpha_\delta = \sup\{\alpha_\gamma \mid \gamma < \delta\}$ and $f_\delta = \varprojlim\{f_\gamma \mid \gamma < \delta\}$. If $\delta = \delta' + 1$, let $\alpha_\delta = \alpha_{\delta'} + 1$ and find an embedding $i_\delta : r_\delta(D) \to (I^{\omega_1})^{\alpha_\delta} \subset \tilde{X}_{\alpha_\delta}$ such that $\tilde{p}_{\alpha_\delta\alpha_{\delta'}} i_\delta = i_{\delta'}$ and $\tilde{p}_{\alpha_\delta\alpha_{\delta'}} i_\delta(d_\delta) = i_{\delta'}(d_{\delta'})$. Put $f_\delta = i_\delta r_\delta$.

Finally, let $\alpha = \sup\{\alpha_\gamma \mid \gamma < \omega_1\}$ and $f = \varprojlim\{f_\gamma \mid \gamma < \omega_1\}$. Claim is thus proved. $\square$

We now return to the proof of the theorem. Without loss of generality, we assume that $D \subset (I^{\omega_1})^\alpha \subset \tilde{X}_\alpha$ and D is affinely independent in $\tilde{X}_\alpha$. Recall that $h(D)$ denotes the closed convex hull of D in $\tilde{X}_\alpha$. We are going to show that the space $(\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha}))^{-1}(h(D))$ does not satisfy the Souslin condition. There exist two maps $s_1, s_2 : D \to \tilde{X}_{\alpha+1}$ such that $\tilde{p}_{\alpha+1,\alpha}s_1 = \tilde{p}_{\alpha+1,\alpha}s_2 = 1_D$ and $s_1(D) \cap s_2(D) = \emptyset$. Let U_1, U_2 be neighborhoods of $s_1(D)$ and $s_2(D)$, respectively, such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$.

For every isolated point $y \in D$ let V_y be a neighborhood of y in $\tilde{X}_\alpha$ such that $\bar{V}_y \cap h(D \setminus \{y\}) = \emptyset$.

Let

$$
W_y = \big\langle \tilde{X}_{\alpha+1} \setminus \big(\bar{U}_2 \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(D \setminus \{y\})\big), U_2 \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(\bar{V}_y) \big\rangle.
$$

We are going to show that $\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D)) \cap W_y \neq \emptyset$. To this end, consider the set $B = h(s_1(D \setminus \{y\}) \cup \{s_2(y)\})$. Obviously, $B \in \mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D))$ and $s_2(y) \in B \cap U_2 \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(\bar{V}_y)$. In addition, for every $z \in D \setminus \{d_{\omega_1}\}$, $z \neq y$, we have $B \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(z) = \{s_1(z)\}$, therefore $B \subset \tilde{X}_{\alpha+1} \setminus (\bar{U}_2 \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(D \setminus \{y\}))$. We conclude that $B \in W_y$.

It remains to prove that for every $y, z \in D \setminus \{d_{\omega_1}\}$, $y \neq z$, we have $W_y \cap W_z \cap \mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D)) \neq \emptyset$. Indeed, otherwise, for any $A \in W_y \cap W_z \cap \mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D))$ we would have $A \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(y) \subset p^{-1}_{\alpha+1,\alpha}(y) \setminus \bar{U}_2$ and, on the other hand, $A \cap \tilde{p}^{-1}_{\alpha+1,\alpha}(y) \subset U_2$, a contradiction. We therefore conclude that

$$
\big\{W_y \cap \mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}\big(h(D)\big) \mid y \in D \setminus \{d_{\omega_1}\}\big\}
$$

is a family of nonempty disjoint open subsets in $\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D))$. Since the space $\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D))$ does not satisfy the Souslin condition, we obtain that $\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})^{-1}(h(D)) \notin \mathrm{AR}$ and hence the map $\mathrm{cc}(\tilde{p}_{\alpha+1,\alpha})$ is not a soft map. This contradiction demonstrates that $w(X) \leqslant \omega_1$.

We are going to show that X is openly generated. Since $\mathrm{cc}(X)$ is an AR of weight ω_1, there exists an inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ consisting of compact metrizable convex spaces and affine maps such that $\mathrm{cc}(X) = \varprojlim \mathrm{cc}(\mathcal{S})$.

Applying Shchepin's Spectral Theorem, we may additionally assume that all the maps $\mathrm{cc}(p_{\alpha\beta})$, $\beta \leqslant \alpha < \omega_1$, are soft. By Lemma 3.3, the maps $p_{\alpha\beta}$, $\beta \leqslant \alpha < \omega_1$, are soft and therefore open. □

Theorem 4.2. *Let X be a convex compact subset of a locally convex space. The space $\mathrm{cc}(X)$ is homeomorphic to I^{ω_1} if and only if X is homeomorphic to I^{ω_1}.*

Proof. We use the following characterization of the Tychonov cube I^τ, $\tau > \omega$, due to Shchepin [13]: a compact Hausdorff space X of weight $\tau > \omega$ is homeomorphic to the Tychonov cube I^τ if and only if X is a character homogeneous absolute retract. Recall that a space is called *character homogeneous* if the characters of all of its points are equal.

If the weight of X is ω_1, then it easily follows from the Shchepin Spectral Theorem [13] that X can be represented as $\varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ is an inverse system consisting of convex compact metrizable subsets in locally convex spaces and affine continuous maps. Since the functor cc is continuous (see, e.g., [10]), we obtain that $\mathrm{cc}(X) = \varprojlim\{\mathrm{cc}(X_\alpha), \mathrm{cc}(p_{\alpha\beta}); \omega_1\}$. Since $\mathrm{cc}(X_\alpha)$ is an absolute retract (see [15]) and, by Proposition 3.1, the map $\mathrm{cc}(p_{\alpha\beta})$ is soft for every $\alpha, \beta < \omega_1, \alpha \geqslant \beta$, we apply a result of Shchepin (see [13]) to derive that $\mathrm{cc}(X)$ is an absolute retract.

If X is character homogeneous, then we can in addition assume that no projection $p_{\alpha\beta}$ possesses one-point preimages. By Lemma 3.2, the maps $\mathrm{cc}(p_{\alpha\beta})$ do not possess one-point preimages and therefore $\mathrm{cc}(X)$ is character homogeneous. By the mentioned result of Shchepin, $\mathrm{cc}(X)$ is homeomorphic to I^{ω_1}.

If $\mathrm{cc}(X)$ is homeomorphic to I^{ω_1}, then there exists an inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ consisting of compact metrizable convex spaces and open affine maps such that $\mathrm{cc}(X) = \varprojlim \mathrm{cc}(\mathcal{S})$. Applying Shchepin's Spectral Theorem, we may additionally assume that all the maps $\mathrm{cc}(p_{\alpha\beta})$, $\beta \leqslant \alpha < \omega_1$, are soft and do not possess points with one-point preimage. It is then evident that the maps $p_{\alpha\beta}$, $\beta \leqslant \alpha < \omega_1$, do not possess points with one-point preimage. Applying Lemma 3.3 we conclude that the maps $p_{\alpha\beta}$, $\beta \leqslant \alpha < \omega_1$, are open and therefore, by the Michael Selection Theorem, soft. Then X is a character homogeneous AR of weight ω_1. By the cited characterization theorem for I^{ω_1}, the space X is homeomorphic to I^{ω_1}. □

5. Cone over Tychonov cube

Define the *cone functor* cone in the category **Conv** as follows. Given an object X in **Conv**, i.e. a compact convex subset X in a locally convex space L, let $\mathrm{cone}(X)$ be the convex hull of the set $X \times \{0\} \cup \{(0, 1)\}$ in $L \times \mathbb{R}$. For a morphism $f : X \to Y$ in **Conv** define $\mathrm{cone}(f) : \mathrm{cone}(X) \to \mathrm{cone}(Y)$ as the only affine continuous map that extends $f \times \{0\} : X \times \{0\} \to Y \times \{0\}$ and sends $(0, 1) \in \mathrm{cone}(X)$ to $(0, 1) \in \mathrm{cone}(Y)$.

We will need the following notion. A map $f : X \to Y$ is called a *trivial Q-bundle* if there exists a homeomorphism $g : X \to Y \times Q$ such that $f = \mathrm{pr}_1 g$. The following statement is a characterization theorem for the space $\mathrm{cone}(I^{\omega_1})$ among the convex compact spaces.

Proposition 5.1. *A convex compactum X is homeomorphic to the space $\mathrm{cone}(I^{\omega_1})$ if and only if X satisfies the following properties*:

(1) *X is an AR*;
(2) *$\mathrm{w}(X) = \omega_1$; and*
(3) *there exists a unique point $x \in X$ of countable character.*

Proof. Obviously, if a convex compactum X is homeomorphic to $\mathrm{cone}(I^{\omega_1})$, then X satisfies properties (1)–(3).

Suppose now that X satisfies (1)–(3). Then X is homeomorphic to the limit of a continuous inverse system $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ in **Conv** which satisfies the properties

  (i) X_α is a convex metrizable compactum for every α;
 (ii) $p_{\alpha\beta}$ is an open affine map for every $\alpha \geqslant \beta$; and
(iii) $\{x_\beta\} = \{y \in X_\beta \mid |p_{\alpha\beta}^{-1}(y)| = 1\}$.

Passing, if necessary, to a subsystem of $\mathcal{S}$, one can assume that for every α and every compact subset K of $X_\alpha \setminus \{x_\alpha\}$ the map

$$p_{\alpha+1,\alpha}|p_{\alpha+1,\alpha}^{-1}(K) : p_{\alpha+1,\alpha}^{-1}(K) \to K$$

satisfies the condition of fibrewise disjoint approximation. The Toruńczyk–West characterization theorem [14] implies that, if K is an AR, the map $p_{\alpha+1,\alpha}|p_{\alpha+1,\alpha}^{-1}(K)$ is a trivial Q-bundle and therefore the map

$$p_{\alpha+1,\alpha}|p_{\alpha+1,\alpha}^{-1}\big(X_\alpha \setminus \{x_\alpha\}\big) = p_{\alpha+1,\alpha}|\big(X_{\alpha+1} \setminus \{x_{\alpha+1}\}\big),$$

being a locally trivial Q-bundle, is a trivial Q-bundle (see [2]). Therefore, the map $p_{\alpha+1,\alpha}$ is homeomorphic to the projection map $\mathrm{pr}_{23}: Q \times Q \times [0,1) \to Q \times [0,1)$ (that $X_\alpha \setminus \{x_\alpha\}$ is homeomorphic to $Q \times [0,1)$ follows from the fact that the spaces Q and $\mathrm{cone}(Q)$ are homeomorphic—see [3]). Passing to the one-point compactifications of these maps we obtain the commutative diagram
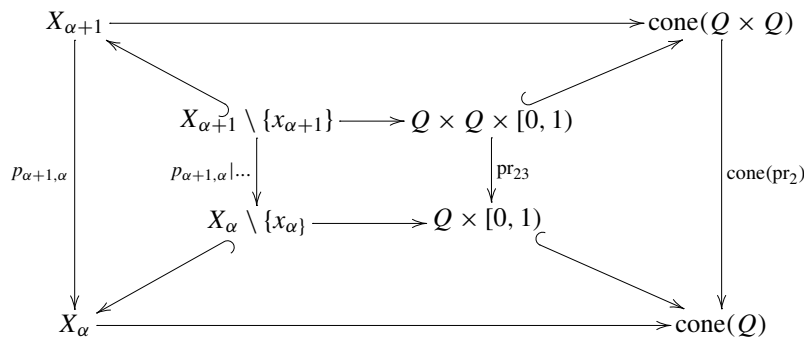
in which the horizontal arrows are homeomorphisms. Therefore X and $\mathrm{cone}(I^{\omega_1})$ are homeomorphic. □

Theorem 5.2. *Let X be an object of the category* **Conv***. The space* $\mathrm{cc}(X)$ *is homeomorphic to the cone over the Tychonov cube,* $\mathrm{cone}(I^{\omega_1})$*, if and only if X is homeomorphic to the space* $\mathrm{cone}(I^{\omega_1})$*.*

Proof. Suppose that a convex compact space X is an absolute retract of weight ω_1 with exactly one point x, of countable character. It follows from the Shchepin Spectral Theorem ([13]; see also [4]) that X can be represented as $\varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ is an inverse system in which every X_α is a metrizable convex compactum and every $p_{\alpha\beta}$, $\alpha \geqslant \beta$, is an affine map. Denote by $p_\alpha: X \to X_\alpha$ the limit projections and let $x_\alpha = p_\alpha(x)$. Passing, if necessary, to a subsystem of $\mathcal{S}$, one can assume additionally that for every $\alpha \geqslant \beta$ we have $\{x_\beta\} = \{y \in X_\beta \mid |p_{\alpha\beta}^{-1}(y)| = 1\}$.

Then for every $\alpha \geqslant \beta$, the map $\mathrm{cc}(p_{\alpha\beta})$ is a soft map and similarly as in the proof of Lemma 3.2, one can show that

$$\big\{\{x_\beta\}\big\} = \big\{A \in \mathrm{cc}(X_\beta) \mid \big|\mathrm{cc}(p_{\alpha\beta})^{-1}(A)\big| = 1\big\}.$$

We conclude that the space $\mathrm{cc}(X) = \varprojlim(\mathcal{S})$ satisfies the conditions of Proposition 5.1 and therefore is homeomorphic to the space $\mathrm{cone}(I^{\omega_1})$.

Now, if $\mathrm{cc}(X)$ is homeomorphic to $\mathrm{cone}(I^{\omega_1})$, it follows from Theorem 4.1 that X is an AR of weight ω_1. Note that for every point x of countable character in X, the point $\{x\}$ is of countable character in $\mathrm{cc}(X)$. We therefore conclude that there is a unique point of countable character in X. By Proposition 5.1, X is homeomorphic to $\mathrm{cone}(I^{\omega_1})$. □

6. Remarks and open problems

Problem 6.1. Let $f: X \to Y$ be an affine continuous map of compact metrizable compacta in locally convex spaces such that $\dim f^{-1}(y) \geqslant 2$, for every $y \in Y$. Is the map $\mathrm{cc}(f): \mathrm{cc}(X) \to \mathrm{cc}(Y)$ homeomorphic to the projection map $\mathrm{pr}_1: Q \times Q \to Q$?

Note that there is an open map $f: X \to Y$ of metrizable compacta with infinite fibers such that the map $P(f): P(X) \to P(Y)$ is not homeomorphic to the projection map $\mathrm{pr}_1: Q \times Q \to Q$ (see [6]). (Recall that P denotes the probability measure functor.)

Problem 6.2. Does every compact convex AR of weight $\tau \geqslant \omega_1$ contain an affine copy of the Tychonov cube I^τ?

It is known that every compact Hausdorff AR of weight $\tau \geqslant \omega_1$ contains a topological copy of the Tychonov cube I^τ (see [12]).

The theory of nonmetrizable noncompact absolute extensors which is, in some sense, parallel to that of compact absolute extensors, was elaborated by Chigogidze [4,5]. One can also consider the hyperspaces of compact subsets in the spaces $\mathbb{R}^\tau$ and conjecture that for noncountable τ, the hyperspace $cc(\mathbb{R}^\tau)$ is homeomorphic to $\mathbb{R}^\tau$ if and only if $\tau = \omega_1$.

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