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Existence and nonexistence of solutions for p(x)-curl systems arising in electromagnetism

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ABSTRACT

In this paper, we study the existence and the nonexistence of solutions for a new class of $p(x)$-curl systems arising in electromagnetism. This work generalizes some results obtained in the $p-$curl case. There seems to be no results on the nonexistence of solutions for curl systems with variable exponent.

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1. Introduction

The study of PDE's involving variable exponents has become very attractive in recent decades since differential operators involving variable exponent growth conditions can serve in describing nonhomogeneous phenomena which can occur in different branches of science, e.g.: electrorheological fluids and nonlinear Darcy's law in porous media, see [1,2].

The literature on equations with $p(x)$-Laplacian or $p-$curl operators is quite large, see e.g. [3–17] and the references therein. To the best of our knowledge, the only results involving the $p(x)$-curl operators can be found in [18,19]. In [18], the authors introduced a suitable variable exponent Sobolev space and obtained the existence of local or global weak solutions for equation with $p(x,t)-$curl operator by using Galerkin's method. In [19], the authors used for the first time the variational methods for equations involving $p(x)$-curl operator.

In this paper, our aim is to study equations in which a variable exponent curl operator is present. More precisely, we study the existence and nonexistence of solutions. To the best of the authors knowledge, this is one of the first works devoted to the studies of the nonexistence of solutions in variable exponent curl operator.

Let Ω be a bounded simply connected domain of $\mathbb{R}^3$ with a $C^{1,1}$ boundary denoted by $\partial\Omega$. To introduce our problem more precisely, we first give some notations. Let $u =$

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(u_1, u_2, u_3) be a vector function on Ω. The divergence of u is denoted by

$$\nabla.u = \partial_{x_1}u_1 + \partial_{x_2}u_2 + \partial_{x_3}u_3$$

and the curl of u is defined by

$$\nabla \times u = (\partial_{x_2}u_3 - \partial_{x_3}u_2, \partial_{x_3}u_1 - \partial_{x_1}u_3, \partial_{x_1}u_2 - \partial_{x_2}u_1).$$

We consider the following p(x)-curl systems:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) = \lambda g(x,u) - \mu f(x,u), & \nabla.u = 0 \text{ in } \Omega, \\ |\nabla \times u|^{p(x)-2}\nabla \times u \times \mathbf{n} = 0, & u.\mathbf{n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda, \mu > 0$, $p \in C(\overline{\Omega})$ with $\frac{6}{5} < p^- = \min_{x\in\overline{\Omega}} p(x) \le p^+ = \max_{x\in\overline{\Omega}} p(x) < 3$ and there exists $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\forall x, y \in \overline{\Omega},\ |x-y| < 1,\ |p(x)-p(y)| \le w(|x-y|), \text{ and } \lim_{s\to 0^+} w(s)log\left(\frac{1}{s}\right) = C < \infty \quad (P).$$

Throughout this paper, we shall always make the following assumptions:

(F_1) $F : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is differentiable with respect to $u \in \mathbb{R}^3$ and $f = \partial_u F(x,u)$: $\Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function.

(F_2) There exist $\alpha, \beta > 0$ and $q \in C(\overline{\Omega})$ such that $p^+ < q(x) < p^*(x) = \frac{3p(x)}{3-p(x)}$ in $\overline{\Omega}$ and

$$F(x,u) \ge \alpha |u|^{q(x)} \text{ and } |f(x,u)| \le \beta(1 + |u|^{q(x)-1}),\ \forall (x,u) \in \overline{\Omega} \times \mathbb{R}^3.$$

(G_1) There exist a nonnegative function $g \in L^\infty(\Omega)$ and $r \in C(\overline{\Omega})$ such that

$$p^+ < r^- \le r(x) < q^- \quad \text{and } G(x,u) = g(x)|u|^{r(x)},$$

for all $(x,u) \in \overline{\Omega} \times \mathbb{R}^3$.

(G_2) $G : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is differentiable with respect to $u \in \mathbb{R}^3$ and $g = \partial_u G(x,u)$: $\Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function.

(G_3) There exist $\gamma, \mu > 0$, $L > 1$ and $k, r \in C(\overline{\Omega})$ such that $1 < k < p^-$ and $1 < r(x) < p^*(x)$,

$$|g(x,u)| \le \mu(1 + |u|^{r(x)-1}), \quad \forall (x,u) \in \overline{\Omega} \times \mathbb{R}^3,$$
$$\limsup_{u\to 0} \frac{G(x,u)}{|u|^{p^+}} = 0 \text{ uniformly in } x \in \Omega$$

and

$$\sup_{u\in E} \int_\Omega G(x,u)\mathrm{d}x > 0,\ |G(x,u)| \le \gamma |u|^{k(x)}, \quad \forall x \in \mathbb{R}^3, \forall |u| > L.$$

Our main results are the following two theorems.

Theorem 1.1: *Assume that hypotheses $(F_1) - (F_2)$ and $(G_1) - (G_2)$ hold. Then:*

(i) *There exist* $\lambda_1, \mu_1 > 0$ *such that, if* $0 < \lambda < \lambda_1$ *and* $\mu > \mu_1$, *then problem* (1.1) *does not have any nontrivial weak solution.*
(ii) *For each* $\mu > 0$, *there exists* $\lambda_\mu > 0$ *such that if* $\lambda > \lambda_\mu$, *then problem* (1.1) *has at least one nontrivial weak solution.*

Theorem 1.2: *Assume that* $(F_1) - (F_2)$ *and* $(G_2) - (G_3)$ *hold. Then there exist* $\lambda_2, \lambda_3, r > 0$ *such that, if* $\lambda \in [\lambda_2, \lambda_3]$, *there exists* $\mu_2 > 0$ *with the following property: for each* $\mu \in [0, \mu_2]$, *equation* (1.1) *has at least three solutions whose norms are less than* r.

We have divided this paper into 3 sections. In Section 2, we give some notations and we recall some necessary definitions. In Section 3, we prove our main results.

2. Function spaces with variable exponent and preliminary results

In this section we recall some basic definitions and properties concerning the basic function spaces with variable exponent and the space $\mathbf{W}^{p(x)}(\Omega)$ of divergence free vector functions belonging to $\mathbf{L}^{p(x)}(\Omega)$ with curl in $\mathbf{L}^{p(x)}(\Omega)$. We refer to [7–9,13,16,18,19] and the references therein.

Consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ define

$$p^+ = \sup_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\overline{\Omega}} p(x),$$

and the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{u;\ u \text{ is measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,\mathrm{d}x \le 1\right\}.$$

The function space $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}\,. \tag{2.1}$$

Moreover, if $p_j \in C_+(\overline{\Omega})$ $(j = 1, 2, 3)$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1$$

then for all $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$, $w \in L^{p_3(x)}(\Omega)$

$$\left|\int_\Omega uvw \, \mathrm{d}x\right| \leq \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right) |u|_{p_1(x)} |v|_{p_2(x)} |w|_{p_3(x)} . \tag{2.2}$$

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \leq p_2$ in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Proposition 2.1: *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x, \ \ \forall u \in L^{p(x)}(\Omega),$$

then

(i) $|u|_{p(x)} < 1 (resp. = 1; > 1) \Leftrightarrow \rho(u) < 1 (resp. = 1; > 1)$;
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$; *and*
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.

Proposition 2.2: *If $u, u_n \in L^{p(x)}(\Omega)$ and $n \in \mathbb{N}$, then the following statements are equivalent:*

(1) $\lim_{n\to+\infty} |u_n - u|_{p(x)} = 0$.
(2) $\lim_{n\to+\infty} \rho(u_n - u) = 0$.
(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \rho(u_n) = \rho(u)$.

If k is a positive integer and $p \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \quad \text{for all } |\alpha| \leq k\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}} .$$

On $W^{k,p(x)}(\Omega)$ we consider the following norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}.$$

Then $W^{k,p(x)}(\Omega)$ is a reflexive and separable Banach space. Let $W_0^{k,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Theorem 2.3: *Let $q \in C(\overline{\Omega})$ such that $1 \leq q(x) < \frac{3p(x)}{3-p(x)}$ in $\overline{\Omega}$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Let $\mathbf{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}(\Omega)$ and define

$$E = \mathbf{W}^{p(x)}(\Omega) = \{v \in \mathbf{L}^{p(x)}(\Omega) : \nabla \times v \in \mathbf{L}^{p(x)}(\Omega), \nabla . v = 0, v.\mathbf{n}|\partial\Omega = 0\},$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$. Equip $\mathbf{W}^{p(x)}(\Omega)$ with the following norm

$$\|v\| = \|v\|_{\mathbf{L}^{p(x)}(\Omega)} + \|\nabla \times v\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, then by Theorem 2.1 of [18], $E = \mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_{\mathbf{n}}^{1,p(x)}(\Omega)$, where

$$\mathbf{W}_{\mathbf{n}}^{1,p(x)}(\Omega) = \{v \in \mathbf{W}^{1,p(x)}(\Omega), v.\mathbf{n}|\partial\Omega = 0\}$$

and

$$\mathbf{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega).$$

Thus we have the following theorem.

Theorem 2.4: *Assume that $1 < p^- \leq p^+ < \infty$ and p satisfies (P). Then $\boldsymbol{W}^{p(x)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_{\boldsymbol{n}}^{1,p(x)}(\Omega)$. Moreover, if $p^- > \frac{6}{5}$, then $\|\nabla \times .\|$ is a norm in $\boldsymbol{W}^{p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$ such that*

$$\|v\|_{\boldsymbol{W}^{1,p(x)}(\Omega)} \leq C\|\nabla \times v\|_{\boldsymbol{L}^{p(x)}(\Omega)}.$$

Corollary 2.5: *By Theorems 2.3 and 2.4, the embedding $\boldsymbol{W}^{p(x)}(\Omega) \hookrightarrow \boldsymbol{L}^{q(x)}(\Omega)$ is compact, with $1 < p^- \leq p^+ < 3$, $q \in C(\overline{\Omega})$ and $1 \leq q(x) < \frac{3p(x)}{3-p(x)}$ in $\overline{\Omega}$. Moreover, $(\boldsymbol{W}^{p(x)}(\Omega), \|.\|)$ is a uniformly convex and reflexive Banach space.*

Define for any $\lambda, \mu > 0$ and $u \in E$,

$$\phi(u) = \int_\Omega |\nabla \times u|^{p(x)} \mathrm{d}x, J(u) = \int_\Omega G(x, u) \mathrm{d}x$$

$$\psi(u) = \int_\Omega -F(x, u) \mathrm{d}x \text{ and } I(u) = \phi(u) - \lambda J(u) - \mu\psi(u).$$

It is easy to see, under assumptions (P), $(F_1) - (F_2)$ and $(G_1) - (G_3)$, that $I, \phi, J, \psi \in C^1(E, \mathbb{R})$.

Definition 2.6: For every $\lambda, \mu > 0$, we say that $u \in E$ is a weak solution of problem (1.1), if

$$\int_\Omega |\nabla \times u|^{p(x)-2} \nabla \times u.\nabla \times v \mathrm{d}x - \lambda \int_\Omega g(x, u(x)).v \mathrm{d}x + \mu \int_\Omega f(x, u(x)).v \mathrm{d}x = 0, \ \forall v \in E.$$

For more details, we refer the reader to [19].

3. Proofs of main results

3.1. Proof of Theorem 1.1

Lemma 3.1: *Suppose that the assumptions $(F_1) - (F_2)$ and $(G_1) - (G_2)$ are fulfilled. Then there exist positive constants λ_1, μ_1 such that, for every $0 < \lambda < \lambda_1$ and $\mu_1 < \mu$, problem (1.1) does not have any nontrivial weak solutions.*

Proof: Assume that u is a nontrivial weak solution of equation (1.1).

Case 1: We suppose that $\|u\| < 1$. Then, by Proposition 2.1, u satisfies the following inequality

$$\|u\|^{p^+} \leq \int_\Omega |\nabla \times u|^{p(x)} \mathrm{d}x = \lambda \int_\Omega G(x,u)\mathrm{d}x - \mu \int_\Omega F(x,u)\mathrm{d}x. \tag{3.1}$$

Now, since $r(x) < q(x)$ in Ω, applying the Young inequality we can deduce that

$$\lambda \int_\Omega g(x)|u|^{r(x)}\mathrm{d}x \leq \frac{q^+ - r^-}{q^-} \int_\Omega |\lambda g|^{\frac{q(x)}{q(x)-r(x)}} \mathrm{d}x + \frac{r^+}{q^-} \int_\Omega |u|^{q(x)}\mathrm{d}x. \tag{3.2}$$

Invoking inequalities (3.1) and (3.2), and conditions $(F_1) - (F_2)$ and (G_1), for λ small enough, we obtain

$$\begin{aligned} 0 < \|u\|^{p^+} &\leq \frac{(q^+ - r^-)\lambda^{\frac{q^-}{q^+-r^-}}}{q^-} \int_\Omega |g|^{\frac{q(x)}{q(x)-r(x)}} \mathrm{d}x + \left(\frac{r^+}{q^-} - \mu\alpha\right) \int_\Omega |u|^{q(x)}\mathrm{d}x \\ &\leq \frac{(q^+ - r^-)\lambda^{\frac{q^-}{q^+-r^-}}}{q^-} \int_\Omega |g|^{\frac{q(x)}{q(x)-r(x)}} \mathrm{d}x = \lambda^{\frac{q^-}{q^+-r^-}} A < \infty, \end{aligned} \tag{3.3}$$

where $A = \frac{q^+-r^-}{q^-} \int_\Omega |g|^{\frac{q(x)}{q(x)-r(x)}} \mathrm{d}x$ and $\mu > \mu_1 = \frac{r^+}{\alpha q^-}$.

Thanks to Corollary 2.5, there exists a constant $\beta > 0$ such that

$$\beta |u|^{p^+}_{r(x)} \leq \|u\|^{p^+}, \ \ \forall u \in E. \tag{3.4}$$

Thus, in view of (3.1), (3.4), and Proposition 2.1, we get

$$\beta |u|^{p^+}_{r(x)} \leq \lambda \|g\|_\infty \max\left(|u|^{r^-}_{r(x)}, |u|^{r^+}_{r(x)}\right). \tag{3.5}$$

Having in mind $p^+ < r^- < r^+$ and $\|u\|_{r(x)} > 0$, by (3.3) and (3.5), we have

$$\beta \max\left(\left(\frac{\beta}{\lambda\|g\|_\infty}\right)^{\frac{p^+}{r^- - p^+}}, \left(\frac{\beta}{\lambda\|g\|_\infty}\right)^{\frac{p^+}{r^+ - p^+}}\right) \leq \|u\|^{p^+} \leq \lambda^{\frac{q^-}{q^+-r^-}} A. \tag{3.6}$$

Case 2: We suppose that $\|u\| > 1$. It suffices to replace p^+ by p^- in the proof of Case 1.

This concludes the proof. □

Lemma 3.2: *Assume that assumptions $(F_1) - (F_2)$ and $(G_1) - (G_2)$ hold. Then*

(a) *I is a coercive functional; and*
(b) *I is a weakly lower semicontinuous functional.*

***Proof*:**

(a) Let $\lambda, \mu > 0$ and $u \in E$ with $\|u\| > 1$. Combining Proposition 2.1, Young inequality, and assumptions (F_2) and (G_1), one obtains the following inequalities

$$
\begin{aligned}
I(u) &\geq \frac{1}{p^+}\|u\|^{p^-} + \alpha\mu \int_\Omega |u(x)|^{q(x)}\mathrm{d}x - \lambda \int_\Omega g(x)|u|^{r(x)}\mathrm{d}x \\
&\geq \frac{1}{p^+}\|u\|^{p^-} + \frac{\mu\alpha}{2} \int_\Omega |u(x)|^{q(x)}\mathrm{d}x - c_{\lambda,\mu,\alpha} \int_\Omega |g(x)|^{\frac{q(x)}{q(x)-r(x)}}\mathrm{d}x,
\end{aligned}
$$

where $c_{\lambda,\mu,\alpha}$ is a positive constant. This demonstrates the coercivity of the functional I.

(b) Let (u_n) be a sequence such that $u_n \rightharpoonup u$ in E. Using the fact that (u_n) is bounded in E, Corollary 2.5 and Proposition 2.2, up to a subsequence, still denoted by (u_n), we can infer that

$$
u_n \to u \text{ a.e in } \Omega \text{ and } \lim_{n\to+\infty} \int_\Omega g(x)|u_n|^{r(x)}\mathrm{d}x = \int_\Omega g(x)|u|^{r(x)}\mathrm{d}x. \tag{3.7}
$$

By the weak lower semicontinuity of the norm $\|.\|$, we have

$$
\|u\| \leq \liminf_{n\to+\infty} \|u_n\|. \tag{3.8}
$$

Furthermore, Fatou's lemma and (F_2) yield the following inequality

$$
\int_\Omega \liminf_{n\to+\infty} F(x,u_n)\mathrm{d}x \leq \liminf_{n\to+\infty} \int_\Omega F(x,u_n)\mathrm{d}x. \tag{3.9}
$$

Combining (3.7)–(3.9), we have thus proved the claim.

□

Completion of the proof of Theorem 1.1:

(i) Evidently, by Lemma 3.1, assertion (i) of Theorem 1.1 holds.

(ii) Fix $\mu > 0$. Using Lemma 3.2, for every $\lambda > 0$, we can find $u \in E$ such that

$$
I(u) = \inf_{v\in E} I(v).
$$

Hence, for every $\lambda > 0$ and $\mu > 0$, u is a weak solution of problem (1.1). It remains to show that u is nontrivial weak solution of system (1.1). Invoking assumption (G_1), we can find $w \in E$ such that

$$
\int_\Omega G(x,w)\mathrm{d}x = 1.
$$

It follows that

$$
I(w) = \int_\Omega \frac{|\nabla \times w|^{p(x)}}{p(x)}\mathrm{d}x + \mu \int_\Omega F(x,w)\mathrm{d}x - \lambda = \lambda_\mu - \lambda,
$$

where $\lambda_\mu = \int_\Omega \frac{|\nabla \times w|^{p(x)}}{p(x)}\mathrm{d}x + \mu \int_\Omega F(x,w)\mathrm{d}x$. Thus, $I(w) < 0$ for any $\lambda > \lambda_\mu$. This completes the proof.

3.2. Proof of Theorem 1.2

The main tool in the proof of Theorem 1.2 is the variant of the three critical points theorem established by Ricceri [20]. Before stating his theorem, we need the following definition.

Definition 3.3: If X is a real Banach space, we denote by R_X the class of all functional $\phi : X \to \mathbb{R}$ possessing the following property: If (u_n) is a sequence in X, converging weakly to $u \in X$, and $\liminf_{n\to+\infty} \phi(u_n) \le \phi(u)$, then (u_n) has a subsequence strongly converging to u.

Theorem 3.4: *Let X be a separable and reflexive real Banach space, $\phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to R_X, bounded on each bounded subset of X and with the derivative admitting a continuous inverse on X^*; $J : X \to \mathbb{R}$, a C^1 functional with compact derivative. Assume that ϕ has a strict local minimum x_0 with $\phi(x_0) = J(x_0) = 0$. Finally, setting*

$$\alpha = \max\left\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\phi(x)}, \lim_{\|x\|\to x_0} \frac{J(x)}{\phi(x)}\right\}$$

and

$$\beta = \sup_{x\in\phi^{-1}(0,+\infty)} \frac{J(x)}{\phi(x)},$$

assume that $\alpha < \beta$. Then for each compact interval $[a, b] \subset (\frac{1}{\beta}, \frac{1}{\alpha})$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$), there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\phi^{'}(x) = \lambda J^{'}(x) + \mu\psi^{'}(x)$$

has at least three solutions whose norm is less than r.

Completion of the proof of Theorem 1.2:

Standard arguments can be used to show that $J^{'}$ and $\psi^{'}$ are compact, while ϕ is a coercive, sequentially weakly lower semicontinuous and $\phi^{'}$ is a homeomorphism between E and its dual. Clearly, $\phi \in R_E$, since E is uniformly convex.

Fix $\epsilon, s > 0$ such that $p^+ + s < p^*(x)$. By virtue of assumption (G_3), there exists a constant η with $0 < \eta < L$, such that

$$G(x, u) \le \epsilon|u|^{p^+}, \quad \forall x \in \Omega, \forall |u| \in [-\eta, \eta].$$

Again, by assumption (G_3), it follows that

$$\begin{aligned} J(u) &\le \int_{\{x\in\Omega,|u(x)|\le\eta\}} G(x, u)\mathrm{d}x + \int_{\{x\in\Omega,\eta\le|u(x)|\le L\}} G(x, u)\mathrm{d}x + \int_{\{x\in\Omega,|u(x)|\ge L\}} G(x, u)\mathrm{d}x \\ &\le c\left(\epsilon\int_{\{x\in\Omega,|u(x)|\le\eta\}} |u|^{p^+}\mathrm{d}x + \int_{\{x\in\Omega,\eta\le|u(x)|\le L\}} |u|^{p^++s}\mathrm{d}x + \gamma\int_{\{x\in\Omega,|u(x)|\ge L\}} |u|^{k(x)}\mathrm{d}x\right) \\ &\le c\left(\epsilon\int_{\{x\in\Omega,|u(x)|\le\eta\}} |u|^{p^+}\mathrm{d}x + \int_{\{x\in\Omega,\eta\le|u(x)|\le L\}} |u|^{p^++s}\mathrm{d}x + \int_{\{x\in\Omega,|u(x)|\ge L\}} |u|^{p^++s}\mathrm{d}x\right) \\ &\le c\left(\epsilon\|u\|^{p^+} + \|u\|^{p^++s}\right), \end{aligned}$$

for some positive constant c. This, along with Proposition 2.1, yields, for $\|u\| < 1$

$$\frac{J(u)}{\phi(u)} \leq c\frac{\epsilon\|u\|^{p^+} + \|u\|^{p^+ + s}}{\|u\|^{p^+}},$$

hence,

$$\limsup_{u\to 0} \frac{J(u)}{\phi(u)} \leq c\epsilon. \tag{3.10}$$

Taking $u \in E$ with $\|u\| > 1$, from $(G_2) - (G_3)$, we get

$$\begin{aligned} J(u) &\leq \int_{\{x\in\Omega, |u(x)|\leq L\}} G(x,u)\mathrm{d}x + \int_{\{x\in\Omega, |u(x)|\geq L\}} G(x,u)\mathrm{d}x \\ &\leq c + \int_{\{x\in\Omega, |u(x)|\geq L\}} |u|^{k(x)}\mathrm{d}x \\ &\leq c(1 + \|u\|^{k^-} + \|u\|^{k^+}). \end{aligned}$$

This implies that

$$\limsup_{u\to +\infty} \frac{J(u)}{\phi(u)} \leq \limsup_{u\to +\infty} \frac{c(1 + \|u\|^{k^-} + \|u\|^{k^+})}{\|u\|^{p^-}} = 0 \tag{3.11}$$

Therefore, by (3.10) and (3.11), $\alpha = 0$. In view of assumption (G_3), we have $\beta > 0$. Thus all hypotheses of Theorem 3.4 are satisfied. The proof is therefore complete.

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