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Polyhedral approximations of strictly convex compacta

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ABSTRACT

We consider polyhedral approximations of strictly convex compacta in finite-dimensional Euclidean spaces (such compacta are also uniformly convex). We obtain the best possible estimates for errors of considered approximations in the Hausdorff metric. We also obtain new estimates of an approximate algorithm for finding the convex hulls.

1. Introduction

We begin by some definitions for a finite-dimensional Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ over $\mathbb{R}$ with an inner product $(\cdot,\cdot)$. Let $B_r(a) = \{x \in \mathbb{R}^n \mid \|x-a\| \leqslant r\}$. Let $\operatorname{cl} A$ denote the *closure* and $\operatorname{int} A$ the *interior* of the subset $A \subset \mathbb{R}^n$. The *diameter* of the subset $A \subset \mathbb{R}^n$ is defined as $\operatorname{diam} A = \sup_{x,y\in A} \|x-y\|$. The *distance* from the point $x \in \mathbb{R}^n$ to the set $A \subset \mathbb{R}^n$ is given by the formula $\varrho(x, A) = \inf_{a\in A} \|x-a\|$. We shall denote the *convex hull* of a set $A \subset \mathbb{R}^n$ by $\operatorname{co} A$, the convex hull of a function $f : \mathbb{R}^n \to \mathbb{R}$ by $\operatorname{co} f$ (cf. [1,9,13]).

The *Hausdorff distance* between two subsets $A, B \subset \mathbb{R}^n$ is defined as follows

$$h(A, B) = \max\Big\{ \sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\| \Big\} = \inf\big\{ r > 0 \bigm| A \subset B + B_r(0),\ B \subset A + B_r(0)\big\}.$$

The *supporting function* of the subset $A \subset \mathbb{R}^n$ is defined as follows

$$s(p, A) = \sup_{x\in A} (p, x), \quad \forall p \in \mathbb{R}^n. \tag{1.1}$$

The supporting function of any set A is always lower semicontinuous, positively uniform and convex. If the set A is bounded then the supporting function is Lipschitz continuous [1,9].

It follows from the separation theorem that for any convex compacta A, B in $\mathbb{R}^n$ (cf. [9, Lemma 1.11.4])

$$h(A, B) = \sup_{\|p\|=1} \big| s(p, A) - s(p, B) \big|. \tag{1.2}$$

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A convex compactum in $\mathbb{R}^n$ is called *strictly* convex if its boundary contains no nontrivial line segments.

Definition 1.1. (See Polyak [11].) Let E be a Banach space and let a subset $A \subset E$ be convex and closed. *The modulus of convexity* $\delta_A : [0, \operatorname{diam} A) \to [0, +\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup\left\{\delta \geqslant 0 \,\middle|\, B_\delta\left(\frac{x_1 + x_2}{2}\right) \subset A,\ \forall x_1, x_2 \in A\colon \|x_1 - x_2\| = \varepsilon\right\}.$$

Definition 1.2. (See Polyak [11].) Let E be a Banach space and let a subset $A \subset E$ be convex and closed. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0, \operatorname{diam} A)$, then we call the set A *uniformly convex* (*with modulus* $\delta_A(\cdot)$).

We proved in [3] that every uniformly convex set is bounded and if the Banach space E contains a nonsingleton uniformly convex set then it admits a uniformly convex equivalent norm. We also proved that the function $\varepsilon \to \delta_A(\varepsilon)/\varepsilon$ is increasing (see also [6, Lemma 1.e.8]), and for any uniformly convex set A there exists a constant $C > 0$ such that $\delta_A(\varepsilon) \leqslant C\varepsilon^2$.

The class of strictly convex compacta coincides with the class of uniformly convex compacta with moduli of convexity $\delta_A(\varepsilon) > 0$ for all permissible $\varepsilon > 0$ in the finite-dimensional case (cf. [3]).

Definition 1.3. (See [8,9].) A *grid* $\mathbb{G}$ with *step* $\Delta \in (0, \frac{1}{2})$ is a finite collection of unit vectors $\{p_i\} \subset \mathbb{R}^n$, $i \in \overline{1, I} = \{1, \ldots, I\}$, such that for any vector $p \neq 0$, $p \in \mathbb{R}^n$, with $\frac{p}{\|p\|} \notin \mathbb{G}$ there exist a set of indexes $I_p \subset \overline{1, I}$ and numbers $\alpha_i > 0$, $i \in I_p$, with the property

$$p = \sum_{i \in I_p} \alpha_i p_i, \quad p_i \in \mathbb{G}, \tag{1.3}$$

$$\|p_i - p_j\| < \Delta, \quad \forall i, j \in I_p. \tag{1.4}$$

It is well known [1,9,13] that for any convex closed subset $A \subset \mathbb{R}^n$ we have

$$A = \bigl\{x \in \mathbb{R}^n \bigm| (p, x) \leqslant s(p, A),\ \forall p \in \partial B_1(0)\bigr\}.$$

We shall consider external polyhedral approximation of the compact $A \subset \mathbb{R}^n$ on the grid $\mathbb{G}$ from Definition 1.3

$$\hat{A} = \bigl\{x \in \mathbb{R}^n \bigm| (p, x) \leqslant s(p, A),\ \forall p \in \mathbb{G}\bigr\}.$$

From the inclusion $\mathbb{G} \subset \partial B_1(0)$ we easily see that $A \subset \hat{A}$. For an arbitrary convex compact set $A \subset \mathbb{R}^n$ we have $h(A, \hat{A}) \leqslant 2h(\{0\}, A)\Delta$ (cf. [8,9]). If $A = \bigcap_{x \in X} B_R(x) \neq \emptyset$, then $h(A, \hat{A}) \leqslant 2R\Delta^2$ (cf. [8,9]). Further we shall consider the approximation of an arbitrary strictly = uniformly convex compact set $A \subset \mathbb{R}^n$. Our further goals are

(1) Estimate the error $h(A, \hat{A})$ via the geometric properties of the set A.
(2) Suppose that we know a presupporting function $f(p)$ of the convex compact A, i.e. the function f is positively uniform, continuous and $\operatorname{co} f(p) = s(p, A)$. Let

$$\tilde{A} = \bigl\{x \in \mathbb{R}^n \bigm| (p, x) \leqslant f(p),\ \forall p \in \mathbb{G}\bigr\}.$$

Estimate the error $h(A, \tilde{A})$ via the properties of the function f and geometric properties of the set A. In this case we do not know the supporting function of the set A, but we can find information about some properties of the set A: diameter, modulus of convexity, etc.
(3) We shall consider an algorithm for calculating the convex hull of a positively uniform function defined on the grid and discuss estimates for the errors of such algorithms.

2. Approximation by supporting functions

Lemma 2.1. *For a given grid* $\mathbb{G}$ (*Definition* 1.3) *with step* $\Delta \in (0, \frac{1}{2})$ *in the representation of any vector* $p \neq 0$, $\frac{p}{\|p\|} \notin \mathbb{G}$ *by formulae* (1.3), (1.4) *the following estimates hold*

$$\|\hat{p} - p_j\| < \Delta, \quad \forall j \in I_p,\ 1 \geqslant \|\hat{p}\| \geqslant 1 - \frac{1}{2}\Delta^2,$$

where

$$\hat{p} = \frac{p}{\alpha} = \sum_{i \in I_p} \hat{\alpha}_i p_i, \quad \alpha = \sum_{i \in I_p} \alpha_i,\ \hat{\alpha}_i = \frac{\alpha_i}{\alpha}. \tag{2.5}$$

Proof. From the definitions of $\hat{p}$ and $\hat{\alpha}_i$ we obtain that

$$\hat{p} = \sum_{i\in I_p} \hat{\alpha}_i p_i, \qquad \sum_{i\in I_p} \hat{\alpha}_i = 1.$$

Hence

$$\sum_{i\in I_p} \hat{\alpha}_i (\hat{p} - p_i) = 0.$$

By the triangle inequality we get

$$\|\hat{p} - p_j\| \leqslant \sum_{i\in I_p} \hat{\alpha}_i \|p_i - p_j\| < \Delta, \quad \forall j \in I_p,$$
$$\|\hat{p}\| \leqslant \sum_{i\in I_p} \hat{\alpha}_i \|p_i\| = 1. \tag{2.6}$$

The condition $\|p_i - p_j\| < \Delta$ is equivalent to the condition $(p_i, p_j) = \frac{1}{2}(\|p_i\|^2 + \|p_j\|^2 - \|p_i - p_j\|^2) \geqslant 1 - \Delta^2/2$. Thus $1 \geqslant \|\hat{p}\| \geqslant \|\hat{p}\|^2 = \sum_{i,j} \hat{\alpha}_i \hat{\alpha}_j (p_i, p_j) \geqslant (1 - \frac{1}{2}\Delta^2) \sum_{i,j} \hat{\alpha}_i \hat{\alpha}_j = 1 - \frac{1}{2}\Delta^2$. $\square$

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively uniform function. Let $\mathbb{G}$ be a grid with step $\Delta \in (0, \frac{1}{2})$. Define the grid operators

$$\mathcal{C}f(p) = \begin{cases} f(p), & \frac{p}{\|p\|} \in \mathbb{G}, \\ +\infty, & \frac{p}{\|p\|} \notin \mathbb{G}, \end{cases} \qquad \mathcal{U}f(p) = \begin{cases} f(p), & \frac{p}{\|p\|} \in \mathbb{G}, \\ \sum_{i\in I_p} \alpha_i f(p_i), & \frac{p}{\|p\|} \notin \mathbb{G}. \end{cases}$$

Indices I_p and numbers α_i are from Definition 1.3.

Lemma 2.2. *(See [8, Lemma 5], [9, Lemma 2.6.2].) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively uniform function.*

(1) *If the function f is convex then $\mathcal{C}f \geqslant f$, $\mathrm{co}\,\mathcal{C}f(p) = f(p)$, $\forall p \in \mathbb{G}$.*
(2) *If the function f is convex then $f \leqslant \mathrm{co}\,\mathcal{U}f$.*
(3) $\mathrm{co}\,\mathcal{C}f = \mathrm{co}\,\mathcal{U}f$, $\forall f$.
(4) $\mathrm{co}\, f \leqslant \mathrm{co}\,\mathcal{C}f$, $\forall f$.

The next lemma is a modification of Lemma 2.2 from [3].

Lemma 2.3. *Let $A \subset \mathbb{R}^n$ be compact and uniformly convex set with the modulus of convexity δ. Let $\varepsilon \in (0, \operatorname{diam} A)$, $\Delta \in (0, \frac{1}{2})$. Let $p_1, p_2 \in \mathbb{R}^n$, $\|p_1\| = 1$, $1 - \frac{1}{2}\Delta^2 \leqslant \|p_2\| \leqslant 1$. Let $x_i = \arg\max_{x\in A}(p_i, x)$, $i = 1, 2$. If $\|p_1 - p_2\| < (4 - \Delta^2)\frac{\delta(\varepsilon)}{\varepsilon}$, then $\|x_1 - x_2\| < \varepsilon$.*

Proof. Suppose that $\|x_1 - x_2\| \geqslant \varepsilon$. Let $t = \delta(\|x_1 - x_2\|) \geqslant \delta(\varepsilon)$. By the condition

$$B_t\left(\frac{x_1 + x_2}{2}\right) \subset A$$

we have that $(p_1, x_1) \geqslant (p_1, \frac{x_1+x_2}{2}) + t$,

$$(p_1, x_1 - x_2) \geqslant 2t, \tag{2.7}$$
$$(p_2, x_2) \geqslant \left(p_2, \frac{x_1 + x_2}{2}\right) + t\|p_2\| \geqslant \left(p_2, \frac{x_1 + x_2}{2}\right) + t\left(1 - \frac{1}{2}\Delta^2\right),$$
$$(p_2, x_2 - x_1) \geqslant \left(2 - \Delta^2\right)t. \tag{2.8}$$

By formulae (2.7), (2.8) we obtain that

$$(p_1 - p_2, x_1 - x_2) \geqslant \left(4 - \Delta^2\right)t,$$

and

$$\|p_1 - p_2\| \geqslant \left(4 - \Delta^2\right)\frac{\delta(\|x_1 - x_2\|)}{\|x_1 - x_2\|}.$$

By Lemma 2.1 of [3] we have the inequality $\frac{\delta(\|x_1-x_2\|)}{\|x_1-x_2\|} \geqslant \frac{\delta(\varepsilon)}{\varepsilon}$ and

$$\|p_1 - p_2\| \geqslant \left(4 - \Delta^2\right)\frac{\delta(\varepsilon)}{\varepsilon}.$$

Contradiction. □

Corollary 2.1. *Let under the conditions of Lemma 2.3 $\varepsilon(\Delta)$ be a solution of the equation $\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$. If $\|p_1 - p_2\| < \Delta$ then $\|x_1 - x_2\| < \varepsilon(\Delta)$.*

Proof. The proof follows from Lemma 2.3 and strict monotonicity of the function $\frac{\delta(\varepsilon)}{\varepsilon}$ [4, Lemma 1.2]. □

Theorem 2.1. *Let $p_1, p_2 \in \mathbb{R}^n$, $\|p_1\| = 1$, $1 - \frac{1}{2}\Delta^2 \leqslant \|p_2\| \leqslant 1$, $\|p_1 - p_2\| < \Delta$. Let $A \subset \mathbb{R}^n$ be compact and uniformly convex set with modulus of convexity δ and $\Delta \in (0, \frac{1}{2})$, $\delta(\operatorname{diam} A)/\operatorname{diam} A > \frac{\Delta}{4-\Delta^2}$. Let $x_i = \arg\max_{x\in A}(p_i, x)$, $i = 1, 2$. Then*

$$s(p_1, A) - s(p_2, A) = (x_2, p_1 - p_2) + \varepsilon_1\left(\|p_1 - p_2\|\right)\|p_1 - p_2\|, \tag{2.9}$$

$$s(p_2, A) - s(p_1, A) = (x_1, p_2 - p_1) + \varepsilon_2\left(\|p_1 - p_2\|\right)\|p_1 - p_2\|, \tag{2.10}$$

and

$$\max\left\{\left|\varepsilon_1\left(\|p_1 - p_2\|\right)\right|\|p_1 - p_2\|, \left|\varepsilon_1\left(\|p_1 - p_2\|\right)\right|\|p_1 - p_2\|\right\} \leqslant \varepsilon(\Delta)\Delta,$$

where $\varepsilon(\Delta)$ is a solution of the equation $\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$.

Proof. Eqs. (2.9) and (2.10) are equivalent to the condition of continuous gradient for (convex) supporting function at the points p_2 and p_1, respectively. It is a well-known fact that the supporting function of the strictly convex compact is continuously differentiable [9,13].

By Corollary 2.1 we have the estimate

$$\left|(p_1 - p_2, x_1 - x_2)\right| \leqslant \|p_1 - p_2\|\|x_1 - x_2\| \leqslant \Delta\varepsilon(\Delta).$$

From the equalities $(p_i, x_i) = s(p_i, A)$, $i = 1, 2$, we conclude that

$$\varepsilon_1\left(\|p_1 - p_2\|\right)\|p_1 - p_2\| = (p_1, x_1 - x_2), \qquad \varepsilon_2\left(\|p_1 - p_2\|\right)\|p_1 - p_2\| = (p_2, x_2 - x_1).$$

By the formulae

$$(p_1, x_1 - x_2) + (p_2, x_2 - x_1) = (p_1 - p_2, x_1 - x_2), \tag{2.11}$$

and $(p_1, x_1 - x_2) \geqslant 0$, $(p_2, x_2 - x_1) \geqslant 0$ we have

$$\max\left\{\left|(p_1, x_1 - x_2)\right|, \left|(p_2, x_2 - x_1)\right|\right\} \leqslant (p_1 - p_2, x_1 - x_2) \leqslant \varepsilon(\Delta)\Delta. \qquad \square$$

It is well known that the external polyhedral approximation $\hat{A}$ of the convex compact set $A \subset \mathbb{R}^n$ with supporting function $s(p, A)$ on the grid $\mathbb{G}$ satisfies the formula $s(p, \hat{A}) = \operatorname{co}\mathcal{C}s(p, A)$ [8,9].

Theorem 2.2. *Let $A \subset \mathbb{R}^n$ be a convex compact set with the modulus of convexity $\delta(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A]$. Let $\mathbb{G}$ be a grid with the step $\Delta \in (0, \frac{1}{2})$, $\delta(\operatorname{diam} A)/\operatorname{diam} A > \frac{\Delta}{4-\Delta^2}$. Then*

$$h(A, \hat{A}) \leqslant \frac{8}{7}\varepsilon(\Delta)\Delta,$$

where $\varepsilon(\Delta)$ is a solution of the equation $\frac{\delta(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$.

Proof. From the inclusion $A \subset \hat{A}$, formula $s(p, \hat{A}) = \operatorname{co}\mathcal{C}s(p, A) = \operatorname{co}\mathcal{U}s(p, A)$ (see Lemma 2.2) and from Definition 2.1 it follows that (in terms of Definition 1.3)

$$0 \leqslant s(p, \hat{A}) - s(p, A) \leqslant \mathcal{U}s(p, A) - s(p, A) = \alpha \sum_{i\in I_p} \hat{\alpha}_i\left(s(p_i, A) - s(\hat{p}, A)\right), \quad \forall p \in \mathbb{R}^n. \tag{2.12}$$

Let $\hat{x} = \arg\max_{x\in A}(\hat{p}, x)$.

$$s(p_i, A) - s(\hat{p}, A) = (\hat{x}, p_i - \hat{p}) + \varepsilon_i\left(\|p_i - \hat{p}\|\right)\|p_i - \hat{p}\|,$$

and by properties of vector $\hat{p}$ (Lemma 2.1) and Theorem 2.1 we conclude that

$$\left|\varepsilon_i\left(\|p_i-\hat{p}\|\right)\|p_i-\hat{p}\|\right| \leqslant \varepsilon(\Delta)\Delta, \quad \forall i \in I_p.$$

Finally, using the equality $\sum_{i\in I_p}\hat{\alpha}_i p_i = \hat{p}$, we obtain that

$$\sum_{i\in I_p}\hat{\alpha}_i\left(s(p_i,A)-s(\hat{p},A)\right) = \sum_{i\in I_p}\hat{\alpha}_i\left((\hat{x},p_i-\hat{p})+\varepsilon_i\left(\|p_i-\hat{p}\|\right)\|p_i-\hat{p}\|\right)$$
$$\leqslant \max_{i\in I_p}\left|\varepsilon_i\left(\|p_i-\hat{p}\|\right)\|p_i-\hat{p}\|\right| \leqslant \varepsilon(\Delta)\Delta.$$

By formula (2.12) it follows

$$0 \leqslant s(p,\hat{A})-s(p,A) \leqslant \alpha\varepsilon(\Delta)\Delta = \frac{\|p\|}{\|\hat{p}\|}\varepsilon(\Delta)\Delta \leqslant \frac{\varepsilon(\Delta)\Delta}{1-\frac{1}{2}\Delta^2}\|p\| \leqslant \frac{8}{7}\varepsilon(\Delta)\Delta\|p\|.$$

By formula (1.2) we get

$$h(A,\hat{A}) \leqslant \frac{8}{7}\varepsilon(\Delta)\Delta. \qquad \square$$

Corollary 2.2. *For any convex compact set A the modulus of convexity δ satisfies the estimate $\delta(\varepsilon) \leqslant C\varepsilon^2$. Thus the typical value of $\varepsilon(\Delta)$ is $\varepsilon(\Delta) \asymp \Delta^s$, $\Delta \to +0$, where $s \in (0,1]$.*

Corollary 2.3. *The estimate of Theorem 2.2 is exact.*

Consider an example. Let $\mathbb{R}^2$ be the Euclidean plane with the standard basis Ox_1x_2. Let $A = \{x_2 \geqslant |x_1|^s\} \cap B_1(0)$, $s \geqslant 2$.

The modulus of convexity for the set A equals $\delta(\varepsilon) = \varepsilon^s/2^s$ for small $\varepsilon > 0$ and it is realized on the segment $[(-\frac{\varepsilon}{2},\frac{\varepsilon^s}{2^s}),(\frac{\varepsilon}{2},\frac{\varepsilon^s}{2^s})]$. Let a and b be two points from ∂A:

$$a = \left(-\frac{\varepsilon}{2},\frac{\varepsilon^s}{2^s}\right), \qquad b = \left(\frac{\varepsilon}{2},\frac{\varepsilon^s}{2^s}\right).$$

Let p_a and p_b be unit normals to the set A at the points a and b, respectively. It is easy to calculate that

$$p_b = \frac{(s(\varepsilon/2)^{s-1},-1)}{\sqrt{s^2(\varepsilon/2)^{2(s-1)}+1}}, \qquad p_a = \frac{(-s(\varepsilon/2)^{s-1},-1)}{\sqrt{s^2(\varepsilon/2)^{2(s-1)}+1}}, \qquad \|p_b-p_a\| = \frac{2s(\varepsilon/2)^{s-1}}{\sqrt{s^2(\varepsilon/2)^{2(s-1)}+1}}.$$

Let $\Delta = \|p_b-p_a\|$ and p_a, p_b be adjacent vectors of some grid $\mathbb{G}$ with the step Δ. Suppose that the grid $\mathbb{G}$ has symmetry with respect to the line Ox_2.

Then $\varepsilon \asymp \Delta^{\frac{1}{s-1}}$ (for small Δ). The tangent line to the graph $x_2 = |x_1|^s$ at the point b is

$$y_{tan}(x_1) = s\left(\frac{\varepsilon}{2}\right)^{s-1}\left(x_1-\frac{\varepsilon}{2}\right)+\left(\frac{\varepsilon}{2}\right)^s.$$

We have $y_{tan}(0) = -(s-1)(\frac{\varepsilon}{2})^s$. The point $c = (0,y_{tan}(0))$ belongs to the set $\hat{A}$ (because approximation $\hat{A}$ has symmetry with respect to the line Ox_2). Hence

$$h(A,\hat{A}) \geqslant \varrho(c,A) = \left|y_{tan}(0)\right| = (s-1)\left(\frac{\varepsilon}{2}\right)^s.$$

So we have $h(A,\hat{A}) \geqslant C\cdot\Delta^{\frac{s}{s-1}}$. The same order $\frac{s}{s-1}$ is given by Theorem 2.2.

3. Approximation by presupporting functions

Suppose that we know a presupporting function f of a convex set $A \subset \mathbb{R}^n$. We want to estimate the difference $\operatorname{co}\mathcal{C}f(p) - \operatorname{co}f(p)$ for all $p \in \mathbb{R}^n$. The last question is equivalent to the question of evaluation of the value $h(A,\tilde{A})$ where $A = \{x \mid (p,x) \leqslant \operatorname{co}f(p),\ \forall p \in \mathbb{R}^n\}$, $\tilde{A} = \{x \mid (p,x) \leqslant \operatorname{co}\mathcal{C}f(p),\ \forall p \in \mathbb{R}^n\}$.

The *geometric difference* of sets $B, A \subset \mathbb{R}^n$ is the set

$$B \overset{*}{-} A = \{x \mid x+A \subset B\} = \bigcap_{a\in A}(B-a).$$

We shall obtain the solution for two particular cases of presupporting function. The first case is $f(p) = s(p, B) - s(p, A)$, where A and B are convex compacta. In the case $B \overset{*}{-} A \neq \emptyset$ the convex hull $\operatorname{co} f(p)$ equals the supporting function $s(p, B \overset{*}{-} A)$ of the geometric difference $B \overset{*}{-} A$ [9, formula (1.11.18)].

The second case is $f(p) = \min\{s(p, A), s(p, B)\}$, where A and B are convex compacta. In the case $A \cap B \neq \emptyset$ the convex hull $\operatorname{co} f(p)$ equals the supporting function $s(p, A \cap B)$ of the intersection $A \cap B$ [9, formula (1.11.17)].

The considered cases have important role for computational geometry [5] and for linear differential games [10,12].

Theorem 3.1. *Let $A, B \subset \mathbb{R}^n$ be convex compacta and suppose that B is uniformly convex with modulus δ_B. Let $f(p) = s(p, B) - s(p, A)$. Let $B_{r_0}(a) \subset B \overset{*}{-} A \subset B_d(a)$. Let $\mathbb{G}$ be a grid with step $\Delta \in (0, \frac{1}{2})$; $\delta_B(\operatorname{diam} B)/\operatorname{diam} B > \frac{\Delta}{4-\Delta^2}$. Then*

$$\operatorname{co} f(p) \leqslant \operatorname{co} \mathcal{C} f(p) \leqslant \operatorname{co} f(p) + \frac{8d}{7r_0} \varepsilon_B(\Delta) \Delta \|p\|, \quad \forall p \in \mathbb{R}^n, \tag{3.13}$$

where $\varepsilon = \varepsilon_B(\Delta)$ is a solution of $\frac{\delta_B(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$.

Proof. The left inequality in (3.13) is obvious. By [9, formula (1.11.18)] we have

$$\operatorname{co} f(p) = \operatorname{co}\big(s(p, B) - s(p, A)\big) = s(p, B \overset{*}{-} A).$$

Let $C = B \overset{*}{-} A$. By the formula

$$\mathcal{C} f(p) + \mathcal{C} s(p, A) = \mathcal{C} s(p, B)$$

we obtain that

$$\operatorname{co}\big(\mathcal{C} f(p) + \mathcal{C} s(p, A)\big) = \operatorname{co} \mathcal{C} s(p, B) = s(p, \hat{B}),$$

where $\hat{B} = \{x \mid (p, x) \leqslant \mathcal{C} s(p, B),\ \forall p \in \mathbb{R}^n\}$. Using inequality $\operatorname{co}(f + g) \geqslant \operatorname{co} f + \operatorname{co} g$ (which is true for any functions f, g) we have

$$\operatorname{co} \mathcal{C} f(p) + \operatorname{co} \mathcal{C} s(p, A) \leqslant s(p, \hat{B}),$$

or

$$\operatorname{co} \mathcal{C} f(p) \leqslant s(p, \hat{B}) - \operatorname{co} \mathcal{C} s(p, A) = s(p, \hat{B}) - s(p, \hat{A}) \leqslant s(p, \hat{B}) - s(p, A).$$

By the last inequality

$$\operatorname{co} \mathcal{C} f(p) \leqslant \operatorname{co}\big(s(p, \hat{B}) - s(p, A)\big) = s(p, \hat{B} \overset{*}{-} A).$$

Let $\|p\| = 1$. Using (1.2) we have

$$\operatorname{co} \mathcal{C} f(p) - \operatorname{co} f(p) \leqslant s(p, \hat{B} \overset{*}{-} A) - s(p, B \overset{*}{-} A) \leqslant h(\hat{B} \overset{*}{-} A, B \overset{*}{-} A).$$

Let $h = h(B, \hat{B})$. Using conditions of the theorem we conclude that

$$\begin{aligned}
B \overset{*}{-} A \subset \hat{B} \overset{*}{-} A &\subset \big(B + B_h(0)\big) \overset{*}{-} A \subset \left(B + \frac{h}{r_0}(C - a)\right) \overset{*}{-} A \\
&= \left(B + \frac{h}{r_0}(B \overset{*}{-} A)\right) \overset{*}{-} A - \frac{h}{r_0} a \subset \left(B + \frac{h}{r_0} B \overset{*}{-} \frac{h}{r_0} A\right) \overset{*}{-} A - \frac{h}{r_0} a \\
&= (B \overset{*}{-} A) + \frac{h}{r_0}\big((B \overset{*}{-} A) - a\big) \subset (B \overset{*}{-} A) + \frac{h}{r_0} B_d(0).
\end{aligned}$$

Hence

$$h(B \overset{*}{-} A, \hat{B} \overset{*}{-} A) \leqslant \frac{d}{r_0} h(B, \hat{B}).$$

Applying Theorem 2.2 we finish the proof. $\square$

Theorem 3.2. *Let $A, B \subset \mathbb{R}^n$ be uniformly convex compacta with moduli δ_A, δ_B. Let $f(p) = \min\{s(p, A), s(p, B)\}$. Let $B_{r_0}(a) \subset A \cap B$, $\max\{\operatorname{diam} \hat{A}, \operatorname{diam} \hat{B}\} \leqslant d$. Let $\mathbb{G}$ be a grid with step $\Delta \in (0, \frac{1}{2})$; $\delta_A(\operatorname{diam} A)/\operatorname{diam} A > \frac{\Delta}{4-\Delta^2}$, $\delta_B(\operatorname{diam} B)/\operatorname{diam} B > \frac{\Delta}{4-\Delta^2}$. Then*

$$\operatorname{co} f(p) \leqslant \operatorname{co} \mathcal{C} f(p) \leqslant \operatorname{co} f(p) + \frac{8}{7}\left(\max\big\{\varepsilon_A(\Delta), \varepsilon_B(\Delta)\big\} + \frac{d}{r_0}\big(\varepsilon_A(\Delta) + \varepsilon_B(\Delta)\big)\right) \Delta \|p\|, \quad \forall p \in \mathbb{R}^n, \tag{3.14}$$

where $\varepsilon = \varepsilon_X(\Delta)$ is a solution of $\frac{\delta_X(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$, $X = A$ or $X = B$.

Proof. Let $C = A \cap B$ and $\hat{C}$ be external polyhedral approximation of the set C on the grid $\mathbb{G}$. By the inclusions $C \subset A$, $C \subset B$ we have $\hat{C} \subset \hat{A}$, $\hat{C} \subset \hat{B}$ and thus $\hat{C} \subset \hat{A} \cap \hat{B}$.

$$\mathcal{C} f(p) = \min\{\mathcal{C}s(p, A), \mathcal{C}s(p, B)\}, \qquad \operatorname{co}\mathcal{C} f(p) = \operatorname{co}\min\{\mathcal{C}s(p, A), \mathcal{C}s(p, B)\}.$$

By the inequality $\operatorname{co}\min\{f, g\} \leqslant \min\{\operatorname{co} f, \operatorname{co} g\}$ (which is valid for any functions f, g) we have

$$\operatorname{co}\mathcal{C} f(p) \leqslant \min\{\operatorname{co}\mathcal{C}s(p, A), \operatorname{co}\mathcal{C}s(p, B)\} = \min\{s(p, \hat{A}), s(p, \hat{B})\}.$$

Hence

$$\operatorname{co}\mathcal{C} f(p) \leqslant \operatorname{co}\bigl(\min\{s(p, \hat{A}), s(p, \hat{B})\}\bigr) = s(p, \hat{A} \cap \hat{B}).$$

Let $\|p\| = 1$. Using (1.2) we have

$$\operatorname{co}\mathcal{C} f(p) - \operatorname{co} f(p) \leqslant s(p, \hat{A} \cap \hat{B}) - s(p, A \cap B) \leqslant h(A \cap B, \hat{A} \cap \hat{B}).$$

Applying Theorem 3.1 [2] we obtain that

$$h(A \cap B, \hat{A} \cap \hat{B}) \leqslant \max\{h(A, \hat{A}), h(B, \hat{B})\} + \frac{d}{r_0}\bigl(h(A, \hat{A}) + h(B, \hat{B})\bigr).$$

Using of Theorem 2.2 ends the proof. □

4. On finding the convex hulls

Theorem 4.1. *Let $A \subset \mathbb{R}^n$ be a uniformly convex compact set with modulus of convexity δ. Let*

$$r_0 = \sup\{r \geqslant 0 \mid \exists a \in \mathbb{R}^n\colon\ B_r(a) \subset A\}.$$

Let a point $a \in \mathbb{R}^n$ be such that $B_{r_0}(a) \subset A$ and $d = \sup_{x \in A}\|x - a\|$. Then

$$d \leqslant \max\left\{2r_0, r_0 + \delta^{-1}\left(\frac{r_0}{2}\right)\right\},$$

where δ^{-1} is the inverse function for modulus δ.

Proof. Suppose that $d > 2r_0$. We shall prove that $d \leqslant r_0 + \delta^{-1}(\frac{r_0}{2})$.

Let $b \in A$ and $\|a - b\| = d$. Let L be any 2-dimensional affine plane which contains points a, b. Let $\mathcal{L} = (a - b)^{\perp}$, $\dim\mathcal{L} = n - 1$.

Our further consideration will take place on the plane L. Let the line l be orthogonal to the line $\operatorname{aff}\{a, b\}$ and $a \in l$. Let $\{x, y\} = l \cap \partial B_{r_0}(a)$. From the triangle xab we have $\|x - b\| \geqslant d - r_0$, from the triangle yab we have $\|y - b\| \geqslant d - r_0$.

Let $z = \frac{a+b}{2}$ and let the line l_1 be parallel to the line l and $z \in l_1$. Let $x_1 = l_1 \cap \operatorname{aff}\{x, b\}$, $y_1 = l_1 \cap \operatorname{aff}\{y, b\}$. By the uniform convexity of the set A we obtain that

$$\left[z, z + \frac{x_1 - z}{\|x_1 - z\|}\left(\frac{r_0}{2} + \delta\bigl(\|x - b\|\bigr)\right)\right] \cup \left[z, z + \frac{y_1 - z}{\|y_1 - z\|}\left(\frac{r_0}{2} + \delta\bigl(\|y - b\|\bigr)\right)\right] \subset A,$$

hence

$$\left[z, z + \frac{y_1 - z}{\|y_1 - z\|}\left(\frac{r_0}{2} + \delta(d - r_0)\right)\right] \cup \left[z, z + \frac{x_1 - z}{\|x_1 - z\|}\left(\frac{r_0}{2} + \delta(d - r_0)\right)\right] \subset A.$$

If $R = \frac{r_0}{2} + \delta(d - r_0) > r_0$ then (due to the previous inclusion being valid for any 2-dimensional plane L with $\{a, b\} \subset L$) we have

$$B_R(z) \cap (\mathcal{L} + z) \subset A$$

and thus

$$\operatorname{co}\bigl(B_{r_0}(a) \cup \bigl(B_R(z) \cap (\mathcal{L} + z)\bigr)\bigr) \subset A.$$

By the last inclusion and by the inequality $\|a - z\| > r_0$ we obtain that a shift of the ball $B_{r_0}(a)$ on a small distance in the direction $b - a$ occurs in the interior of the set A. Hence r_0 is not the maximal radius of balls from A. This contradiction shows that $\frac{r_0}{2} + \delta(d - r_0) \leqslant r_0$. □

Now we describe an algorithm for finding the convex hull of a positively uniform function [9,10].

Suppose that $\mathbb{G}$ is a grid with step $\Delta \in (0, \frac{1}{2})$, $f(p)$ is a positively uniform continuous function and $\tilde{A} = \{x \mid (p, x) \leqslant \mathcal{C}f(p),\ \forall p\}$. We wish to calculate $\operatorname{co}\mathcal{C}f(p)$ for all $p \in \mathbb{G}$. In other words, we wish to find $\mathcal{C}\operatorname{co}\mathcal{C}s(p, \tilde{A})$. The problem can be solved as a collection of problems of linear programming: for all $q \in \mathbb{G}$ to find

$$(q, x) \to \max (p, x) \leqslant \mathcal{C}f(p), \quad \forall p \in \mathbb{G}.$$

We shall describe an approximate algorithm from [7], [9, Theorem 2.6.3], [10] and discuss its error for the case of uniformly convex set $A = \{x \mid (p, x) \leqslant \operatorname{co} f(p),\ \forall p \in \mathbb{R}^n\}$ with modulus of convexity δ.

Suppose that $B_{r_0}(a) \subset A$ is the ball of maximum radius in the set A and $d = \sup_{x \in A} \|x - a\|$.

We often do not know the precise values of a, r_0, but we can easily calculate the ball of maximum radius $B_R(b)$, $r_0 \leqslant R$, from $\tilde{A}$: it suffices to solve the following problem of linear programming

$$R \to \max (p, b) + R \leqslant f(p), \quad \forall p \in \mathbb{G}.$$

The solution $(b, R) \in \mathbb{R}^n \times \mathbb{R}$ gives the center of the ball and its radius. In this case $B_R(b) \subset \tilde{A}$ and $\tilde{A} \subset b + (d + \frac{4d^2}{r_0}\Delta)B_1(0)$, see [9, Corollary 2.6.2]. Thus

$$(p, b) + R\|p\| \leqslant f(p) \leqslant (p, b) + \left(d + \frac{4d^2}{r_0}\Delta\right)\|p\|, \quad \forall p \in \mathbb{R}^n. \tag{4.15}$$

We shall further assume that $b = 0$.

The first step of the approximate algorithm is to calculate for all $q \in \mathbb{G}$ the values

$$s^\circ(q) = \max\left\{\frac{(p, q)}{f(p)} \;\middle|\; p \in \mathbb{G}\right\}.$$

The second step is to define $z(q) = q/s^\circ(q)$, $\forall q \in \mathbb{G}$. Then the polyhedron

$$A_1 = \operatorname{co} \bigcup_{q \in \mathbb{G}} z(q)$$

is an approximation of A. The approximate value for $\operatorname{co}\mathcal{C}f(p)$, $p \in \mathbb{G}$, is

$$\max\big\{\big(p, z(q)\big) \;\big|\; q \in \mathbb{G}\big\}.$$

By [9, Theorem 2.6.3] we have under the assumption (4.15) that $A_1 \subset A$,

$$h(A, A_1) \leqslant \frac{2(d + \frac{4d^2}{r_0}\Delta)^2}{R}\Delta, \tag{4.16}$$

and

$$0 \leqslant \operatorname{co}\mathcal{C}f(p) - \max\big\{\big(p, z(q)\big) \;\big|\; q \in \mathbb{G}\big\} \leqslant \frac{2(d + \frac{4d^2}{r_0}\Delta)^2}{R}\Delta, \quad \forall p \in \mathbb{G}.$$

So the error of the algorithm is proportional to the step Δ and to the value $\frac{1}{r_0}$ in the general case.

Consider the case when the set A has modulus of convexity of the second order: $\delta(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2)$, $\varepsilon \to +0$. Then under assumption $\Delta \in (0, \sqrt{r_0})$ we obtain by Theorem 4.1 that

$$\frac{2(d + \frac{4d^2}{r_0}\Delta)^2}{R} \leqslant \frac{2(d + \frac{4d^2}{r_0}\Delta)^2}{r_0} \leqslant \text{Const.},$$

and the error of the algorithm does not depend on the radius R of an interior ball.

5. Epilogue

1. By Theorem 4.1 we can estimate the value $\frac{d}{r_0}$ in the theorems from Section 3. For example, in Theorem 3.1 in the case when $B_{r_0}(a)$ is the ball of maximum radius from $B \stackrel{*}{-} A$ and $d = \sup_{x \in B \stackrel{*}{-} A} \|x - a\|$, we have (for small $r_0 > 0$)

$$\frac{d}{r_0} \leqslant \frac{r_0 + \delta_B^{-1}(\frac{r_0}{2})}{r_0}.$$

If the modulus δ_B has the second order at zero then $\frac{d}{r_0} \asymp \frac{1}{\sqrt{r_0}}$, $r_0 \to +0$.

2. We want to point out that if the sets A, B are uniformly convex with moduli δ_A, δ_B respectively and $\hat{A}$, $\hat{B}$ are polyhedral approximations of A and B on a grid $\mathbb{G}$ with the step Δ, then

$$h(\hat{A}+\hat{B}, \widehat{A+B}) \leqslant \frac{8}{7}\varepsilon_{A+B}(\Delta)\Delta$$

and in general in spaces of 3 or more dimensions $\hat{A}+\hat{B} \subset \widehat{A+B}$, but $\hat{A}+\hat{B} \neq \widehat{A+B}$. So the sum of approximations does not equal the approximation of sum.

3. The results can easily be reformulated in any finite-dimensional Banach space. The only obstacle for the proofs is in Lemma 2.1 when we estimate $\|\hat{p}\| \geqslant 1 - \frac{1}{2}\Delta^2$. One must demand from the space and the grid that

$$C = \inf_{\|p\|=1} \left\| \sum_{i \in I_p} \hat{\alpha}_i p_i \right\| \in (0, 1).$$

Then one must replace denominator $4 - \Delta^2$ by the new $2 + 2C$ and coefficient $\frac{8}{7}$ by the new $\frac{1}{C}$ in all theorems.

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