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ON LINEAR REALIZATIONS AND LOCAL SELF-SIMILARITY OF THE UNIVERSAL ZARICHNYI MAP

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Abstract. Answering a question of M.Zarichnyi we show that the universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$ is not locally self-similar. We also characterize linear operators homeomorphic to μ and on this base give a simple construction of the universal Zarichnyi map μ.

In this paper we investigate the properties of the universal map $\mu : \mathbb{R}^\infty \to Q^\infty$ constructed by M.Zarichnyi in [15] and subsequently studied in [16] and [17]. Answering a question from [17] we prove that the map μ is not locally self-similar. We also characterize linear operators homeomorphic to μ and on this base give a simple construction of the universal Zarichnyi map μ.

1. Strongly universal maps

All topological spaces considered in this paper are Tychonov, all compact spaces are metrizable, and all maps are continuous; $\omega = \{0, 1, 2, \dots\}$ stands for the set of all finite ordinals.

Given a class $\mathcal{C}$ of compacta, we denote by $\mathcal{C}^\infty$ the class of topological spaces X admitting a countable cover $\mathcal{U}$ by subsets of the class $\mathcal{C}$, generating the topology of X in the sense that a subset $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K for every $K \in \mathcal{U}$. In our subsequent considerations $\mathcal{C} = \mathcal{K}$ or $\mathcal{K}_{fd}$, where $\mathcal{K}$ ($\mathcal{K}_{fd}$) is the class of all (finite-dimensional) metrizable compacta. In this case $\mathcal{C}^\infty$ can be equivalently defined as the class of the direct limits of towers consisting of compacta in $\mathcal{C}$.

Given a space X with a fixed point $*$ let X^∞ denote the set

$$X^\omega_f = \{(x_i)_{i\in\omega} \in X^\omega : x_i = * \text{ for almost all } i\}$$

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endowed with the strongest topology inducing the product topology on each space $X^n = \{(x_i)_{i\in\omega} \in X : x_i = * \text{ for all } i \geq n\}$, $n \in \omega$, that is, X^∞ is the direct limit of the tower $X^1 \subset X^2 \subset \dots$.

If the space X is topologically homogeneous (like the real line $\mathbb{R}$ or the Hilbert cube $Q = [0,1]^\omega$), then the topology of the space X^∞ does not depend on the particular choice of a fixed point $* \in X$.

Among the spaces X^∞ the spaces $\mathbb{R}^\infty$ and Q^∞ occupy the special place: they are universal for the classes $\mathcal{K}^\infty_{fd}$ and $\mathcal{K}^\infty$ in the sense that each space from the class $\mathcal{K}^\infty$ (resp. $\mathcal{K}^\infty_{fd}$) is homeomorphic to a closed subspace of Q^∞ (resp. $\mathbb{R}^\infty$). Topological copies of the spaces $\mathbb{R}^\infty$ and Q^∞ very often appear in topological algebra and functional analysis, see [2]—[7], [12]—[14]. In particular, every infinite-dimensional linear topological space $X \in \mathcal{K}^\infty_{fd}$ is homeomorphic to $\mathbb{R}^\infty$ [2] while each locally convex space $Y \in \mathcal{K}^\infty$ with uncountable Hamel basis is homeomorphic to Q^∞ [3].

A topological characterization of the spaces $\mathbb{R}^\infty$ and Q^∞ was given by K.Sakai [11]: *Up to a homeomorphism, $\mathbb{R}^\infty$ (resp. Q^∞) is the unique strongly $\mathcal{K}_{fd}$-universal (resp. strongly $\mathcal{K}$-universal) space in the class $\mathcal{K}^\infty_{fd}$ (resp. $\mathcal{K}^\infty$).*

A topological space X is defined to be *strongly $\mathcal{C}$-universal* if every embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{C}$ can be extended to an embedding $\bar{f} : A \to X$. Replacing the words "embedding" by "map" we obtain the definition of an *absolute extensor* for the class $\mathcal{C}$ (briefly, AE($\mathcal{C}$)).

In [15] the notion of the strong universality was generalized to maps. A map $\pi : X \to Y$ between topological spaces is defined to be *strongly $\mathcal{C}$-universal* if for every embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{C}$ and a map $g : A \to Y$ with $\pi \circ f = g|B$ there exists an embedding $\bar{f} : A \to X$ such that $\bar{f}|B = f$ and $\pi \circ \bar{f} = g$. Replacing the words "embedding" by "map" we obtain the definition of a *$\mathcal{C}$-soft* map. Observe that a space X is strongly $\mathcal{C}$-universal (resp. is an AE($\mathcal{C}$)) if and only if the constant map $X \to \{*\}$ is strongly $\mathcal{C}$-universal (resp. $\mathcal{C}$-soft).

The following theorem was proved in [1].

Uniqueness Theorem 1.1. *If $\pi : X \to Y$ and $\pi' : X' \to Y$ are strongly $\mathcal{C}$-universal maps from spaces $X, X' \in \mathcal{C}^\infty$, then there is a homeomorphism $h : X \to X'$ such that $\pi' \circ h = \pi$.*

In case of the one-point space Y we obtain the Uniqueness Theorem for strongly $\mathcal{C}$-universal spaces (see [11]): *Any two strongly $\mathcal{C}$-universal spaces $X, X' \in \mathcal{C}^\infty$ are homeomorphic.*

Thus (up to a homeomorphism) there is at most one strongly $\mathcal{K}_{fd}$-universal map from a space $X \in \mathcal{K}_{fd}^{\infty}$ onto a given space Y. For which spaces Y does such a map exist? If $Y \in \mathcal{K}_{fd}^{\infty}$, then the answer is easy: just consider the projection $\pi : Y \times \mathbb{R}^{\infty} \to Y$. If $Y \notin \mathcal{K}_{fd}^{\infty}$ (for example, if $Y = Q^{\infty}$) the situation is not so obvious. Nonetheless, applying certain non-trivial results of A.Dranishnikov [8], M.Zarichnyi has constructed in [15] a strongly $\mathcal{K}_{fd}$-universal map $\mu : \mathbb{R}^{\infty} \to Q^{\infty}$.

Afterwards, he proved that this map μ is homeomorphic to a group homomorphism [17] and to an affine map between suitable spaces of probability measures [16], thus giving an alternative and simpler constructions of the map μ. The Zarichnyi map $\mu : \mathbb{R}^{\infty} \to Q^{\infty}$ contains any map $f : A \to B$ from a finite-dimensional metrizable compactum A into a metrizable compactum B in the sense that there are two embeddings $e_A : A \to \mathbb{R}^{\infty}$ and $e_B : B \to Q^{\infty}$ such that $\mu \circ e_A = e_B \circ f$.

2. Characterizing linear operators homeomorphic to the map μ

We define two maps $\pi : X \to Y$ and $\pi' : X' \to Y'$ to be *homeomorphic* if $\pi' \circ h = H \circ \pi$ for some homeomorphisms $h : X \to X'$ and $H : Y \to Y'$. In this section we characterize linear operators homeomorphic to the universal Zarichnyi map and classify $\mathcal{K}_{fd}$-invertible linear operators from a linear topological space $X \in \mathcal{K}_{fd}^{\infty}$ onto a locally convex space $Y \in \mathcal{K}^{\infty}$. By a "linear operator" we understand a linear continuous operator between linear topological spaces.

We define a map $\pi : X \to Y$ to be $\mathcal{C}$*-invertible* if for every map $g : A \to Y$ of a space $A \in \mathcal{C}$ there is a map $f : A \to X$ such that $\pi \circ f = g$. It is clear that each strongly $\mathcal{C}$-universal or $\mathcal{C}$-soft map is $\mathcal{C}$-invertible. In [17, 2.1] M.Zarichnyi proved that a continuous group homomorphism $h : G \to H$ is $\mathcal{K}_{fd}$-soft if and only if h is $\mathcal{K}_{fd}$-invertible and its kernel Ker$h$ is an AE($\mathcal{K}_{fd}$). Since each linear topological space is an AE($\mathcal{K}_{fd}$) (see, e.g. [2]), we get

Theorem 2.1. *A linear operator between linear topological spaces is $\mathcal{K}_{fd}$-soft if and only if it is $\mathcal{K}_{fd}$-invertible.*

Next, we find conditions under which a given linear operator is strongly $\mathcal{K}_{fd}$-universal.

Theorem 2.2. *A linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}_{fd}^{\infty}$ to a linear topological space Y is strongly $\mathcal{K}_{fd}$-universal if and only if the operator T is $\mathcal{K}_{fd}$-invertible and has infinite-dimensional kernel.*

In the proof we will exploit two lemmas.

Lemma 2.3. *If $X \in \mathcal{K}_{fd}^{\infty}$ is an infinite-dimensional linear topological space, then for every compactum $K \subset X$ there is a non-zero point $x \in X$ such that $([-1,1] \cdot x) \cap K \subset \{0\}$.*

Proof. Replacing K by $[-1,1] \cdot K$, if necessary, we may assume that $K = [-1,1] \cdot K$. Since K is a compact subset of the space $X \in \mathcal{K}_{fd}^{\infty}$, $\dim K < n$ for some $n \in \mathbb{N}$. The linear space X, being infinite-dimensional, contains an n-dimensional linear space $\mathbb{R}^n$. We claim that there is a point x on the unit sphere S of $\mathbb{R}^n$ such that $([-1,1] \cdot x) \cap K = \{0\}$. Assuming the converse, for every $x \in S$ let $k(x) \in \mathbb{N}$ be the smallest number such that $([-1,1] \cdot x) \cap K \supset [0, \frac{1}{k(x)}] \cdot x$. It can be shown that for every $k \in \mathbb{N}$ the set $S_k = \{x \in S : k(x) \leq k\}$ is closed in S. Since $S = \bigcup_{k \in \mathbb{N}} S_k$, the Baire Theorem guarantees that S_k has non-empty interior in S for some k. Then $\dim S_k = n - 1$. Since $K \supset [0, 1/k] \cdot S_k$ and $n > \dim K \geq \dim([0,1/k] \cdot S_k) = n$, we get a contradiction. $\square$

Lemma 2.4. *If $T : X \to Y$ is a linear operator with infinite-dimensional kernel from a linear topological space $X \in \mathcal{K}_{fd}^{\infty}$, then for every compact subset $C \subset X$ there exists an embedding $e : C \times [0,1] \to X$ such that $e(c,0) = c$ and $T \circ e(c,t) = T(c)$ for all $c \in C$ and $t \in [0,1]$.*

Proof. Let $L = \mathrm{Ker} T$ and $K = L \cap (C - C)$. By Lemma 2.3, there exists a non-zero point $x_0 \in L$ such that $([-1,1] \cdot x_0) \cap K \subset \{0\}$. Define the map $e : C \times [0,1] \to X$ letting $e(c,t) = c + tx_0$ for $(c,t) \in C \times [0,1]$. It is clear that $e(c,0) = c$ and $T \circ e(c,t) = T(c)$ for every $c \in C$ and $t \in [0,1]$. To show that the map e is injective, fix two points $(c,t), (c',t') \in C \times [0,1]$ with $e(c,t) = e(c',t')$. Then $c - c' = (t' - t)x_0$ and $T(c - c') = 0$ which implies $c - c' \in K$. Since $K \ni c - c' = (t' - t)x_0 \in [-1,1]x_0$ and $K \cap ([-1,1]x_0) \subset \{0\}$, we get $c - c' = 0$ and $t' - t = 0$, i.e., $(c,t) = (c',t')$. $\square$

Proof. (of Theorem 2.2) The "only if" part of Theorem 2.2 is trivial. To prove the "if" part, assume that $T : X \to Y$ is an $\mathcal{K}_{fd}$-invertible operator, $X \in \mathcal{K}_{fd}^{\infty}$, and $\dim \mathrm{Ker} T = \infty$. To prove the strong $\mathcal{K}_{fd}$-universality of the map T, fix an embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \to Y$ such that $T \circ f = g|B$. By Theorem 2.1, the operator T is $\mathcal{K}_{fd}$-soft and hence there is a map $\tilde{f} : A \to X$ such that $\tilde{f}|B = f$ and $T \circ \tilde{f} = g$.

Denote by A/B the quotient space and let $q : A \to A/B$ be the quotient map. It is clear that the space A/B is finite-dimensional and thus admits an embedding $i : A/B \to [0,1]^n$ for some $n \in \mathbb{N}$ such that $i(\{B\}) = 0^n$ where $0^n = (0, \dots, 0) \in [0,1]^n$. Applying Lemma 2.4 n times, construct an embedding $e : \tilde{f}(A) \times [0,1]^n \to X$ such that $e(x, 0^n) = x$ and $T \circ e(x,t) = T(x)$ for any

$x \in \tilde{f}(A)$ and $t \in [0,1]^n$. It is easy to verify that the map $\bar{f} : A \to X$ defined by $\bar{f}(a) = e(\tilde{f}(a), i \circ q(a))$ for $a \in A$ is an embedding satisfying the conditions $\bar{f}|B = f$ and $T \circ \bar{f} = g$. □

We apply Theorem 2.2 to prove the following theorem characterizing linear operators homeomorphic to the universal Zarichnyi map μ.

Theorem 2.5. *A linear operator $T : X \to Y$ between linear topological spaces is homeomorphic to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map μ if and only if $X \in \mathcal{K}_{fd}^\infty$, Y is homeomorphic to Q^∞, and the operator T is $\mathcal{K}_{fd}$-invertible.*

Proof. This theorem will follow from Theorem 2.2 and Uniqueness Theorem for strongly $\mathcal{K}_{fd}$-universal map as soon as we prove that each $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a linear topological space Y containing a Hilbert cube has infinite-dimensional kernel. Assume to the contrary that Ker$T$ is finite-dimensional. By Theorem 2.1, the operator T, being $\mathcal{K}_{fd}$-invertible, is $\mathcal{K}_{fd}$-soft. Fix a copy $Q \subset Y$ of the Hilbert cube in Y and an open surjective map $g : A \to Q$ of a finite-dimensional compactum A onto Q (such a map exists according to [8]). Since the operator T is $\mathcal{K}_{fd}$-invertible, there is a map $f : A \to X$ such that $T \circ f = g$. Let $B \subset \mathrm{Ker}T$ be any compact neighborhood of the origin in the finite-dimensional linear space Ker$T$. Next, consider the compact set $K = f(A) + B \subset X$. Since $X \in \mathcal{K}_{fd}^\infty$, $\dim K < n$ for some $n \in \mathbb{N}$. Let $e : I^n \to Q \subset Y$ be any embedding of the n-dimensional cube I^n into Q. By the $\mathcal{K}_{fd}$-softness of the map T, there is a map $i : I^n \to X$ such that $T \circ i = e$ and $i(0^n) \in f(A)$. It is clear that i is an embedding.

We claim that $K \cap i(I^n)$ is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$. Assuming that it is not true, we would find a sequence $(x_k)_{k=1}^\infty \in i(I^n) \setminus K$ tending to x_0. Then the sequence $(T(x_k))_{k=1}^\infty$ converges to $T(x_0)$. Let $a_0 \in A$ be any point with $f(a_0) = x_0$. Since the map $g : A \to Q$ is open and $g(a_0) = T \circ f(a_0) = T(x_0) = \lim_{k\to\infty} T(x_k)$, there exists a sequence $(a_k)_{k=1}^\infty \subset A$ tending to a_0 such that $g(a_k) = T(x_k)$ for each n. Then the sequence $(f(a_k))_{k=1}^\infty$ converges to $f(a_0) = x_0$ and has the property: $T \circ f(a_k) = g(a_k) = T(x_k)$ for every k. Hence $x_k - f(a_k) \in \mathrm{Ker}T$ for every k. Since $\lim_{k\to\infty} x_k = x_0 = \lim_{k\to\infty} f(a_k)$, we get $\lim_{k\to\infty}(x_k - f(a_k)) = 0$ and thus $x_m - f(a_m) \in B$ for some m. Then $x_m \in B + f(a_m) \subset B + f(A) = K$, a contradiction with the choice of the sequence $(x_k)_{k=1}^\infty$. Thus K is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$ what is not possible since $\dim K < n = \dim(U)$ for any neighborhood $U \subset i(I^n)$ of $i(0^n)$. This contradiction shows that the kernel of T is infinite-dimensional. □

We recall that a topological space Y is called a *k-space* if a subset $F \subset Y$ is closed in Y if and only if for every compact subset $K \subset Y$ the intersection $F \cap K$ is closed in K.

Theorem 2.6. *A $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}^\infty_{fd}$ onto a locally convex k-space Y is homeomorphic either to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for some $n, m \in \omega \cup \{\infty\}$ or to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$. The latter case occurs if and only if the space Y has uncountable Hamel basis.*

Proof. First, we show that $Y \in \mathcal{K}^\infty$. Since $X \in \mathcal{K}^\infty_{fd}$, the space X contains a countable collection $\{X_n : n \in \omega\}$ of compact subsets, fundamental in the sense that every compact subset of X lies in some X_n. By [9, 3.1.22], for every n the image $T(X_n)$ of the compact set X_n is a metrizable compactum. The operator T, being $\mathcal{K}_{fd}$-invertible, is surjective. Then $Y = \bigcup_{n\in\omega} T(X_n)$. We claim that the collection $\{T(X_n) : n \in \omega\}$ generates the topology of Y, i.e., $Y \in \mathcal{K}^\infty$.

Assuming the converse, we could find a non-closed subset $F \subset Y$ such that $F \cap T(X_n)$ is closed in $T(X_n)$ for every n. Since Y is a k-space, there is a compactum $K \subset Y$ such that $F \cap K$ is not closed in K. The compactum $K = \bigcup_{n\in\omega} K \cap T(X_n)$, being the countable union of metrizable compacta, is metrizable, [9, 3.1.20]. Consequently, there is a sequence $(y_n)_{n=1}^\infty \subset F \cap K$ converging to a point $y_0 \in K \setminus F$. Since the map T is $\mathcal{K}_{fd}$-invertible, there is a sequence $(x_n)_{n=1}^\infty \subset X$ converging to a point $x_0 \in X$ such that $T(x_n) = y_n$ for all $n \in \omega$. The subset $\{x_n : n \in \omega\} \subset X$, being compact, lies in the compactum X_m for some m. Then $\{y_n : n > 0\} \subset T(X_m) \cap F$. Since the intersection $T(X_m) \cap F$ is closed in $T(X_m)$, we get $y_0 = \lim_{n\to\infty} y_n \in T(X_m) \cap F$, a contradiction with $y_0 \notin F$.

Thus the locally convex space Y belongs to the class $\mathcal{K}^\infty$. By [3], Y is homeomorphic either to Q^∞ or to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$. Moreover, the last case occurs if and only if the algebraic dimension of Y is at most countable. If Y is homeomorphic to Q^∞, then by Theorem 2.5, the operator T is homeomorphic to the universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$.

If Y is homeomorphic to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$, then the algebraic dimension of Y is at most countable and Y carries the strongest linear topology, see [2]. In this case there is a linear continuous operator $S : Y \to X$ such that $T \circ S = \mathrm{id}$ and the map $h : X \to Y \times \mathrm{Ker}T$ defined by $h(x) = (T(x), x - S \circ T(x))$ for $x \in X$ is a linear homeomorphism (with inverse $h^{-1}(y, l) = S(y) + l$) such that $\mathrm{pr} \circ h = T$, where $\mathrm{pr} : Y \times \mathrm{Ker}T \to Y$ is the projection. Hence the operator T is homeomorphic to the projection $\mathrm{pr} : Y \times \mathrm{Ker}T \to Y$. Since $\mathrm{Ker}T$ is a linear topological space from the class $\mathcal{K}_{fd}^{\infty}$ we can apply [2] to conclude that Ker$T$ is homeomorphic to $\mathbb{R}^m$ for some $m \in \omega \cup \{\infty\}$. Therefore T is homeomorphic to the projection $\mathrm{pr} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. □

Next, we show that linear operators classified by Theorem 2.5 do exist.

Theorem 2.7. *For every linear topological space $Y \in \mathcal{K}^{\infty}$ there is a $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}_{fd}^{\infty}$.*

Proof. Since $Y \in \mathcal{K}^{\infty}$, the space Y possesses a countable family $\{F_n : n \in \omega\}$ of compact subsets such that every compact subset of Y lies in some F_n. For every $n \in \omega$ denote by $\mathcal{K}_n$ the subclass of $\mathcal{K}_{fd}$ consisting of all at most n-dimensional compacta. Using the Dranishnikov Theorem [8, Theorem 1.2], for every n we can find an $\mathcal{K}_n$-invertible map $f_n : K_n \to F_n$ of a finite-dimensional metrizable compactum K_n onto F_n. Now consider the discrete sum $K = \sqcup_{n\in\omega} K_n$ and the map $f = \sqcup_{n\in\omega} f_n : K \to Y$. It is clear that $K \in \mathcal{K}_{fd}^{\infty}$ and the map f is $\mathcal{K}_{fd}$-invertible. Let $L(K) \supset K$ be the free linear topological space over K (see [6, §2.4]) and $T : L(K) \to Y$ be the unique linear operator extending the $\mathcal{K}_{fd}$-invertible map f. It is clear that the operator T is $\mathcal{K}_{fd}$-invertible. By [2], $L(K) \in \mathcal{K}_{fd}^{\infty}$. □

Note that Theorems 2.6 and 2.7 allow us to give an alternative construction of the universal map of Zarichnyi.

Question 1. *Let $h : G \to H$ be a $\mathcal{K}_{fd}$-invertible continuous homomorphism between topological groups. Is h strongly $\mathcal{K}_{fd}$-universal if its kernel $h^{-1}(1)$ is strongly $\mathcal{K}_{fd}$-universal?*

3. The universal Zarichnyi map is not self-similar

We define a space X to be *locally self-similar* if every point of X has a basis of neighborhoods homeomorphic to X. It is well-known that the spaces $\mathbb{R}^{\infty}$ and Q^{∞} (like many other model spaces of infinite-dimensional topology) are locally self-similar.

In [17] M.Zarichnyi extended the notion of the local self-similarity onto maps and asked if the map $\mu : \mathbb{R}^{\infty} \to Q^{\infty}$ is locally self-similar. He defined a map $\pi : X \to Y$ to be *locally self-similar* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|V : V \to \pi(V)$ is homeomorphic to π. Observe that for locally self-similar spaces X, Y the projection $\mathrm{pr} : X \times Y \to Y$ is locally self-similar.

The following result shows that the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map is close to being locally self-similar.

Theorem 3.1. *If $T : X \to Y$ is a linear operator homeomorphic to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$, then for every nonempty open convex subset $U \subset X$ the map $T|U : U \to T(U)$ is homeomorphic to μ.*

Proof. Fix any non-empty open convex subset $U \subset X$. Since X is homeomorphic to $\mathbb{R}^\infty$, the space U, being a contractible $\mathbb{R}^\infty$-manifold, is homeomorphic to $\mathbb{R}^\infty$ according to the Classification Theorem for $\mathbb{R}^\infty$-manifolds, see [11].

The map T, being strongly $\mathcal{K}_{fd}$-universal, is open. Consequently, the set $T(U)$ is open in Y and thus is homeomorphic to Q^∞, being open contractible subspace of Y, the topological copy of Q^∞. By Uniqueness Theorem, to show that the map $T|U : U \to T(U)$ is homeomorphic to μ it suffices to verify that $T|U$ is a strongly $\mathcal{K}_{fd}$-universal map.

First we show that the map $T|U : U \to T(U)$ is $\mathcal{K}_{fd}$-invertible. Fix any map $g : A \to T(U)$ from a compactum $A \in \mathcal{K}_{fd}$. Since the map T is $\mathcal{K}_{fd}$-invertible, there is a map $f : A \to X$ such that $T \circ f = g$. For every point $x \in A$ find a point $a_x \in \mathrm{Ker}T$ such that $f(x) + a_x \in U$. Next, let $U(x) = f^{-1}(U - a_x)$. Using the compactness of A find a finite subcover $\{U(x_1), \dots, U(x_n)\}$ of the open cover $\{U(x) : x \in A\}$ and let $\{\lambda_i : A \to [0,1]\}_{i=1}^n$ be a partition of unity such that $\lambda_i^{-1}(0,1] \subset U(x_i)$ for every $i \le n$. Consider the map $\alpha : A \to \mathrm{Ker}T$ defined by $\alpha(x) = \sum_{i=1}^n \lambda_i(x) a_{x_i}$ for $x \in K$. It can be shown that the map $h = f + \alpha : A \to X$ has the properties: $T \circ h = T \circ f = g$ and $h(A) \subset U$. Therefore, the map $T|U$ is $\mathcal{K}_{fd}$-invertible.

To show that it is strongly $\mathcal{K}_{fd}$-universal, fix an embedding $f : B \to U$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \to T(U)$ such that $T \circ f = g$. Since the map $T|U : U \to T(U)$ is $\mathcal{K}_{fd}$-invertible, there is a map $h_1 : K \to U$ such that $T \circ h_1 = g$. Next, using the strong $\mathcal{K}_{fd}$-universality of the operator T, find an embedding $h_2 : A \to X$ such that $h_2|B = f$ and $T \circ h_2 = g$. Let $\lambda : A \to [0,1]$ be a continuous map such that $\lambda(B) = \{0\}$ and $\lambda(A \setminus W) = \{1\}$, where $W = h_2^{-1}(U)$. Next, consider the map $h : A \to X$ defined by $h(a) = \lambda(a)h_1(a) + (1 - \lambda(a))h_2(a)$ for $a \in A$. It is easy to see that $T \circ h = g$ and $h(A) \subset U$.

Let $q : A \to A/B$ be the quotient map. Since the quotient space A/B is finite-dimensional, there is an embedding $i : A/B \to [0,1]^n$ for some $n \in \omega$ such that $i(\{B\}) = 0^n$. The space $h(A)$ belongs to the class $\mathcal{K}_{fd}$, being a compact subset of the space $X \in \mathcal{K}_{fd}^\infty$. Then by the strong $\mathcal{K}_{fd}$-universality of the operator T, there is an embedding $e : h(A) \times [0,1]^n \to X$ such that $e(x, 0^n) = x$ and $T \circ e(x,t) = T(x)$ for every $x \in h(A)$ and $t \in [0,1]^n$. Since $h(A) \subset U$, there exists $\varepsilon > 0$ such that $e(h(A) \times [0,\varepsilon]^n) \subset U$. Finally consider the map $\bar{f} : A \to X$ defined by $\bar{f}(a) = e(h(a), \varepsilon \cdot i \circ q(a))$ for $a \in A$. It can be easily shown that $\bar{f}|B = f$,

$\bar{f}(A) \subset U$ and $T \circ \bar{f} = g$. Therefore, the map $T|U$ is strongly $\mathcal{K}_{fd}$-universal. The space U, being an open subset of the space $X \in \mathcal{K}_{fd}^{\infty}$, belongs to the class $\mathcal{K}_{fd}^{\infty}$. If the space $T(U)$ is homeomorphic to Y, then by Uniqueness Theorem the maps $T|U$ and T are homeomorphic. □

Thus the local self-similarity of the Zarichnyi map μ would be proven if we could find a linear operator between locally convex spaces, homeomorphic to μ. Unfortunately, no such operators exist. This is because each locally convex space $X \in \mathcal{K}_{fd}^{\infty}$ has at most countable Hamel basis, see [2]. Consequently, each linear image of X also has at most countable Hamel basis and thus cannot be homeomorphic to Q^{∞}. But this is not a unique reason why we cannot find a linear operator between locally convex spaces, homeomorphic to the universal Zarichnyi map μ.

Theorem 3.2. *Each open $\mathcal{K}_{fd}$-invertible map $f : \mathbb{R}^{\infty} \to Q^{\infty}$ is not locally self-similar.*

Let us define a map $\pi : X \to Y$ to be *locally $\mathcal{K}_{fd}$-invertible* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|V : V \to \pi(V)$ is $\mathcal{K}_{fd}$-invertible. It is clear that each locally self-similar $\mathcal{K}_{fd}$-invertible map is locally $\mathcal{K}_{fd}$-invertible.

We define a space X to be *almost finite-dimensional* if there is $n \in \mathbb{N}$ such that $\dim F \le n$ for every finite-dimensional closed subset F of X. Let us note that there exist infinite-dimensional almost finite-dimensional compact spaces, see [10, 5.2.23].

Theorem 3.2 will be derived from

Lemma 3.3. *Each compactum K admitting an open surjective locally $\mathcal{K}_{fd}$-invertible map $f : X \to K$ from a space $X \in \mathcal{K}_{fd}^{\infty}$, is almost finite-dimensional.*

To prove this lemma we need

Lemma 3.4. *For every compact space K and a closed subset $A \subset K \times Q$ with $\dim A < \dim K$ there is a closed subset $F \subset K \times Q$ such that $F \cap A = \emptyset$ but $F \cap s(K) \neq \emptyset$ for every section $s : K \to K \times Q$ of the projection* $\mathrm{pr} : K \times Q \to K$.

Proof. Find any $n \in \omega$ with $\dim A \le n < \dim K$. By Hurewicz-Wallman Theorem [10, 1.9.3], there exists a map $f : L \to S^n$ from a closed subset L of K into the n-dimensional sphere which has no continuous extension $\bar{f} : K \to S^n$.

Let $B = L \times Q$. Since $\dim A \le n$, we can apply the Hurewicz-Wallman Theorem to find a continuous map $p : A \to S^n$ such that $p|A \cap B = f \circ pr|A \cap B$. Next, since S^n is an ANR, there is a continuous map $\bar{p} : V \to S^n$ defined on an open neighborhood V of the closed set $A \cup B$ in $K \times Q$ such that $\bar{p}|A = p$ and $\bar{p}|B = f \circ \mathrm{pr}|B$.

We claim that the closed set $F = (K \times Q) \setminus V$ misses A but meets the image $s(K)$ of any section $s : K \to K \times Q$ of the projection pr. Assuming the converse, we would find a section $s : K \to K \times Q$ of the projection pr such that $s(K) \cap F = \emptyset$. Then $s(K) \subset V$ and we can consider the map $g = \bar{p} \circ s : K \to S^n$. Observe that $s(L) \subset L \times Q = B$ and hence $g|L = f$, which contradicts to the choice of f. $\square$

Proof of Lemma 3.3. Suppose a compactum K admits an open surjective locally $\mathcal{K}_{fd}$-invertible map $\pi : X \to K$ of a space $X \in \mathcal{K}_{fd}^\infty$. Assume that the compactum K is not almost finite-dimensional. The compactum K, being a countable union of metrizable compacta, is metrizable. By the compactness of K there is a point $y \in K$ having no almost finite dimensional neighborhood in K. Fix a countable base $\{U_n : n \in \omega\}$ of neighborhoods of y in K and for every $n \in \omega$ find a finite dimensional compactum $K_n \subset U_n$ with $\dim K_n > n$.

Fix any point $x \in X$ with $\pi(x) = y$ and let $\{X_n : n \in \omega\}$ be an increasing collection of finite-dimensional compact subsets of X generating its topology. Without loss of generality, $x \in X_0$ and $\dim X_n \le n$ for every $n \in \omega$. The space X, having a countable network of the topology, admits an injective continuous map $i : X \to Q$ into the Hilbert cube. Now consider the injective map $e = (\pi, i) : X \to K \times Q$ defined by $e(x) = (\pi(x), i(x))$ for $x \in X$. For every $n \in \omega$ let $A_n = e(X_n) \cap (K_n \times Q)$. Since $\dim A_n \le \dim X_n \le n < \dim K_n$, we can apply Lemma 3.4 to find a closed subset $F_n \subset K_n \times Q$ such that $A_n \cap F_n = \emptyset$ but $F_n \cap s(K_n) \neq \emptyset$ for every section $s : K_n \to K_n \times Q$ of the projection $\mathrm{pr} : K_n \times Q \to K_n$.

Consider the set $F = \bigcup_{k\in\omega} e^{-1}(F_k)$. Since each set F_k is closed in $K \times Q$ and $X_n \cap F = X_n \cap (\bigcup_{k=0}^{n-1} e^{-1}(F_k))$ for each $n \in \omega$, we get that F is a closed subset of X. Next, since $X_0 \cap F = \emptyset$, the set $U = X \setminus F$ is an open neighborhood of the point x in X.

Let us show that for every open neighborhood $V \subset U$ of x the map $\pi|V : V \to \pi(V)$ is not $\mathcal{K}_{fd}$-invertible. Indeed, since the map π is open, $\pi(V)$ is an open neighborhood of the point $\pi(x) = y$ and thus $f(V) \supset K_n$ for some $n \in \omega$. Assuming that the map $\pi|V$ is $\mathcal{K}_{fd}$-invertible we would find a map $g : K_n \to V \subset U$ such that $\pi \circ g = \mathrm{id}$. Then the map $s = e \circ g : K_n \to K \times Q$ has the properties: $\mathrm{pr} \circ s = \mathrm{id}$ and $s(K_n) \cap F_n = \emptyset$, a contradiction with the choice of the set F_n. $\square$

Proof of Theorem 3.2. Assume that $f : \mathbb{R}^\infty \to Q^\infty$ is an open $\mathcal{K}_{fd}$-invertible locally self-similar map. Then f is locally $\mathcal{K}_{fd}$-invertible. Let $K \subset Q^\infty$ be a topological copy of the Hilbert cube and $X = f^{-1}(K)$. Then $X \in \mathcal{K}_{fd}^\infty$ and

$f|K : X \to K$ is an open surjective locally $\mathcal{K}_{fd}$-invertible map, a contradiction to Lemma 3.3. $\square$

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