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Low Perturbations for a Class of Nonuniformly Elliptic Problems

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Abstract. In this paper, we introduce and study a new functional which was motivated by the work of Bahrouni et al. (Nonlinearity 31:1518–1534, 2018) on the Caffarelli–Kohn–Nirenberg inequality with variable exponent. We also study the eigenvalue problem for equations involving this new functional.

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1. Introduction

The Caffarelli–Kohn–Nirenberg inequality plays an important role in studying various problems of mathematical physics, spectral theory, analysis of linear and nonlinear PDEs, harmonic analysis, and stochastic analysis. We refer to [2,4,7,8] for relevant applications of the Caffarelli–Kohn–Nirenberg inequality.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary. The following Caffarelli–Kohn–Nirenberg inequality [5] establishes that given $p \in (1, N)$ and real numbers a, b, and q, such that:

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = \frac{Np}{N-p(1+a-b)},$$

there is a positive constant $C_{a,b}$, such that for every $u \in C_c^1(\Omega)$:

$$\left(\int_\Omega |x|^{-bq}|u|^q \, \mathrm{d}x\right)^{p/q} \leq C_{a,b} \int_\Omega |x|^{-ap}|\nabla u|^p \, \mathrm{d}x\,. \tag{1}$$

This inequality has been extensively studied (see, e.g., [1–3,6,11] and the references therein).

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In particular, Bahrouni et al. [3] gave a new version of the Caffarelli–Kohn–Nirenberg inequality with variable exponent. They proved the next theorem under the following assumptions: let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary and suppose that the following hypotheses are satisfied:

(A) $a : \overline{\Omega} \to \mathbb{R}$ is a function of class C^1 and there exist $x_0 \in \Omega$, $r > 0$, and $s \in (1, +\infty)$, such that:
  (1) $|a(x)| \neq 0$, for every $x \in \overline{\Omega} \setminus \{x_0\}$;
  (2) $|a(x)| \geq |x - x_0|^s$, for every $x \in B(x_0, r)$;

(P) $p : \overline{\Omega} \to \mathbb{R}$ is a function of class C^1 and $2 < p(x) < N$ for every $x \in \Omega$.

Theorem 1.1. (Bahrouni et al. [3]) *Suppose that hypotheses* (A) *and* (P) *are satisfied. Let* $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *be a bounded domain with smooth boundary. Then, there exists a positive constant* β*, such that:*

$$\begin{aligned}\int_\Omega |a(x)|^{p(x)}|u(x)|^{p(x)}\mathrm{d}x \leq \beta \int_\Omega |a(x)|^{p(x)-1}|\nabla a(x)||u(x)|^{p(x)}\mathrm{d}x \\ + \beta\left(\int_\Omega |a(x)|^{p(x)}|\nabla u(x)|^{p(x)}\mathrm{d}x \right.\\ \left. + \int_\Omega |a(x)|^{p(x)}|\nabla p(x)||u(x)|^{p(x)+1}\mathrm{d}x\right) \\ + \beta\int_\Omega |a(x)|^{p(x)-1}|\nabla p(x)||u(x)|^{p(x)-1}\mathrm{d}x.\end{aligned}$$

for every $u \in C_c^1(\Omega)$.

Motivated by [3], we introduce and study in the present paper a new functional $T : E_1 \to \mathbb{R}$ via the Caffarelli–Kohn–Nirenberg inequality, in the framework of variable exponents. More precisely, we study the eigenvalue problem in which functional T is present. Our main result is Theorem 4.2 and we prove it in Sect. 5.

2. Function Spaces with Variable Exponent

We recall some necessary properties of variable exponent spaces. We refer to [10,12,13,15–17] and the references therein. Consider the set:

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \mid \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, let

$$p^+ = \sup_{x \in \overline{\Omega}} p(x) \qquad \text{and} \qquad p^- = \inf_{x \in \overline{\Omega}} p(x),$$

and define the *variable exponent Lebesgue space* as follows:

$$L^{p(x)}(\Omega) = \left\{ u \mid \ u \text{ is measurable real-valued function, such that } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty \right\},$$

with the *Luxemburg norm*:

$$|u|_{p(x)} = \inf\left\{\mu > 0 \mid \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,\mathrm{d}x \leq 1\right\}.$$

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces if and only if $1 < p^- \leq p^+ < \infty$, and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, then the following Hölder-type inequality holds:

$$\left| \int_\Omega uv \, \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \,. \tag{2}$$

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $p(.)-$modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by:

$$\rho(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x.$$

Proposition 2.1. (See [17]) *The following properties hold:*

(i) $|u|_{p(x)} < 1 (\textit{resp.}, = 1; > 1) \Leftrightarrow \rho(u) < 1 (\textit{resp.}, = 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$; *and*

(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.

Proposition 2.2. (See [17]) *If* $u, u_n \in L^{p(x)}(\Omega)$ *and* $n \in \mathbb{N}$, *then the following statements are equivalent:*

1. $\lim_{n \to +\infty} |u_n - u|_{p(x)} = 0$.
2. $\lim_{n \to +\infty} \rho(u_n - u) = 0$.
3. $u_n \to u$ *in measure in* Ω *and* $\lim_{n \to +\infty} \rho(u_n) = \rho(u)$.

We define the *variable exponent Sobolev space* by:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid \ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$, we consider the following norm:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Then, $W^{1,p(x)}(\Omega)$ is a reflexive separable Banach space.

3. Functional T

We shall introduce a new functional $T : E_1 \to \mathbb{R}$ motivated by the Caffarelli–Kohn–Nirenberg inequality obtained in [3]. We denote by E_1 the closure of $C_c^1(\Omega)$ under the norm:

$$\begin{aligned} \|u\| = &\, ||B(x)|^{\frac{1}{p(x)}} \nabla u(x)|_{p(x)} + |A(x)^{\frac{1}{p(x)}} u(x)|_{p(x)} + \\ &\, ||D(x)|^{\frac{1}{p(x)+1}} u(x)|_{p(x)+1} + ||C(x)|^{\frac{1}{p(x)-1}} u(x)|_{p(x)-1}, \end{aligned}$$

where the potentials A, B, C, and D are defined by:

$$\begin{cases} A(x) = |a(x)|^{p(x)-1} |\nabla a(x)| \\ B(x) = |a(x)|^{p(x)} \\ C(x) = |a(x)|^{p(x)-1} |\nabla p(x)| \\ D(x) = B(x) |\nabla p(x)|. \end{cases} \tag{3}$$

We now define $T : E_1 \to \mathbb{R}$ as follows:

$$T(u) = \int_\Omega \frac{B(x)}{p(x)}|\nabla u(x)|^{p(x)}\mathrm{d}x + \int_\Omega \frac{A(x)}{p(x)}|u(x)|^{p(x)}\mathrm{d}x + \int_\Omega \frac{D(x)}{p(x)+1}|u(x)|^{p(x)+1}\mathrm{d}x + \int_\Omega \frac{C(x)}{p(x)-1}|u(x)|^{p(x)-1}\mathrm{d}x.$$

The following properties of T will be useful in the sequel.

Lemma 3.1. *Suppose that hypotheses* (A) *and* (P) *are satisfied. Then, the functional* T *is well defined on* E_1*. Moreover,* $T \in C^1(E_1, \mathbb{R})$ *with the derivative given by:*

$$\langle L(u), v\rangle = \langle T'(u), v\rangle = \int_\Omega B(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)\mathrm{d}x + \int_\Omega A(x)|u(x)|^{p(x)-2}u(x)v(x)\mathrm{d}x + \int_\Omega D(x)|u(x)|^{p(x)-1}u(x)v(x)\mathrm{d}x + \int_\Omega C(x)|u(x)|^{p(x)-3}u(x)v(x)\mathrm{d}x,$$

for every $u, v \in E_1$.

Proof. The proof is standard, see [17]. □

Lemma 3.2. *Suppose that hypotheses* (A) *and* (P) *are satisfied. Then, the following properties hold*

(i) $L : E_1 \to E_1^*$ *is a continuous, bounded, and strictly monotone operator;*
(ii) L *is a mapping of type* (S_+)*, i.e., if* $u_n \rightharpoonup u$ *in* E_1 *and:*

$$\limsup_{n\to+\infty}\langle L(u_n) - L(u), u_n - u\rangle \le 0,$$

then $u_n \to u$ *in* E_1.

Proof. (i) Evidently, L is a bounded operator. Recall the following Simon inequalities: [18]:

$$\begin{cases} |x-y|^p \le c_p\left(|x|^{p-2}x - |y|^{p-2}y\right).(x-y) & \text{for } p \ge 2 \\ |x-y|^p \le C_p\left[\left(|x|^{p-2}x - |y|^{p-2}y\right).(x-y)\right]^{\frac{p}{2}}\left(|x|^p + |y|^p\right)^{\frac{2-p}{2}} & \text{for } 1 < p < 2, \end{cases} \tag{4}$$

for every $x, y \in \mathbb{R}^N$, where:

$$c_p = (\frac{1}{2})^{-p} \quad \text{and} \quad C_p = \frac{1}{p-1}.$$

Using inequalities (4) and recalling that $2 < p^-$, we can prove that L is a strictly monotone operator.

(ii) The proof is identical to the proof of Theorem 3.1 in [9].

□

4. Main Theorem

We recall the Compactness Lemma from [3].

Lemma 4.1. (Bahrouni el al. [3]) *Suppose that hypotheses* (A) *and* (P) *are satisfied and that* $p^- > 1+s$. *Then,* E_1 *is compactly embeddable in* $L^q(\Omega)$ *for each* $q \in (1, \frac{Np^-}{N+sp^+})$. *Moreover, the same conclusion holds if we replace* $L^q(\Omega)$ *by* $L^{q(x)}(\Omega)$, *provided that* $q^+ < \frac{Np^-}{N+sp^+}$.

We are concerned with the following nonhomogeneous problem:

$$\begin{cases} -\mathrm{div}\,(B(x)|\nabla u|^{p(x)-2}\nabla u) + (A(x)|u|^{p(x)-2} + C(x)|u|^{p(x)-3})u \\ \quad = (\lambda |u|^{q(x)-2} - D(x)|u|^{p(x)-1})u \quad \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\lambda > 0$ is a real number and q is continuous on $\overline{\Omega}$. We suppose that q satisfies the following basic inequalities:

$$(Q) \quad 1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} (p(x)-1) < \max_{x\in\overline{\Omega}} q(x) < \frac{Np^-}{N+sp^+}.$$

We can now state the main result of this paper.

Theorem 4.2. *Suppose that all hypotheses of Lemma 4.1 are satisfied and that inequalities* (Q) *hold. Then, there exists* $\lambda_0 > 0$, *such that every* $\lambda \in (0, \lambda_0)$ *is an eigenvalue for problem* (5).

To prove Theorem 4.2 (which will be done in the Sect. 5), we shall need some preliminary results. We begin by defining the functional $I_\lambda : E_1 \to \mathbb{R}$:

$$I_\lambda(u) = \int_\Omega \frac{B(x)}{p(x)} |\nabla u(x)|^{p(x)} \mathrm{d}x + \int_\Omega \frac{A(x)}{p(x)} |u(x)|^{p(x)} \mathrm{d}x + \int_\Omega \frac{C(x)}{p(x)-1} |u(x)|^{p(x)-1} \mathrm{d}x \\ + \int_\Omega \frac{D(x)}{p(x)+1} |u(x)|^{p(x)+1} \mathrm{d}x - \lambda \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} \mathrm{d}x.$$

Standard argument shows that $I_\lambda \in C^1(E_1, \mathbb{R})$ and:

$$\langle I'_\lambda(u), v\rangle = \int_\Omega B(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)\mathrm{d}x + \int_\Omega A(x)|u(x)|^{p(x)-2}u(x)v(x)\mathrm{d}x \\ + \int_\Omega D(x)|u(x)|^{p(x)-1}u(x)v(x)\mathrm{d}x + \int_\Omega C(x)|u(x)|^{p(x)-3}u(x)v(x)\mathrm{d}x \\ - \lambda \int_\Omega |u(x)|^{q(x)-2}u(x)v(x),$$

for every $u, v \in E_1$. Thus, the weak solutions of problem (5) coincide with the critical points of I_λ.

Lemma 4.3. *Suppose that all hypotheses of* Theorem 4.2 *are satisfied. Then, there exists* $\lambda_0 > 0$, *such that for any* $\lambda \in (0, \lambda_0)$, *there exist* $\rho, \alpha > 0$, *such that:*

$$I_\lambda(u) \geq \alpha \quad \textit{for any} \quad u \in E_1 \quad \textit{with} \quad \|u\| = \rho.$$

Proof. By Lemma 4.1, there exists $\beta > 0$, such that:

$$|u|_{r(x)} \leq \beta \|u\|, \quad \text{for every} \quad u \in E_1 \quad \text{and} \quad r^+ \in \left(1, \frac{Np^-}{N + sp^+}\right).$$

We fix $\rho \in (0, \min(1, \frac{1}{\beta}))$. Invoking Proposition 2.1, for every $u \in E_1$ with $\|u\| = \rho$, we can get:

$$|u|_{q(x)} < 1.$$

Combining the above relations and Proposition 2.1, for any $u \in E_1$ with $\|u\| = \rho$, we can then deduce that:

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+}\left(\int_\Omega B(x)|\nabla u(x)|^{p(x)}\mathrm{d}x + \int_\Omega A(x)|u(x)|^{p(x)}\mathrm{d}x\right) \\
&\quad + \frac{1}{p^+ + 1}\int_\Omega D(x)|u(x)|^{p(x)+1}\mathrm{d}x \\
&\quad + \frac{1}{p^+ - 1}\int_\Omega C(x)|u(x)|^{p(x)-1}\mathrm{d}x - \frac{\lambda}{q^-}\int_\Omega |u(x)|^{q(x)}\mathrm{d}x \\
&\geq \frac{1}{4^{p^+}(p^+ + 1)}\|u\|^{p^+ + 1} - \lambda\frac{\beta^{q^-}}{q^-}\|u\|^{q^-} \\
&\geq \frac{1}{4^{p^+}(p^+ + 1)}\rho^{p^+ + 1} - \lambda\frac{\beta^{q^-}}{q^-}\rho^{q^-} \\
&= \rho^{q^-}\left(\frac{1}{4^{p^+}(p^+ + 1)}\rho^{p^+ + 1 - q^-} - \lambda\frac{\beta^{q^-}}{q^-}\right).
\end{aligned} \tag{6}$$

Put $\lambda_0 = \frac{\rho^{p^+ + 1 - q^-}}{4^{p^+}(2p^+ + 2)}\frac{q^-}{\beta^{q^-}}$. It now follows from (6) that for any $\lambda \in (0, \lambda_0)$:

$$I_\lambda(u) \geq \alpha \quad \text{with} \quad \|u\| = \rho,$$

and $\alpha = \frac{\rho^{p^+ + 1}}{4^{p^+}(2p^+ + 2)} > 0$. This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that all hypotheses of* Theorem 4.2 *are satisfied. Then, there exists $\varphi \in E_1$, such that $\varphi > 0$ and $I_\lambda(t\varphi) < 0$, for small enough t.*

Proof. By virtue of hypotheses (P) and (Q), there exist $\epsilon_0 > 0$ and $\Omega_0 \subset \Omega$, such that:

$$q(x) < q^- + \epsilon_0 < p^- - 1, \quad \text{for every} \quad x \in \Omega_0. \tag{7}$$

Let $\varphi \in C_0^\infty(\Omega)$, such that $\overline{\Omega_0} \subset \operatorname{supp}(\varphi)$, $\varphi = 1$ for every $x \in \overline{\Omega_0}$ and $0 \leq \varphi \leq 1$ in Ω. It then follows that for $t \in (0, 1)$:

$$\begin{aligned}
I_\lambda(t\varphi) &= \int_\Omega \frac{t^{p(x)}B(x)}{p(x)}|\nabla\varphi(x)|^{p(x)}\mathrm{d}x \\
&\quad + \int_\Omega \frac{t^{p(x)}A(x)}{p(x)}|\varphi(x)|^{p(x)}\mathrm{d}x + \int_\Omega \frac{t^{p(x)-1}C(x)}{p(x)-1}|\varphi|^{p(x)-1}\mathrm{d}x \\
&\quad + \int_\Omega \frac{t^{p(x)+1}D(x)}{p(x)+1}|\varphi(x)|^{p(x)+1}\mathrm{d}x - \lambda\int_\Omega t^{q(x)}\frac{|\varphi(x)|^{q(x)}}{q(x)}\mathrm{d}x
\end{aligned}$$

$$\leq \frac{t^{p^- -1}}{p^- - 1}\Bigg(\int_\Omega \frac{B(x)}{p(x)}|\nabla\varphi(x)|^{p(x)}\mathrm{d}x + \int_\Omega \frac{A(x)}{p(x)}|\varphi(x)|^{p(x)}\mathrm{d}x$$
$$+ \int_\Omega \frac{C(x)}{p(x)-1}|\varphi|^{p(x)-1}\mathrm{d}x$$
$$+ \int_\Omega \frac{D(x)}{p(x)+1}|\varphi(x)|^{p(x)+1}\mathrm{d}x \Bigg) - \lambda t^{q^- + \epsilon_0} \int_\Omega \frac{|\varphi(x)|^{q(x)}}{q(x)}\mathrm{d}x. \quad (8)$$

Combining (7) and (8), we finally arrive at the desired conclusion. This completes the proof of Lemma 4.4. □

5. Proof of Theorem 4.2

In the last section, we shall prove the main theorem of this paper. Let λ_0 be defined as in Lemma 4.3 and choose any $\lambda \in (0, \lambda_0)$. Again, invoking Lemma 4.3, we can deduce that:

$$\inf_{u \in \partial B(0,\rho)} I_\lambda(u) > 0. \quad (9)$$

On the other hand, by Lemma 4.4, there exists $\varphi \in E_1$, such that $I_\lambda(t\varphi) < 0$ for every small enough $t > 0$. Moreover, by Proposition 2.1, when $\|u\| < \rho$, we have:

$$I_\lambda(u) \geq \frac{1}{4^{p^+}(p^+ + 1)}\|u\|^{p^+ + 1} - c\|u\|^{q^-},$$

where c is a positive constant. It follows that:

$$-\infty < m = \inf_{u \in B(0,\rho)} I_\lambda(u) < 0.$$

Applying Ekeland's variational principle to the functional $I_\lambda : B(0,\rho) \to \mathbb{R}$, we can find a (PS) sequence $(u_n) \in B(0,\rho)$, that is:

$$I_\lambda(u_n) \to m \ \text{ and } \ I_\lambda^{'}(u_n) \to 0.$$

It is clear that (u_n) is bounded in E_1. Thus, there exists $u \in E_1$, such that, up to a subsequence, $(u_n) \rightharpoonup u$ in E_1. Using Theorem 4.1, we see that (u_n) strongly converges to u in $L^{q(x)}(\Omega)$. Therefore, by the Hölder inequality and Proposition 2.2, we can obtain the following:

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q(x)-2}u_n(u_n - u)\mathrm{d}x = \lim_{n\to+\infty} \int_\Omega |u|^{q(x)-2}u(u_n - u)\mathrm{d}x = 0.$$

On the other hand, since (u_n) is a (PS) sequence, we can also infer that:

$$\lim_{n\to+\infty} \langle I_\lambda^{'}(u_n) - I_\lambda^{'}(u), u_n - u\rangle = 0.$$

Combining the above pieces of information with Lemma 3.2, we can now conclude that $u_n \to u$ in E_1. Therefore:

$$I_\lambda(u) = m < 0 \ \text{ and } \ I_\lambda^{'}(u) = 0.$$

We have thus shown that u is a nontrivial weak solution for problem (5) and that every $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (5). This completes the proof of Theorem 4.2. □

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