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# On Pliś Metric on the Space of Strictly Convex Compacta*

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We consider a certain metric on the space of all convex compacta in $\mathbb{R}^n$, introduced by Pliś. The set of strictly convex compacta is a complete metric subspace of the metric space of convex compacta with respect to this metric. We present some applications of this metric to the problems of set-valued analysis, in particular we estimate the distance between two compact sets with respect to this metric and the Hausdorff metric.

*Keywords:* Metric space, strictly convex compactum, modulus of convexity, set-valued mapping, strict convexity, uniform convexity, supporting function, Demyanov distance, Hausdorff distance

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## 1. Introduction

We begin by some definitions for a finite-dimensional Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ over $\mathbb{R}$ with the inner product $(\cdot,\cdot)$. Let $B_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| \le r\}$. Let $\operatorname{cl} A$ denote the *closure* and $\operatorname{int} A$ the *interior* of the subset $A \subset \mathbb{R}^n$. The *diameter* of the subset $A \subset \mathbb{R}^n$ is defined by $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$. The *distance* from the point $x \in \mathbb{R}^n$ to the set $A \subset \mathbb{R}^n$ is given by the formula $\varrho(x, A) = \inf_{a\in A} \|x - a\|$. We shall denote the *convex hull* of a set $A \subset \mathbb{R}^n$ by $\operatorname{co} A$. We shall denote the *conic hull* of a set $A \subset \mathbb{R}^n$ by $\operatorname{cone} A$ (cf. [1, 9, 13]).

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The *Hausdorff distance* between two subsets $A, B \subset \mathbb{R}^n$ is defined as follows

$$h(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$
$$= \inf\{r>0 \mid A\subset B+B_r(0),\ B\subset A+B_r(0)\}.$$

The *supporting function* of the subset $A \subset \mathbb{R}^n$ is defined as follows

$$s(p,A) = \sup_{x\in A}(p,x), \quad \forall p\in\mathbb{R}^n. \tag{1}$$

The supporting function of any set $A$ is always lower semicontinuous, positively uniform and convex. If the set $A$ is bounded then the supporting function is Lipschitz continuous [1, 9].

It follows from the Separation Theorem (cf. [9, Lemma 1.11.4]) that for any convex compacta $A, B$ in $\mathbb{R}^n$

$$h(A,B) = \sup_{\|p\|=1}|s(p,A)-s(p,B)|. \tag{2}$$

Let $(T,\varrho)$ be a metric space. We say that a set-valued mapping $F : (T,\varrho) \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$ is *upper semicontinuous* at the point $t = t_0$ if

$$\forall\varepsilon>0\ \exists\delta>0\ \forall t:\ \varrho(t,t_0)<\delta\ \ F(t)\subset F(t_0)+B_\varepsilon(0),$$

and *lower semicontinuous* at the point $t = t_0$ if

$$\forall\varepsilon>0\ \exists\delta>0\ \forall t:\ \varrho(t,t_0)<\delta\ \ F(t_0)\subset F(t)+B_\varepsilon(0).$$

We say that a set-valued mapping $F : (T,\varrho) \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$ is *continuous* at the point $t = t_0$ if $F$ is upper and lower semicontinuous at the point $t = t_0$.

We say that a set-valued mapping $F : (T,\varrho) \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$ is *(upper, lower) (semi)continuous* on the set $T$, if $F$ is (upper, lower) (semi)continuous at any point $t_0 \in T$.

For any convex compact set $A \subset \mathbb{R}^n$ and any vector $p \in \mathbb{R}^n$, the subset $A(p) = \{x \in A \mid (p,x) = s(p,A)\}$ is the subdifferential of the supporting function $s(p,A)$ at the point $p$. The set-valued mapping $\mathbb{R}^n \ni p \to A(p)$ is always upper semicontinuous (cf. [1, 13]).

A convex compactum in $\mathbb{R}^n$ is called *strictly* convex if its boundary contains no nondegenerate line segments.

**Definition 1.1 ([10]).** Let $E$ be a Banach space and let a subset $A \subset E$ be convex and closed. *The modulus of convexity* $\delta_A : [0,\operatorname{diam} A) \to [0,+\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup\left\{\delta\geq 0 \;\middle|\; B_\delta\left(\frac{x_1+x_2}{2}\right)\subset A,\ \forall x_1,x_2\in A:\ \|x_1-x_2\|=\varepsilon\right\}.$$

**Definition 1.2 ([10]).** Let $E$ be a Banach space and let a subset $A \subset E$ be convex and closed. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0,\operatorname{diam} A)$, then we call the set $A$ *uniformly convex (with modulus* $\delta_A(\cdot)$*)*.

For any uniformly convex set $A$ the modulus $\delta_A$ is a strictly increasing function on the segment $[0, \operatorname{diam} A)$. In the finite-dimensional case the class of strictly convex compacta coincides with the class of uniformly convex compacta with moduli of convexity $\delta_A(\varepsilon) > 0$ for all permissible $\varepsilon > 0$ (cf. [3]).

We shall use $*$ for objects from conjugate space $E^*$: $\|\cdot\|_*$ is the norm in $E^*$, $B_1^*(0)$ is the unit closed ball in $E^*$ and so on.

Following Pliś [8] we define the metric $\rho$ which is the main objective of the present paper.

**Definition 1.3 ([8, Formula (3)]).** The metric $\rho$ on the space of convex compacta in $\mathbb{R}^n$ is defined by the formula

$$\rho(A, B) = \sup_{\|p\|=1} h(A(p), B(p)), \tag{3}$$

for any convex compacta $A, B \subset \mathbb{R}^n$.

Definition 1.3 coincides with the definition of the *Demyanov metric* (see its definition in [4, Formula (4.1)]) – this was proved in [6].

The Hausdorff metric is the most natural metric for various questions of set-valued analysis and its applications. Nevertheless, there are some limitations for using this metric. For example, if we have a sequence $\{A_k\}_{k=1}^{\infty}$ of strictly convex compact sets and $h(A_k, A) \to 0$, then the limit set $A$ needs not be strictly convex. Indeed, consider on the Euclidean plane the following ellipsoids

$$A_k = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + k^2 x_2^2 \leq 1\}.$$

Each set $A_k$ is strictly convex, but the limit set $A = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in [-1, 1]\}$ is not strictly convex. Note, that strict convexity of the set means differentiability of the supporting function of this set. This fact is very useful for applications. Below we give some sufficient conditions for the limit of a sequence of strictly convex compacta to be also strictly convex.

We say that a sequence of convex compacta $\{A_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ is *uniformly convex with modulus* $\delta$ if $\inf_k \operatorname{diam} A_k > 0$ and the function $\delta(\varepsilon)$, $\varepsilon \in [0, \inf_k \operatorname{diam} A_k)$, is continuous and has the property $0 < \delta(\varepsilon) \leq \delta_{A_k}(\varepsilon)$ for all $\varepsilon \in (0, \inf_k \operatorname{diam} A_k)$ and for all $k$.

**Lemma 1.4.** *Let a sequence $\{A_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ of convex compacta converge to a convex compactum $A$ in the Hausdorff metric. If the sequence $\{A_k\}_{k=1}^{\infty}$ is uniformly convex with modulus $\delta$, $\delta : (0, \varepsilon_0] \to (0, +\infty)$, then the compactum $A$ is a uniformly convex set with the modulus $\delta_A(\varepsilon) \geq \delta(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$. In particular, this implies strict convexity of the set $A$.*

**Proof.** Choose arbitrary points $x, y \in A$ with $\|x - y\| < \varepsilon_0$. There are two sequences $\{x_k\} \subset A_k$, $\{y_k\} \subset A_k$ such that $x_k \to x$, $y_k \to y$, $k \to \infty$. For all sufficiently large $k$ we have $\|x_k - y_k\| < \varepsilon_0$. Due to the uniform convexity of the sequence $\{A_k\}$ we obtain that

$$\frac{x_k + y_k}{2} + B_{\delta(\|x_k - y_k\|)}(0) \subset A,$$

and

$$\left(p, \frac{x_k + y_k}{2}\right) + \delta(\|x_k - y_k\|)\|p\| \leq s(p, A_k), \quad \forall p \in \mathbb{R}^n.$$

Taking the limit $k \to \infty$, using (2) and the continuity of the function $\delta$ we get

$$\left(p, \frac{x + y}{2}\right) + \delta(\|x - y\|)\|p\| \leq s(p, A), \quad \forall p \in \mathbb{R}^n,$$

i.e.

$$s\left(p, \frac{x + y}{2} + B_{\delta(\|x-y\|)}(0)\right) \leq s(p, A), \quad \forall p \in \mathbb{R}^n.$$

By the Separation Theorem [9, 13] we obtain the following

$$\frac{x + y}{2} + B_{\delta(\|x-y\|)}(0) \subset A.$$

□

## 2. The main properties of metric $\rho$

In general, the subdifferential of a convex function is only upper semicontinuous [1, 13]. For a (not strictly) convex compactum $A$ the sets $A(p)$ are also upper semicontinuous with respect to $p$. This leads to the fact that in the formula (3) from Definition 1.3 one cannot replace sup by max.

**Example 2.1.** Consider in $\mathbb{R}^3$ two sets:

$$A = \operatorname{co}\left\{\{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 = 1;\ x_3 = 0\} \cup \{(0,0,1)\}\right\},$$

$$B = \operatorname{co}\left\{\{(x_1, x_2, x_3) \mid (x_1 - 1)^2 + x_2^2 + x_2^8 = 1;\ x_3 = 0\} \cup \{(0,0,1)\}\right\}.$$

It is easy to see that $B \subset A$, $\operatorname{diam} B = \operatorname{diam} A = \sqrt{5}$, and $\operatorname{diam} A$ and $\operatorname{diam} B$ are attained only on the line segment $[(0,0,1),(2,0,0)] \subset B$.

Let $a_k \in \{(x_1, x_2, x_3) \mid (x_1-1)^2+x_2^2 = 1;\ x_2 < 0;\ x_3 = 0\}$ be such that $a_k \to (2,0,0)$. The line segment $[(0,0,1), a_k]$ is a generatrix of the cone $A$ for all $k$.

Let $H_k$ be a supporting plane of the set $A$ such that $[(0,0,1), a_k] \subset H_k$. Let $p_k$ be a unit normal vector to the plane $H_k$ such that $(p_k, a_k) > 0$. It is easy to see that $p_k \to p_0 = \frac{1}{\sqrt{5}}(1,0,2)$.

For any $k$ we have $B(p_k) = \{(0,0,1)\}$ and $A(p_k) = H_k \cap A = [(0,0,1), a_k]$. By Definition 1.3 it follows that

$$\begin{aligned}\rho(A,B) &\geq h(A(p_k), B(p_k)) = h\left(\{(0,0,1)\}, \{(0,0,1), a_k\}\right) \\ &= \|(0,0,1) - a_k\| = \sqrt{\|a_k\|^2 + 1},\end{aligned}$$

and $\sqrt{\|a_k\|^2 + 1} \to \sqrt{5}$, $\sqrt{\|a_k\|^2 + 1} < \sqrt{5}$ for all $k$. However, the only line segment which realizes $\operatorname{diam} A = \operatorname{diam} B = \sqrt{5}$ is the line segment $[(0,0,1),(2,0,0)] \subset A \cap B$. Thus $\rho(A,B) = \lim_{k\to\infty} h(A(p_k), B(p_k)) = \sqrt{5}$, but for all $p$, $\|p\| = 1$, $h(A(p), B(p)) < \sqrt{5}$. □

**Lemma 2.2.** *Let $A \subset \mathbb{R}^n$ be a convex compactum. If the set-valued mapping $\partial B_1(0) \ni p \to A(p)$ is lower semicontinuous, then the compactum $A$ is strictly convex.*

**Proof.** Suppose that there exists $p \in \partial B_1(0)$ such that the set $A(p)$ is not a singleton. Let $\{x, y\} \subset A(p)$, $x \neq y$, and $q = \frac{y-x}{\|y-x\|}$. Obviously, $q$ is orthogonal to $p$.

Consider a sequence $\{q_k\}_{k=1}^{\infty} \subset \text{cone}\{p, q\}$ such that $q_k \to p$, $k \to \infty$, $\|q_k\| = 1$ and $q_k \neq p$ for all $k$. Let $H_p^- = \{z \in \mathbb{R}^n \mid (p, z) \leq s(p, A)\}$, $H_{q_k}^+ = \{z \in \mathbb{R}^n \mid (q_k, z) \geq (y, q_k)\}$, $H_q^+ = \{z \in \mathbb{R}^n \mid (q, z) \geq (y, q)\}$.

By lower semicontinuity of $A(\cdot)$ we have for any $\varepsilon > 0$ and for all sufficiently large $k$

$$A(p) \subset A(q_k) + B_\varepsilon(0). \tag{4}$$

On the other hand,

$$A(q_k) \subset H_{q_k}^+ \cap H_p^- \subset H_q^+ \cap H_p^-. \tag{5}$$

Due to the inclusion (5) we obtain that

$$\varrho(x, A(q_k)) \geq \varrho(x, H_q^+ \cap H_p^-) = \|x - y\| > 0. \tag{6}$$

Inequality (6) implies that for all $k$

$$x \notin A(q_k) + \frac{\|x - y\|}{2} B_1(0).$$

This contradicts the inclusion (4). □

**Lemma 2.3.** *Consider a sequence $F_k : (T, \varrho) \to 2^{\mathbb{R}^n} \backslash \{\emptyset\}$ of set-valued mappings which are upper (lower) semicontinuous with compact images. Let the sequence $\{F_k(t)\}_{k=1}^{\infty}$ uniformly converge to the set-valued mapping $F : (T, \varrho) \to 2^{\mathbb{R}^n} \backslash \{\emptyset\}$ in the Hausdorff metric, i.e.*

$$\forall \varepsilon > 0 \; \exists k_\varepsilon \; \forall k > k_\varepsilon \; \forall t \in T \quad h(F_k(t), F(t)) < \varepsilon.$$

*Then the set-valued mapping $F$ is upper (lower) semicontinuous on $T$.*

**Proof.** The proof is a standard argument of uniform convergence. □

We shall write $F_k \rightrightarrows F$, $t \in T$, in the case of uniform convergence on the set $T$ of the sequence $F_k$ to the mapping $F$.

**Theorem 2.4.** *The metric space of convex compacta in $\mathbb{R}^n$ is complete with respect to the metric $\rho$.*

**Proof.** Let $\{A_k\}_{k=1}^{\infty}$ be a fundamental sequence of convex compacta with respect to the metric $\rho$. This means that

$$\forall \varepsilon > 0 \; \exists M \; \forall k, m > M \; \forall p \in \partial B_1(0) \quad h(A_m(p), A_k(p)) < \varepsilon.$$

By convexity of compact sets $A_m(p)$ and completeness of the space of convex compacta with respect to the Hausdorff metric (see [9, Theorem 1.3.2]) we obtain that $A_m(p) \rightrightarrows A_p$, $p \in \partial B_1(0)$, and the set $A_p$ is convex and compact for all $p \in \mathbb{R}^n$, $\|p\| = 1$.

Put

$$A = \operatorname{cl}\operatorname{co} \bigcup_{\|p\|=1} A_p.$$

For any $q \in \partial B_1(0)$ and any $x(q) \in A_q$ there exists a sequence $\{x_m(q)\}_{m=1}^{\infty}$ such that $x_m(q) \in A_m(q)$ for all $m$ and $x_m(q) \to x(q)$. Taking the limit $m \to \infty$ in the inequality $(p, x_m(p)) \geq (p, x_m(q))$, we obtain $(p, x(p)) \geq (p, x(q))$. Hence $(p, x(p)) \geq s(p, A)$ and $x(p) \in A(p)$, i.e. $A_p \subset A(p)$.

The converse inclusion $A(p) \subset A_p$ can be proved with the help of separation theorem. □

**Corollary 2.5.** *The metric subspace of strictly convex compacta in $\mathbb{R}^n$ is complete with respect to the metric $\rho$.*

**Proof.** The proof is analogous to the proof of Theorem 2.4 except that all sets $A_m(p)$, $A_p$ are singletons. □

Suppose that $A$, $B$ are convex compacta. By formula (2) we have

$$\rho(A, B) = \sup_{\|p\|=1} \sup_{\|q\|=1} |s(q, A(p)) - s(q, B(p))|,$$

and hence

$$\rho(A, B) \geq \sup_{\|p\|=1} |s(p, A(p)) - s(p, B(p))| = \sup_{\|p\|=1} |s(p, A) - s(p, B)| = h(A, B). \quad (7)$$

Thus $\rho(A_k, A) \to 0$ implies that $h(A_k, A) \to 0$.

**Theorem 2.6.** *The metric space of strictly convex compacta in $\mathbb{R}^n$ is not locally compact with respect to the metric $\rho$.*

**Proof.** Choose a sequence $\{A_k\}_{k=1}^{\infty}$ of strictly convex compacta such that $A_k \subset B_R(0)$ for all $k$ and there exists a nonstrictly convex compactum $A$ with $h(A_k, A) \to 0$. Suppose that a subsequence $\{A_{k_m}\}_{m=1}^{\infty}$ converges to a compactum $B$ in the metric $\rho$.

From the estimate $\rho(A_{k_m}, B) \geq h(A_{k_m}, B)$ and $h(A_{k_m}, A) \to 0$ we get equality $B = A$. So $\rho(A_{k_m}, A) \to 0$. This means that $A_{k_m}(p) \rightrightarrows A(p)$, $\|p\| = 1$. But the set $A_{k_m}(p)$ is a singleton for all $m$ and $p$. By the choice of $A$ there exists $p_0 \in \partial B_1(0)$ such that the set $A(p_0)$ is not a singleton. Contradiction. □

Next we shall obtain the estimate of the distance $\rho(A, B)$ via $h(A, B)$ for some convex closed sets in a Banach space. In a Banach space $E$ for closed convex bounded sets $A, B \subset E$ we define

$$\rho(A, B) = \sup_{\|p\|_*=1} h(A(p), B(p)).$$

If the space $E$ is reflexive then $A(p) \neq \emptyset$, $B(p) \neq \emptyset$ for all $p \in E^*$.

Note that if the space $E$ contains a uniformly convex nonsingleton set then such space $E$ has an equivalent uniformly convex norm [3, Theorem 2.3]. In particular, such space $E$ is reflexive.

Note also that for any uniformly convex set $A$ we have that $\operatorname{diam} A < +\infty$ and the modulus of convexity $\delta_A(\varepsilon)$ is a strictly increasing function when $\varepsilon \in [0, \operatorname{diam} A)$ [3].

**Theorem 2.7.** *Let $E$ be a Banach space. Let $A, B \subset E$ be convex closed bounded sets. Let the set $A$ be a nonsingleton and a uniformly convex set with the modulus of convexity $\delta_A$. Let $\Delta = \lim_{t\to \operatorname{diam} A - 0} \delta_A(t)$. Then*

$$\rho(A,B) \leq \begin{cases} h(A,B) + \delta_A^{-1}(h(A,B)), & h(A,B) < \Delta, \\ h(A,B)\left(1 + \dfrac{\operatorname{diam} A}{\Delta}\right), & h(A,B) \geq \Delta, \end{cases} \tag{8}$$

*where the function $\delta_A^{-1}$ is the inverse function of $\delta_A$. Furthemore, if the set $A$ is a singleton then $\rho(A,B) = h(A,B)$.*

**Proof.** Let $h = h(A,B)$. Suppose that $A$ is not a singleton. Fix $p \in \partial B_1^*(0)$. Let $A(p) = \{a(p)\}$. Fix an arbitrary point $b(p) \in B(p)$.

*Case 1.* $h < \Delta$. Choose $t > 1$ such that $th < \Delta$.

*Subcase 1.1.* $s(p,A) \geq s(p,B)$. By formula (2) we have $0 \leq (p, a(p)) - (p, b(p)) \leq h$. Let $a \in A$ be a point such that $a \in b(p) + B_{\mathrm{th}}(0)$.

Define $H_A(p) = \{z \in E \mid (p,z) = s(p,A)\}$, $H_A^-(p) = \{z \in E \mid (p,z) \leq s(p,A)\}$, $H_B(p) = \{z \in E \mid (p,z) = s(p,B)\}$.

We have $\varrho(b(p), H_A(p)) = (p, a(p) - b(p)) \leq h$, $\varrho(a, H_A(p))) \leq \|a - b(p)\| + \varrho(b(p), H_A(p)) \leq (1+t)h$ and $A \cup B \subset H_A^-(p)$. Hence the line segment $[a(p), a]$ belongs to the set $H_A(p)^-$. Let $w = \frac{a(p)+a}{2}$, $\varrho(w, H_A(p)) = \frac{1}{2}\varrho(a, H_A(p)) \leq \frac{1+t}{2}h$. By the inclusion

$$w + \delta_A(\|a(p) - a\|)B_1(0) \subset A \subset H_A^-(p)$$

we get

$$\delta_A(\|a(p) - a\|) \leq \varrho(w, H_A(p)) \leq \frac{1+t}{2}h.$$

Hence $\|a(p) - a\| \leq \delta_A^{-1}\left(\frac{1+t}{2}h\right)$. Thus we obtain that

$$\|a(p) - b(p)\| \leq \|a(p) - a\| + \|a - b(p)\| \leq \delta_A^{-1}\left(\frac{1+t}{2}h\right) + th,$$

i.e. $b(p) \in a(p) + \left(\delta_A^{-1}\left(\frac{1+t}{2}h\right) + th\right) B_1(0)$. Due to the arbitrary choice of the point $b(p) \in B(p)$ we have

$$B(p) \subset a(p) + \left(\delta_A^{-1}\left(\frac{1+t}{2}h\right) + th\right) B_1(0)$$

and
$$h(A(p),B(p)) = h(\{a(p)\},B(p)) \le \delta_A^{-1}\left(\frac{1+t}{2}h\right) + th.$$

Taking the limit $t \to 1+0$, we obtain that
$$h(A(p),B(p)) = h(\{a(p)\},B(p)) \le \delta_A^{-1}(h) + h.$$

*Subcase 1.2.* $s(p,A) < s(p,B)$. Then all arguments of the Subcase *1.1* still apply except that
$$\varrho(a,H_A(p)) \le \varrho(a,H_B(p)) \le \|a-b(p)\| \le th,$$
$\varrho(w,H_A(p)) \le \frac{t}{2}h$, $\|a(p)-a\| \le \delta_A^{-1}\left(\frac{t}{2}h\right)$. Hence
$$h(A(p),B(p)) = h(\{a(p)\},B(p)) \le \delta_A^{-1}\left(\frac{t}{2}h\right) + th.$$

Taking the limit $t \to 1+0$, we obtain that
$$h(A(p),B(p)) \le \delta_A^{-1}\left(\frac{1}{2}h\right) + h.$$

So again when $h < \Delta$ we have for all $p \in \partial B_1^*(0)$
$$h(A(p),B(p)) \le \delta_A^{-1}(h) + h.$$

Hence $\rho(A,B) = \sup_{\|p\|_*=1} h(A(p),B(p)) \le \delta_A^{-1}(h) + h$.

*Case 2.* $h \ge \Delta$. Then for any $t > 1$ we have
$$\rho(A,B) \le \operatorname{diam} A + th \le \frac{h}{\Delta}\operatorname{diam} A + th \le h\left(t + \frac{\operatorname{diam} A}{\Delta}\right), \quad \forall t > 1.$$

Taking the limit $t \to 1+0$, we get
$$\rho(A,B) \le h\left(1 + \frac{\operatorname{diam} A}{\Delta}\right).$$

In the case when $A$ is a singleton the equality $\rho(A,B) = h(A,B)$ follows by Definition 1.3. □

For a set $A \subset \mathbb{R}^n$, $A \subset B_R(a)$ for some $a \in \mathbb{R}^n$ and $R > 0$, we define the *$R$-strongly convex hull* of the set $A$, as the intersection of all closed balls of radius $R$ each of which contains the set $A$. We shall denote the $R$-strongly convex hull of the set $A$ by $\operatorname{strco}_R A$ (cf. [2]).

**Example 2.8.** The estimate (8) is exact. Consider two sets $A$ and $B$ on the Euclidean plane $\mathbb{R}^2$. Let $0 < \varepsilon < R$, $a\left(\sqrt{2R\varepsilon - \varepsilon^2}, 0\right) \in \mathbb{R}^2$ and
$$A = \operatorname{strco}_R\{B_\varepsilon((0,0)) \cup \{a\}\} + B_R(0), \qquad B = \operatorname{strco}_R\{B_\varepsilon(a) \cup \{(0,0)\}\} + B_R(0).$$

Let $p = (0,1)$. It is easy to see that $h(A,B) = \varepsilon$, $A(p) = (\varepsilon + R)p$, $B(p) = \left(\sqrt{2R\varepsilon - \varepsilon^2}, \varepsilon\right) + Rp$. Hence

$$\rho(A,B) \geq h(A(p),B(p)) = \|a\| = \sqrt{2R\varepsilon - \varepsilon^2} = \sqrt{2Rh(A,B) - h^2(A,B)}.$$

The sets $A$ and $B = a - A$ are intersections of closed balls of radius $R+\varepsilon$ and $\delta_A(s) = \delta_B(s) \geq (R+\varepsilon)\delta_{\mathcal{H}}\left(\frac{s}{R+\varepsilon}\right)$, where $\delta_{\mathcal{H}}(s) = 1 - \sqrt{1 - \frac{s^2}{4}}$ is the modulus of convexity for the Hilbert space (see [7, p. 63]). Thus $\delta_A(s) = \delta_B(s) \geq \frac{s^2}{8(R+\varepsilon)}$, and $\delta_A^{-1}(t) \leq 2\sqrt{2(R+\varepsilon)t}$. So the order of $h(A,B)$ in formula (8) is exact. □

**Remark 2.9.** The result of Theorem 2.7 was proved for $p$-convex sets in [8, Formula (5)]. Note that any $p$-convex set, $p > 0$, in the paper [8] is in fact the intersection of closed balls of radius $R = \frac{1}{2p}$. From the definition of $p$-convex set (inequality (2) of [8]) it follows that for any $p$-convex set $A \subset \mathbb{R}^n$, any point $a \in \partial A$ and any unit vector $w \in \{q \in \mathbb{R}^n \mid (q, x-a) \leq 0,\ \forall x \in A\}$ we have

$$(w, x-a) + p\|x-a\|^2 \leq 0, \quad \forall x \in A,$$

or

$$A \subset B_R(a - Rw), \quad \text{where } R = \frac{1}{2p}.$$

Hence $A = \bigcap_{\|w\|=1} B_{\frac{1}{2p}}(a(w) - \frac{1}{2p}w)$, where $\{a(w)\} = A(w)$.

This also follows by results of [5], [9, Chapter 3].

**Corollary 2.10.** *Suppose that $F_i : (T,\varrho) \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$, $i = 1,2$, are continuous (in the metric $\rho$) set-valued mappings with strictly convex images. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then the set-valued mappings $F_1(t)+F_2(t)$, $LF_1(t)$, $F_2(t) \stackrel{*}{-} F_1(t) = \bigcap_{x\in F_1(t)}(F_2(t) - x)$, $F_1(t)\cap F_2(t)$ (the latter two if nonempty) are continuous in the metric $\rho$.*

**Proof.** The proof is similar for all cases. Let us prove the continuity of $F_1(t)\cap F_2(t)$.

The continuity of set-valued mappings $F_i$ in the metric $\rho$ and formula (7) gives the continuity of set-valued mappings $F_i$ in the Hausdorff metric.

It is well known that the intersection of two continuous in the Hausdorff metric set-valued mappings with compact strictly convex images is also continuous in the Hausdorff metric (cf. [1, 9]). Thus the set-valued mapping $H = F_1 \cap F_2 : (T,\varrho) \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$ is continuous in the Hausdorff metric.

For any point $t = t_0 \in T$ the set $H(t_0)$ is a strictly(=uniformly) convex compactum from $\mathbb{R}^n$ with some modulus of convexity $\delta_{t_0}$. By Theorem 2.7 we have

$$\rho(H(t),H(t_0)) \leq \max\left\{h(H(t),H(t_0)) + \delta_{t_0}^{-1}(h(H(t),H(t_0)));\right.$$
$$\left.\left(1 + \frac{\operatorname{diam} H(t_0)}{\Delta}\right) h(H(t),H(t_0))\right\} \underset{t\to t_0}{\longrightarrow} 0,$$

where $\Delta = \delta_{t_0}(\operatorname{diam} H(t_0))$. If $H(t_0)$ is a singleton then $\rho(H(t), H(t_0)) = h(H(t), H(t_0)) \to_{t \to t_0} 0$. □

## 3. Applications

**3.1.** We prove a theorem about smooth approximation of the extremal problem.

**Theorem 3.1.** *Let $F : (T, \varrho) \to 2^{\mathbb{R}^n} \setminus \emptyset$ be a continuous set-valued mapping with compact convex images and suppose that there exists $r > 0$ such that for all $t \in T$ $F(t) \subset B_r(a(t))$ for some $a(t) \in \mathbb{R}^n$. Let $\operatorname{diam} F(t) \geq d > 0$ for all $t \in T$.*

*For any $t \in T$ and $p \in \mathbb{R}^n$, $\|p\| = 1$, consider the following problem*

$$\max\{(p, x) \mid x \in F(t)\}. \tag{9}$$

*Then for any $\varepsilon \in (0, 1)$ there exists an approximation $F_\varepsilon : (T, \varrho) \to 2^{\mathbb{R}^n} \setminus \emptyset$, $F(t) \subset F_\varepsilon(t)$ for all $t \in T$, $h(F(t), F_\varepsilon(t)) \leq \varepsilon$ for all $t \in T$, such that for each $t \in T$ and $p \in \mathbb{R}^n$, $\|p\| = 1$, the following problem*

$$\max\{(p, x) \mid x \in F_\varepsilon(t)\} \tag{10}$$

*has a unique solution $F_\varepsilon(t, p) = \{f_\varepsilon(t, p)\} = \arg\max_{x \in F_\varepsilon(t)} (p, x)$ which is Hölder continuous with the power $\frac{1}{2}$ with respect to $h(F(t_1), F(t_2))$ for all $t_1, t_2 \in T$. The power $\frac{1}{2}$ is the best possible in the general case.*

**Proof.** Fix $\varepsilon \in (0, 1)$. Let $R = \max\{\frac{r^2}{\varepsilon}, r + 1\}$. Define $F_\varepsilon(t)$ as the intersection of all closed balls of radius $R$, each of which contains the set $F(t)$. This set is nonempty because $F(t) \subset B_R(a(t))$.

By [2, Formulae (5.7), (5.8)] and [9, Theorem 4.4.7] we have for all $t_1, t_2 \in T$

$$h(F_\varepsilon(t_1), F_\varepsilon(t_2)) \leq C(\varepsilon) h(F(t_1), F(t_2)),$$

$$C(\varepsilon) = \max\left\{\sqrt{\frac{R + r}{R - r}},\ 1 + \frac{r^2}{R(R - r)}\right\}.$$

By [2, Theorem 5.4] and [9, Theorem 4.4.6] we have

$$h(F(t), F_\varepsilon(t)) \leq \frac{r^2}{R} \leq \varepsilon, \quad \forall t \in T.$$

By the inequality $\delta_{F_\varepsilon(t)}(s) \geq R\delta_{\mathcal{H}}\left(\frac{s}{R}\right)$, where $\delta_{\mathcal{H}}(s) = 1 - \sqrt{1 - \frac{s^2}{4}}$ is the modulus of convexity for the Hilbert space [7, p. 63], we get

$$\delta_{F_\varepsilon(t)}(s) \geq \frac{s^2}{8R}, \quad \forall s \in (0, \operatorname{diam} F_\varepsilon(t)),$$

and by Theorem 2.7 we obtain for any $p \in \partial B_1(0)$ that

$$\begin{aligned} &\|f_\varepsilon(t_1, p) - f_\varepsilon(t_2, p)\| \\ &\leq \max\Big\{C(\varepsilon) h(F(t_1), F(t_2)) + \sqrt{8RC(\varepsilon) h(F(t_1), F(t_2))}; \\ &\qquad \left(1 + \frac{2r}{\Delta}\right) h(F(t_1), F(t_2))\Big\}, \end{aligned}$$

where $\Delta = R\delta_{\mathcal{H}}\left(\frac{d}{R}\right)$. On the other hand, we have for any convex compact set $A \subset \mathbb{R}^n$ that for some constant $C > 0$ the inequality $\delta_A(\varepsilon) \leq C\varepsilon^2$ holds for all $\varepsilon \in (0, \operatorname{diam} A)$ (see [3]). Taking into account also Example 2.8, we see that the power $\frac{1}{2}$ is the best possible. □

**3.2.** We consider Lipschitz selections and parametrizations of (strictly) convex compact sets with metric $\rho$.

With any convex compact set $A \subset \mathbb{R}^n$ we can associate the *Steiner point*

$$s(A) = \frac{1}{v_1} \int\limits_{\|p\|=1} s(p, A) p \, d\mu_{n-1}, \qquad v_1 = \mu_n B_1(0),$$

where $\mu_n$ is the Lebesgue measure in $\mathbb{R}^n$.

It is well known that the Steiner point is a Lipschitz selection of convex compacta in $\mathbb{R}^n$ with the Hausdorff metric, i.e. for any convex compacta $A, B \subset \mathbb{R}^n$ we have $s(A) \in A$ and

$$\|s(A) - s(B)\| \leq \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} h(A, B).$$

The Lipschitz constant (of the order $\sqrt{n}$) above is the best possible [11]. See also [14, p. 53], [12], [9, Theorem 2.1.2] for details.

Using the Gauss-type formula (see [9, formula (2.1.15)], [12, formula (3.1)]) we obtain that

$$\frac{1}{v_1} \int\limits_{\|p\|=1} s(p, A) p \, d\mu_{n-1} = \frac{1}{v_1} \int\limits_{\|p\|\leq 1} \nabla s(p, A) \, d\mu_n.$$

Note that $\nabla s(p, A)$ exists a.e. on the ball $B_1(0)$.

For any convex compactum $A \subset \mathbb{R}^n$ define $U(A) = \{p \in B_1(0) \mid \exists \nabla s(p, A)\}$. The function $s(p, A)$ is Lipschitz continuous hence $\mu_n U(A) = \mu_n B_1(0)$. Let $a(A, p) = \nabla s(p, A)$ for $p \in U(A)$ and $a(A, p) = 0$ for $p \in B_1(0) \backslash U(A)$.

Let $A, B \subset \mathbb{R}^n$ be convex compacta and $U = U(A) \cap U(B)$, $\mu_n U = \mu_n B_1(0)$. Then

$$\|s(A) - s(B)\| \leq \frac{1}{v_1} \int\limits_U \|a(A, p) - a(B, p)\| \, d\mu_n \leq \frac{1}{v_1} \int\limits_U \rho(A, B) \, d\mu_n = \rho(A, B).$$

Thus the Steiner point is a Lipschitz selection of convex compacta in $\mathbb{R}^n$ with metric $\rho$ with the Lipschitz constant 1.

Let $\mathbb{A}$ be a collection of strictly convex compacta. Then for any $p \in \mathbb{R}^n$, $\|p\| = 1$, the function $a(p) = A(p)$, $A \in \mathbb{A}$, is a Lipschitz selection of the family $\mathbb{A}$ with the Lipschitz constant 1 in the metric $\rho$.

**Theorem 3.2.** *Let a collection of strictly convex compacta $\mathbb{A}$ from $\mathbb{R}^n$ be uniformly bounded, i.e. there exists $M > 0$ such that $\|A\| = h(\{0\}, A) \leq M$ for all $A \in \mathbb{A}$.*

*Then there exists a family of functions*

$$f_{\lambda,p} : \mathbb{A} \to \mathbb{R}^n, \quad (\lambda, p) \in [0,1] \times \partial B_1(0), \tag{11}$$

*such that for any $A \in \mathbb{A}$ we have*

$$A = \{f_{\lambda,p}(A) \mid \lambda \in [0,1],\ p \in \partial B_1(0)\}$$

*and for any $(\lambda, p) \in [0,1] \times \partial B_1(0)$ the function $f_{\lambda,p}$ is a Lipschitz selection on $A \in \mathbb{A}$ in the metric $\rho$ with Lipschitz constant 1.*

*Moreover, the function $[0,1] \times \partial B_1(0) \ni (\lambda, p) \to f_{\lambda,p}(A)$ is continuous for any $A \in \mathbb{A}$ and the function $f_{\lambda,p}$ is additive: $f_{\lambda,p}(A+B) = f_{\lambda,p}(A) + f_{\lambda,p}(B)$, $A, B \in \mathbb{A}$.*

**Proof.** For any $A \in \mathbb{A}$ we define $f_{\lambda,p}(A) = \lambda a(p) + (1-\lambda)s(A) \in A$. Let $A, B \in \mathbb{A}$. Then (note, that $b(p) = B(p)$ for any $p \in \partial B_1(0)$)

$$\begin{aligned} \|f_{\lambda,p}(A) - f_{\lambda,p}(B)\| &\le \lambda\|a(p) - b(p)\| + (1-\lambda)\|s(A) - s(B)\| \\ &\le \lambda\rho(A,B) + (1-\lambda)\rho(A,B) = \rho(A,B). \end{aligned}$$

Choose $\lambda_1, \lambda_2 \in [0,1]$ and $p_1, p_2 \in \partial B_1(0)$ and $A \in \mathbb{A}$.

$$\begin{aligned} \|f_{\lambda_1,p_1}(A) - f_{\lambda_2,p_2}(A)\| &= \|\lambda_1 a(p_1) + (1-\lambda)s(A) - \lambda_2 a(p_2) - (1-\lambda_2)s(A)\| \\ &\le \|\lambda_1 a(p_1) - \lambda_2 a(p_2)\| + |\lambda_1 - \lambda_2|\|s(A)\| \\ &\le |\lambda_1 - \lambda_2|\|a(p_1)\| + |\lambda_2|\|a(p_1) - a(p_2)\| + |\lambda_1 - \lambda_2|\|s(A)\| \\ &\le 2|\lambda_1 - \lambda_2|M + \|a(p_1) - a(p_2)\|. \end{aligned}$$

The gradient $a(p) = \nabla s(p, A)$ for the strictly convex compact set $A$ is uniformly continuous on the unit sphere (see [3, Lemma 2.2]). So the function $[0,1] \times \partial B_1(0) \ni (\lambda, p) \to f_{\lambda,p}(A)$ is uniformly continuous.

By the Moreau-Rockafellar theorem [13] for all $A, B \in \mathbb{A}$ we get $A(p) + B(p) = (A+B)(p)$ for all $p \in \partial B_1(0)$. Using the additive property of the Steiner point [9], [12], [14] we obtain that $f_{\lambda,p}$ is an additive selection for all $\lambda \in [0,1]$ and $\|p\| = 1$. □

**Remark 3.3.** We see from the proof of Theorem 3.2, that the function

$$(\lambda, p) \to f_{\lambda,p}(A)$$

is uniformly continuous for any $A \in \mathbb{A}$. More precisely, $f_{\lambda,p}$ is Lipschitz on $\lambda \in [0,1]$ (with Lipschitz constant $2M$) and uniformly continuous on $p \in \partial B_1(0)$. Note that $f_{\lambda,p}(A)$ is Lipschitz on $p \in \partial B_1(0)$ if and only if the set $A$ is the intersection of closed balls of the same fixed radius. The last assertion follows by results of [5] and by Theorem 4.3.2 of [9]: *a set $A$ is the intersection of closed balls of fixed radius $R > 0$ in Hilbert space if and only if $\|a(p) - a(q)\| \le R\|p - q\|$ for all $p, q \in \partial B_1(0)$.* Here $a(p) = A(p)$.

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