Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 303 (2021) 645–666

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonvariational and singular double phase problems for the Baouendi-Grushin operator

Anouar Bahrouni [a], Vicenţiu D. Rădulescu [b,c,d,∗], Dušan D. Repovš [e]

[a] *Mathematics Department, University of Monastir, Faculty of Sciences, 5019 Monastir, Tunisia*
[b] *Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland*
[c] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*
[d] *'Simion Stoilow' Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania*
[e] *Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia*

Received 19 April 2021; revised 15 September 2021; accepted 21 September 2021

Abstract

In this paper we introduce a new double phase Baouendi-Grushin type operator with variable coefficients. We give basic properties of the corresponding functions space and prove a compactness result. In the second part, using topological argument, we prove the existence of weak solutions of some nonvariational problems in which this new operator is present. The present paper extends and complements some of our previous contributions related to double phase anisotropic variational integrals.

MSC: 35J70; 35P30; 76H05

Keywords: Baouendi-Grushin operator; Double phase problem; Singular term; Existence of solutions

∗ Corresponding author at: 'Simion Stoilow' Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.

E-mail addresses: bahrounianouar@yahoo.fr (A. Bahrouni), radulescu@inf.ucv.ro (V.D. Rădulescu), dusan.repovs@guest.arnes.si (D.D. Repovš).

https://doi.org/10.1016/j.jde.2021.09.033

1. Introduction

The present paper is motivated by recent fundamental enrichment to the mathematical analysis of nonlinear models with unbalanced growth. We mainly refer to the pioneering contributions of Marcellini [19,20] who studied lower semicontinuity and regularity properties of minimizers of certain quasiconvex integrals. Related problems are inspired by models arising in nonlinear elasticity and they describe the deformation of an elastic body, see Ball [1,2].

More precisely, we are concerned with the following nonlinear equations of double phase Baouendi-Grushin type

$$-\Delta_{G,a} u + |u|^{G(z)-2} u = K(z) f(u), \quad z \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $K \in C(\mathbb{R}^N)$, $f \in C(\mathbb{R})$, while $-\Delta_{G,a}$ stands for a new double phase Baouendi-Grushin type operator with variable exponents (see (1.2)).

The main aim of our work is to introduce a new double phase Baouendi-Grushin type operator with variable exponents and its suitable functions space. Our abstract results related to the new function space are motivated by the existence of solutions for nonvariational problems of type (1.1). The present paper complements our previous contributions related to double phase anisotropic variational integrals, see [3–6].

First, we recall the notion of Baouendi-Grushin operator with variable growth. Let $\Omega \subset \mathbb{R}^N$, $N > 1$, be a domain with smooth boundary $\partial\Omega$ and let n, m be nonnegative integers such that $N = n + m$. This means that $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ and so $z \in \Omega$ can be written as $z = (x, y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. In this paper $G : \overline{\Omega} \to (1, \infty)$ is supposed to be a continuous function and $\Delta_{G(x,y)}$ stands for the Baouendi-Grushin operator with variable coefficient, which is defined by

$$\begin{aligned}\Delta_{G(x,y)} u &= \operatorname{div}\left(\nabla_{G(x,y)} u\right)\\ &= \sum_{i=1}^{n} \left(|\nabla_x u|^{G(x,y)-2} u_{x_i}\right)_{x_i} + |x|^{\gamma} \sum_{i=1}^{m} \left(|\nabla_y u|^{G(x,y)-2} u_{y_i}\right)_{y_i},\end{aligned}$$

where

$$\nabla_{G(x,y)} u = \mathcal{A}(x) \begin{bmatrix} |\nabla_x u|^{G(x,y)-2} & \nabla_x u \\ |x|^{\gamma} |\nabla_y u|^{G(x,y)-2} & \nabla_y u \end{bmatrix}$$

and

$$\mathcal{A}(x) = \begin{bmatrix} I_n & 0_{n,m} \\ 0_{m,n} & |x|^{\gamma} I_m \end{bmatrix} \in \mathcal{M}_{N\times N}(\mathbb{R}),$$

with I_n being the identity matrix of size $n \times n$, $O_{n,m}$ is the zero matrix of size $n \times m$ and $\mathcal{M}_{N\times N}$ stands for the class of $(N \times N)$–matrices with real-valued entries. From the representation above it is clear that $\Delta_{G(x,y)}$ is degenerate along the m-dimensional subspace $M := \{0\} \times \mathbb{R}^m$ of $\mathbb{R}^N$.

The differential operator $\Delta_{G(x,y)}$ generalizes the degenerate operator

$$\frac{\partial^2}{\partial x^2}+x^{2r}\frac{\partial^2}{\partial y^2}\quad (r\in\mathbb{N})$$

introduced independently by Baouendi [8] and Grushin [16]. The Baouendi–Grushin operator can be viewed as the Tricomi operator for transonic flow restricted to subsonic regions. On the other hand, a second-order differential operator T in divergence form on the plane, can be written as an operator whose principal part is a Baouendi-Grushin-type operator, provided that the principal part of T is nonnegative and its quadratic form does not vanish at any point, see Franchi & Tesi [14]. For recent contributions to the study of double-phase problems we cite Beck & Mingione [9], Cencelj, Rădulescu & Repovš [10], Eleuteri, Marcellini & Mascolo [12], Papageorgiou, Rădulescu & Repovš [25–27], Pucci et al. [18,29], and Zhang & Rădulescu [35]. We refer to Marcellini [21] and Mingione & Rădulescu [22] for surveys of recent results on elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

Now, we are able to introduce the new Baouendi-Grushin type operator with variable coefficients, which is defined by

$$\begin{aligned}\Delta_{G,a}u&=\operatorname{div}\left(\nabla_{G(x,y)}u\right)\\&=\sum_{i=1}^{n}\left(|\nabla_x u|^{G(x,y)-2}u_{x_i}\right)_{x_i}+a(x)\sum_{i=1}^{m}\left(|\nabla_y u|^{G(x,y)-2}u_{y_i}\right)_{y_i}.\end{aligned}\tag{1.2}$$

The main goal of our recent paper [6] was to study a singular system in the whole space $\mathbb{R}^N$ in which the Baouendi-Grushin operator $(-\Delta_{G(x,y)})$ is present. So, the main difficulty is the lack of compactness corresponding to the whole Euclidean space. To overcome this difficulty, we proved a related compactness property. However, the interval of compactness is too short. So, we are not able to study a large number of equations driven by $-\Delta_{G(x,y)}$ in the whole space $\mathbb{R}^N$. For this reason and in order to get a better compactness result, we introduced the new operator $-\Delta_{G,a}$. Our abstract results are motivated by the existence of solutions of the following class of nonlinear equation

$$-\Delta_{G,a}u=-\operatorname{div}(\alpha_1 u\nabla_x r)-\operatorname{div}(\alpha_2 a^{\frac{1}{G(x,y)}}(x)u\nabla_y r)+f(z,u),\ \ z=(x,y)\in\mathbb{R}^N,\tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ is supposed to be a bounded domain. Another motivation comes from singular problems in the form

$$-\Delta_{G,a}u+|u|^{G(x,y)-2}u=\frac{b(x,y)}{u^{\sigma(x,y)}},\ \ (x,y)\in\mathbb{R}^N,\tag{1.4}$$

where $\sigma(\cdot)\in(0,1)$ and b is positive function.

The paper is organized as follows. In Section 2 we present the basic properties of variable Lebesgue space and introduce the main tools which will be used later. New properties concerning the new operator $(-\Delta_{G,a})$ will be discussed in Section 3. In Section 4, combining these abstract results with the topological argument, we study a nonvariational problem in which $-\Delta_{G,a}$ is present. In last section, we deal with purely singular double phase equation. We refer to the monograph by Papageorgiou, Rădulescu & Repovš [28] as a general reference for the abstract methods used in this paper.

2. Terminology and the abstract setting

In this section we recall some necessary definitions and properties of variable exponent spaces. We refer to the papers of Bahrouni & Repovš [7], Hájek, Montesinos Santalucía, Vanderwerff & Zizler [17], Musielak [23], Rădulescu [30,31], Rădulescu & Repovš [32] and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \;\middle|\; p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}$$

and define for any $p \in C_+(\overline{\Omega})$

$$p^+ := \sup_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- := \inf_{x\in\overline{\Omega}} p(x).$$

Then $1 < p^- \le p^+ < \infty$ for each $p \in C_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \inf\left\{ \mu > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1 \right\}.$$

If $\Omega = \mathbb{R}^N$, we denote $\|u\|_{p(\cdot),\Omega} = \|u\|_{p(\cdot)}$.

It is well known that $L^{p(\cdot)}(\Omega)$ is a reflexive Banach space.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left| \int_\Omega uv\,dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(.)} \|v\|_{q(.)}.$$

Also, if $p_j \in C_+(\overline{\Omega})$ $(j = 1, 2, \ldots, k)$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_k(x)} = 1,$$

then for all $u_j \in L^{p_j(x)}(\Omega)$ $(j = 1, \dots, k)$ we have

$$\left| \int_\Omega u_1 u_2 \cdots u_k \, dx \right| \le \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \cdots + \frac{1}{p_k^-} \right) |u_1|_{p_1(x)} |u_2|_{p_2(x)} \cdots |u_k|_{p_k(x)} . \tag{2.1}$$

Moreover, if $p_1 \le p_2$ in Ω and Ω has finite Lebesgue measure, then there exists the continuous embedding

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega). \tag{2.2}$$

The following two propositions will be useful in the sequel, see Rădulescu & Repovš [32, p. 11].

Proposition 2.1. *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)} \, dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

Then the following hold:

(i) $\|u\|_{p(\cdot),\Omega} < 1$ *(resp.,* $= 1; > 1$*) if and only if* $\rho_1(u) < 1$ *(resp.,* $= 1; > 1$*);*
(ii) $\|u\|_{p(\cdot),\Omega} > 1$ *implies* $\|u\|^{p^-}_{p(\cdot),\Omega} \le \rho_1(u) \le \|u\|^{p^+}_{p(\cdot),\Omega}$*;*
(iii) $\|u\|_{p(\cdot),\Omega} < 1$ *implies* $\|u\|^{p^+}_{p(\cdot),\Omega} \le \rho_1(u) \le \|u\|^{p^-}_{p(\cdot),\Omega}$*.*

Proposition 2.2. *Let*

$$\rho_1(u) = \int_\Omega |u|^{p(x)} \, dx \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

If $u, u_n \in L^{p(\cdot)}(\Omega)$ *and* $n \in \mathbb{N}$*, then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \|u_n - u\|_{p(\cdot),\Omega} = 0$*;*
(ii) $\lim_{n\to+\infty} \rho_1(u_n - u) = 0$*;*
(iii) $u_n(x) \to u(x)$ *in* Ω *and* $\lim_{n\to+\infty} \rho_1(u_n) = \rho_1(u)$*.*

In what follows, we recall Lemma A.1 of Giacomoni, Tiwari & Warnault [15] for variable exponent Lebesgue spaces which is necessary to verify the coercivity in Section 4. A related property can be found in Edmunds & Rákosnik [11, Lemma 2.1].

Lemma 2.3. *Assume that* $h_1 \in L^\infty(\Omega)$ *such that* $h_1 \ge 0$ *and* $h_1 \not\equiv 0$ *a.e. in* Ω*. Let* $h_2 : \Omega \to \mathbb{R}$ *be a measurable function such that* $h_1 h_2 \ge 1$ *a.e. in* Ω*. Then for any* $u \in L^{h_1(\cdot)h_2(\cdot)}(\Omega)$*,*

$$\||u|^{h_1(\cdot)}\|_{h_2(\cdot)} \le \|u\|^{h_1^-}_{h_1(\cdot)h_2(\cdot)} + \|u\|^{h_1^+}_{h_1(\cdot)h_2(\cdot)}.$$

Next, we define the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_W = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

or

$$\|u\|_W = \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \le 1\right\}.$$

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Next, we recall an embedding result regarding variable exponent Sobolev spaces, see Fan, Shen & Zhao [13].

Theorem 2.4. *If $\Omega \subset \mathbb{R}^N$ is bounded domain and $p(x) \in C(\overline{\Omega})$, then for any measurable function $q(x)$ defined in Ω with*

$$p(x) \le q(x),\ \ a.e.\ x \in \overline{\Omega}\ \ and\ \ \operatorname*{ess\,inf}_{x\in\overline{\Omega}}(p^*(x) - q(x)) > 0,\ \ (q^*(\cdot) = \frac{q(\cdot)}{q(\cdot)-1})$$

there is a compact embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

3. Double phase Baouendi-Grushin operators

In this section we prove new results concerning the new Baouendi-Grushin operator defined in (1.2).

First, we give the hypotheses on continuous functions $a, K, G : \mathbb{R}^N \to \mathbb{R}$.
(A) $a(.)$ is a continuous function such that

$$a(x) > 0 \text{ for all } x \in \mathbb{R}^N.$$

(G) G is a function of class C^1 and that

$$G(x, y) \in (2, N) \text{ for every } (x, y) \in \mathbb{R}^N.$$

We need $G > 2$ in the proof of Lemma 4.5, that is, in the first application. So, it is possible to include the case $G = 2$ if we do another kind of applications.
(K) $K \in L^\infty(\mathbb{R}^N)$, $K(x) > 0$ for all $x \in \mathbb{R}^N$ and if $(A_n) \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n| \le R$, for all $n \in \mathbb{N}$ and some $r > 0$, then

$$\lim_{n\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0.$$

In order to treat problem (1.1), let us consider the space:

$$D_a^{1,G}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R}, \ \ u \in L^{G^*}(\mathbb{R}^N) \ \text{ and } \int_{\mathbb{R}^N} (|\nabla_x u|^{G(x,y)} + a(x)|\nabla_y u|^{G(x,y)})dxdy < +\infty\}$$

endowed with the norm

$$\|u\|_D = \|\nabla_x u\|_{G(\cdot,\cdot)} + \left\|a(x)^{\frac{1}{G(\cdot,\cdot)}}\nabla_y u\right\|_{G(\cdot,\cdot)}, \ \text{for all } u \in X.$$

This permits us to construct a suitable space

$$X = D_a^{1,G(\cdot)}(\mathbb{R}^N)\bigcap L^{G(\cdot)}(\mathbb{R}^N),$$

endowed with the norm

$$\|u\|_X = \|u\|_D + \|u\|_{G(\cdot)} \text{ for all } u \in X.$$

Remark 3.1. Note that the norm $\|\cdot\|_X$ on X is equivalent to

$$\begin{aligned}\|u\| &= \inf\left\{\mu \geq 0 \,\middle|\, \rho\left(\frac{u}{\mu}\right) \leq 1\right\}\\ &= \inf\left\{\mu \geq 0 \,\middle|\, \int_{\mathbb{R}^N}\left[\left|\nabla_x\left(\frac{u}{\mu}\right)\right|^{G(x,y)} + a(x)\left|\nabla_y\left(\frac{u}{\mu}\right)\right|^{G(x,y)} + \left(\frac{|u|}{\mu}\right)^{G(x,y)}\right]dx\,dy \leq 1\right\},\end{aligned} \tag{3.1}$$

where

$$\rho(u) = \int_{\mathbb{R}^N}\left[|\nabla_x u|^{G(x,y)} + a(x)\left|\nabla_y u\right|^{G(x,y)} + |u|^{G(x,y)}\right]dx\,dy. \tag{3.2}$$

From now on, we shall denote the duality pairing between X and its dual space X^* by $\langle\cdot,\cdot\rangle_X$. The following lemma will be helpful in the sequel.

Lemma 3.2. *Suppose that conditions (A) and (G) are satisfied. Let $u \in X$, then the following holds:*

(i) For $u \neq 0$ we have: $\|u\| = a$ if and only if $\rho(\frac{u}{a}) = 1$;
(ii) $\|u\| < 1$ implies $\frac{\|u\|^{G^+}}{2^{\frac{1}{G^+-1}}} \leq \rho(u) \leq 2\|u\|^{G^-}$;
(iii) $\|u\| > 1$ implies $\|u\|^{G^-} \leq \rho(u) \leq \|u\|^{G^+}$.

Proof. The proof is similar to that in [5]. □

Lemma 3.3. *Assume that the hypotheses of Lemma 3.2 are fulfilled. Then the following properties hold.*

(i) The functional ρ is of class C^1 and for all $u, v \in X$ we have

$$\langle \rho'(u), v\rangle_X = \int_{\mathbb{R}^N} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + a(x)\left|\nabla_y u\right|^{G(x,y)-2} \nabla_y u \nabla_y v\right] dx\, dy + \int_{\mathbb{R}^N} |u|^{G(z)-2} u v\, dz.$$

(ii) The function $\rho' : X \to X^*$ is coercive, that is, $\frac{\langle \rho'(u), u\rangle_X}{\|u\|_X} \to +\infty$ as $\|u\|_X \to +\infty$.

(iii) ρ' is a mapping of type (S_+), that is, if $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle \rho'(u_n), u_n - u\rangle_X \leq 0$, then $u_n \to u$ in X.

Proof. The proof is similar to that in Bahrouni, Rădulescu & Winkert [5]. □

Now, we establish the following compactness result.

Lemma 3.4. *Assume that (A) and (G) hold. Then $D_a^{1,G}(\mathbb{R}^N)$ is compactly embedded in $L_{loc}^{s(\cdot)}(\mathbb{R}^N)$, for every $s(\cdot) \in (1, G^*(\cdot))$.*

Proof. Let (u_n) be an arbitrary bounded sequence in $D_a^{1,G}(\mathbb{R}^N)$. Fix $R > 0$, $s(\cdot) \in (1, G^*(\cdot))$, and set $B(0, R) = \{x \in \mathbb{R}^N, |x| \leq R\}$.
We note that $u_n \rightharpoonup u$ weakly in $L^{G^*(\cdot)}(\mathbb{R}^N)$. Thus, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} u_n \varphi dx = \int_{\mathbb{R}^N} u \varphi dx. \tag{3.3}$$

Claim. *We prove that $u_n \rightharpoonup u$ in $W_0^{1,G(\cdot)}(B(0, R))$.*

Indeed, denote by $u \restriction B(0, R)$ the restriction of u to $B(0, R)$ and suppose that (u_n) does not converge to $u \restriction B_R$ weakly in $W_0^{1,G}(B(0, R))$.
By condition (A), there exists $x_0 \in B(0, R)$ such that

$$a(x) \geq a(x_0) > 0, \quad \text{for all } \; x \in B(0, R),$$

and so (u_n) is bounded in $W_0^{1,G}(B(0, R))$. Therefore, there exist a subsequence (u_{n_k}) and $\overline{u} \in W^{1,G}(B(0, R))$, with $\overline{u} \neq u \restriction B_R$, such that $u_{n_k} \rightharpoonup \overline{u}$ weakly in $W_0^{1,G}(B(0, R))$. Invoking Theorem 2.4, $u_{n_k} \to \overline{u}$ strongly in $L^{s(\cdot)}(B(0, R))$. Then, taking into account (3.3), we obtain

$$\int_{B(0,R)} u\varphi dx = \lim_{k\to+\infty} \int_{B(0,R)} u_{n_k}\varphi dx = \int_{B(0,R)} \overline{u}\varphi dx,$$

for every $\varphi \in C_0^\infty(B(0,R))$. This implies that $u(x) = \overline{u}(x)$ for almost all $x \in B(0,R)$, against the fact that $\overline{u} \neq u \restriction B_R$. This proves the claim. Hence (u_n) weakly converges to $u \restriction B_R$ in $W_0^{1,G}(B(0,R))$. Applying Theorem 2.4 again, (u_n) strongly converges to u in $L^{s(\cdot)}(B(0,R))$. This completes the proof of Lemma 3.4. □

Now, we are ready to prove our compact embedding result in the whole space $\mathbb{R}^N$. Let us define, for every $s(\cdot) \in C_+(\mathbb{R}^N)$, the following Lebesgue space

$$L_K^{s(\cdot)}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R},\ \text{ u is measurable and } \int_{\mathbb{R}^N} K(z)|u|^{s(z)}dz < +\infty\}.$$

Proposition 3.5. *Let (A), (G) and (K) be satisfied. Then X is compactly embedded in $L_K^{s(\cdot)}(\mathbb{R}^N)$, for every $s(\cdot) \in (G(\cdot), G^*(\cdot))$.*

Proof. Fix $s(\cdot) \in (G(\cdot), G^*(\cdot))$ and $\epsilon > 0$. It is easy to see that

$$\lim_{t\to 0} \frac{|t|^{s(z)}}{|t|^{G(z)}} = \lim_{t\to+\infty} \frac{|t|^{s(z)}}{|t|^{G^*(z)}} = 0 \text{ uniformly for } z \in \mathbb{R}^N.$$

Thus, there exist $0 < t_0 < t_1$ and a positive constant $C > 0$ such that

$$K(z)|t|^{s(z)} \le \epsilon C(|t|^{G(z)} + |t|^{G^*(z)}) + \chi_{[t_0,t_1]}(z)K(z)|t|^{G(z)} \text{ for all } t \in \mathbb{R} \text{ and } z \in \mathbb{R}^N.$$

Set

$$A(u) = \int_{\mathbb{R}^N} |u|^{G(z)}dz + \int_{\mathbb{R}^N} |u|^{G^*(z)}dz \text{ and } R = \{z \in \mathbb{R}^N,\ \ t_0 < |u(z)| < t_1\}.$$

Let $(u_n) \in X$ be a sequence such that $u_n \rightharpoonup u$ in X. It is easy to see that $(A(u_n))_n$ is bounded in $\mathbb{R}$. Denoting $R_n = \{x \in \mathbb{R}^N,\ t_0 < |u_n(x)| < t_1\}$, we get $\sup_{n\in\mathbb{N}} |A_n| < +\infty$. Hence, by (K), there exists a positive radius $r > 0$ such that

$$\begin{aligned}\int_{B_r^c(0)} K(z)|u_n|^{s(z)}dz &\le \epsilon CA(u_n) + \int_{B_r^c(0)} \chi_{[t_0,t_1]}(z)K(z)|u_n|^{G(z)}dz \\ &\le \epsilon CA(u_n) + (t_1^{G^-} + t_1^{G^+}) \int_{B_r^c(0)\frown R_n} K(z)dz \\ &\le (C' + t_1^{G^-} + t_1^{G^+})\epsilon, \text{ for all } n \in \mathbb{N}. \qquad (3.4)\end{aligned}$$

Now, since $s(\cdot) \in (1, G^*(\cdot))$ and $K \in L^\infty(\mathbb{R}^N)$, we deduce, that

$$\lim_{n\to+\infty}\int\limits_{B_r(0)} K(x)|u_n|^{s(z)}dz = \int\limits_{B_r(0)} K(x)|u|^{s(z)}dz. \tag{3.5}$$

Here we used Lemma 3.4. Combining (3.4) and (3.5), we conclude for $\epsilon > 0$ small enough, that

$$\lim_{n\to+\infty}\int\limits_{\mathbb{R}^N} K(z)|u_n|^{s(z)}dz = \int\limits_{\mathbb{R}^N} K(z)|u|^{s(z)}dz.$$

Consequently, using Proposition 2.1, we infer that

$$u_n \to u \ \text{ in } \ L_K^{s(\cdot)}(\mathbb{R}^N) \text{ for every } s(\cdot) \in (G(\cdot), G^*(\cdot)).$$

This completes the proof of Proposition 3.5. □

4. A nonlinear problem driven by $\Delta_{G,a}$

As an application of the previous abstract results, the main result of this section concerns the study of both nonvariational and singular aspects of problem (1.1).

4.1. Nonvariational case

In this paragraph, we work under conditions introduced in Proposition 3.5. We are mainly concerned with the following equation

$$-\Delta_{G,a}u = -\operatorname{div}(\alpha_1 u\nabla_x r) - \operatorname{div}(\alpha_2 a^{\frac{1}{G(x,y)}}(x)u\nabla_y r) + f(z,u), \quad z=(x,y)\in\mathbb{R}^N. \tag{4.1}$$

The hypotheses on functions f and r are the following:

(H_1) $f(z,0)\neq 0$, $f(z,s)\leq (a(z)+b(z)|s|^{\gamma(z)-1})$ and $|f(z,s)|\leq (a(z)+|b(z)||s|^{\gamma(z)-1})$ a.e. $z\in\mathbb{R}^N$ and for all $s\in\mathbb{R}$ where

- $\gamma(\cdot)\in C_+(\mathbb{R}^N)$ and $\gamma(\cdot), \frac{\gamma(\cdot)}{\gamma(\cdot)-1}\in (G(\cdot), G^*(\cdot)$.
- $b\in C_+(\mathbb{R}^N,\mathbb{R}^-)$ and $\frac{b}{K}\in L^\infty(\mathbb{R}^N)$.
- $a\in L^{\frac{G(\cdot)}{G(\cdot)-1}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$.

(H_2) $r:\mathbb{R}^N\to\mathbb{R}$ is some measurable function satisfying

$$\nabla r\in L^{\frac{G(\cdot)\beta(\cdot)}{(\beta(\cdot)-1)(G(\cdot)-1)}}(\mathbb{R}^N),$$

where $\frac{G(\cdot)\beta(\cdot)}{G(\cdot)-1}\in (G, G^*)$.

(H_3) $\alpha_1,\alpha_2\in C_+(\mathbb{R}^N)$ and $\frac{\alpha_1}{K}, \frac{\alpha_2}{K}\in L^\infty(\mathbb{R}^N)$.

Definition 4.1. We say that $u \in X \setminus \{0\}$ is a weak solution of problem (4.1) if for all $v \in X \setminus \{0\}$,

$$\int_{\mathbb{R}^N} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + a(x) \left|\nabla_y u\right|^{G(x,y)-2} \nabla_y u \nabla_y v \right] dx\,dy$$
$$- \int_{\mathbb{R}^N} \alpha_1 u \nabla_x r . \nabla_x v \, dx\,dy - \int_{\mathbb{R}^N} \alpha_2 [a(x)]^{\frac{1}{G(x,y)}} u \nabla_y r . \nabla_y v \, dx\,dy$$
$$- \int_{\mathbb{R}^N} f((x,y),u) v \, dx\,dy = 0.$$

Remark 4.2. Under conditions (A), (G), (K), (H_1)–(H_3) and by virtue of Proposition 3.5, the definition of weak solution of problem (4.1) is well-defined.

The main result of this paragraph reads as follows.

Theorem 4.3. *Assume that (G), (K) and (H_1)–(H_3) hold. Then, problem (4.1) admits at least one nontrivial weak solution.*

The proof of Theorem 4.3 relies on the topological degree theory of (S_+)–type mappings. Define the operator $L : X \to X^*$ by

$$\langle L(u), v \rangle = \int_{\mathbb{R}^N} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + a(x) \left|\nabla_y u\right|^{G(x,y)-2} \nabla_y u \nabla_y v \right] dx\,dy$$
$$- \int_{\mathbb{R}^N} \alpha_1 u \nabla_x r . \nabla_x v \, dx\,dy - \int_{\mathbb{R}^N} \alpha_2 [a(x)]^{\frac{1}{G(x,y)}} u \nabla_y r . \nabla_y v \, dx\,dy$$
$$- \int_{\mathbb{R}^N} f((x,y),u) v \, dx\,dy, \quad u, v \in X.$$

Lemma 4.4. *Suppose that assumptions of Theorem 4.3 are fulfilled. Then L is a mapping of type* (S_+)*, that is, if $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \langle L(u_n), u_n - u \rangle_X \le 0$, then $u_n \to u$ in X.*

Proof. Let $\{u_n\}_{n\ge1} \subseteq X$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \limsup_{n\to+\infty} \langle L(u_n), u_n - u \rangle_X \le 0.$$

This implies that

$$\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u \rangle_X \le 0. \tag{4.2}$$

Claim 1. $\lim_{n\to+\infty} \int_{\mathbb{R}^N} (f(z,u_n) - f(z,u))(u_n - u) dz = 0.$

For $r > 0$, we denote by B_r the open ball centered at the origin and of a radius r. Applying the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^N}(f(z,u_n)-f(z,u))(u_n-u)dz &\le \int_{\mathbb{R}^N}(|f(z,u_n)|+|f(z,u)|)|u_n-u|dz \\ &\le \int_{\mathbb{R}^N}|a(z)||u_n-u|dz+\int_{\mathbb{R}^N}|b(z)||u_n|^{\gamma(z)-1}|u_n-u|dz \\ &+\int_{\mathbb{R}^N}|b(z)||u|^{\gamma(z)-1}|u_n-u|dz \\ &\le \int_{B_r}|a(z)||u_n-u|dz+\int_{B_r^c}|a(z)||u_n-u|dz \\ &+\||b|^{\frac{\gamma(\cdot)-1}{\gamma(\cdot)}}|u_n|^{\gamma(\cdot)-1}\|_{\frac{\gamma(\cdot)}{\gamma(\cdot)-1}}\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)} \\ &+\||b|^{\frac{\gamma(\cdot)-1}{\gamma(\cdot)}}|u|^{\gamma(\cdot)-1}\|_{\frac{\gamma(\cdot)}{\gamma(\cdot)-1}}\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)}.\end{aligned} \tag{4.3}$$

Again, by Hölder's inequality, we obtain

$$\int_{B_r}|a(z)||u_n-u|dz\le\|a\|_{\frac{G(\cdot)}{G(\cdot)-1}(B_r)}\|u_n-u\|_{G(\cdot)}.$$

Using Lemma 3.4, it follows that

$$\lim_{n\to+\infty}\int_{B_r}|a(z)||u_n-u|dz=0. \tag{4.4}$$

Now, using (H_1), we deduce that

$$\int_{B_r^c}|a(z)||u_n-u|dz\le\|a\|_{L^{\frac{G(\cdot)}{G(\cdot)-1}}(B_r^c)}\|u_n-u\|_{L^{G(\cdot)}(B_r^c)}\le C\|a\|_{L^{\frac{G(\cdot)}{G(\cdot)-1}}(B_r^c)}\to 0, \tag{4.5}$$

as $r\to+\infty$ and for some positive constant C.
On the other hand, by (H_1) and Propositions 2.1 and 3.5, we have

$$\begin{aligned}\||b|^{\frac{\gamma(\cdot)-1}{\gamma(\cdot)}}|u_n|^{\gamma(\cdot)-1}\|_{\frac{\gamma(\cdot)}{\gamma(\cdot)-1}}\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)} &\le C\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)} \\ &\le C\left([\int_{\mathbb{R}^N}|b(z)||u_n-u|^{\gamma(z)}dz]^{\frac{1}{\gamma^-}}+[\int_{\mathbb{R}^N}|b(z)||u_n-u|^{\gamma(z)}dz]^{\frac{1}{\gamma^+}}\right)\end{aligned}$$

$$\leq C\left([\int_{\mathbb{R}^N}|K(z)||u_n-u|^{\gamma(z)}dz]^{\frac{1}{\gamma^-}}+[\int_{\mathbb{R}^N}|K(z)||u_n-u|^{\gamma(z)}dz]^{\frac{1}{\gamma^+}}\right),$$

for some positive constant C. Thus, in light of Proposition 3.5, we infer that

$$\lim_{n\to+\infty}\||b|^{\frac{\gamma(\cdot)-1}{\gamma(\cdot)}}|u_n|^{\gamma(\cdot)-1}\|_{\frac{\gamma(\cdot)}{\gamma(\cdot)-1}}\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)}=0. \tag{4.6}$$

In the same way, we prove that

$$\lim_{n\to+\infty}\||b|^{\frac{\gamma(\cdot)-1}{\gamma(\cdot)}}|u|^{\gamma(\cdot)-1}\|_{\frac{\gamma(\cdot)}{\gamma(\cdot)-1}}\||b|^{\frac{1}{\gamma(\cdot)}}|u_n-u|\|_{\gamma(\cdot)}=0 \tag{4.7}$$

Combining (4.3), (4.4), (4.5), (4.6) and (4.7), we get Claim 1.

Claim 2. *In what follows, we show that*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\alpha_1(u_n-u)\nabla_x r.\nabla_x(u_n-u)\,dx\,dy$$
$$=\lim_{n\to+\infty}\int_{\mathbb{R}^N}\alpha_2[a(x)]^{\frac{1}{G(x,y)}}(u_n-u)\nabla_y r.\nabla_y(u_n-u)\,dx\,dy=0.$$

Invoking the Hölder inequality and Proposition 2.1, we obtain

$$\int_{\mathbb{R}^N}\alpha_1(u_n-u)\nabla_x r.\nabla_x(u_n-u)\,dx\,dy\leq\|\alpha_1|u_n-u|\nabla_x r\|_{\frac{G(\cdot)}{G(\cdot)-1)}}\|\nabla_x(u_n-u)\|_{G(\cdot)} \tag{4.8}$$

$$\leq C\left(\int_{\mathbb{R}^N}\alpha_1^{\frac{G(x,y)}{G(x,y)-1}}|u_n-u|^{\frac{G(x,y)}{G(x,y)-1}}|\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}dxdy\right)^{\frac{G^--1}{G^+}}$$
$$+C\left(\int_{\mathbb{R}^N}\alpha_1^{\frac{G(x,y)}{G(x,y)-1}}|u_n-u|^{\frac{G(x,y)}{G(x,y)-1}}|\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}dxdy\right)^{\frac{G^+-1}{G^-}}.$$

Now, from conditions (H_2), (H_3) and the Hölder inequality, we deduce that

$$\int_{\mathbb{R}^N}(\alpha_1|u_n-u|)^{\frac{G(x,y)}{G(x,y)-1}}|\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}dxdy\leq C\int_{\mathbb{R}^N}(K|u_n-u|)^{\frac{G(x,y)}{G(x,y)-1}}|\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}dxdy$$
$$\leq C\|K^{\frac{1}{\beta(\cdot)}}|u_n-u|^{\frac{G(\cdot,\cdot)}{G(\cdot,\cdot)-1}}\|_{\beta(\cdot)}\||\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}\|_{\frac{\beta(\cdot)}{\beta(\cdot)-1}}$$
$$\leq C\|K^{\frac{1}{\beta(\cdot)}}|u_n-u|^{\frac{G(\cdot,\cdot)}{G(\cdot,\cdot)-1}}\|_{\beta(\cdot)},$$

which, by Proposition 3.5, implies that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}(\alpha_1|u_n-u|)^{\frac{G(x,y)}{G(x,y)-1}}|\nabla_x r|^{\frac{G(x,y)}{G(x,y)-1}}dxdy=0. \tag{4.9}$$

Consequently, from (4.8) and (4.9), we conclude that

$$\int_{\mathbb{R}^N}\alpha_1(u_n-u)\nabla_x r.\nabla_x(u_n-u)\,dx\,dy=0.$$

Again, using the same argument, we show that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\alpha_2[a(x)]^{\frac{1}{G(x,y)}}(u_n-u)\nabla_y r.\nabla_y(u_n-u)\,dx\,dy=0.$$

This proves Claim 2.

Finally, from Claim 1, Claim 2 and (4.2), we infer that

$$\limsup_{n\to+\infty}\langle\rho'(u_n)-\rho'(u),u_n-u\rangle_X\le 0.$$

Hence, by Lemma 3.3, we get our desired result. □

Lemma 4.5. *Suppose that assumptions of Theorem 4.3 are fulfilled. Then for $R>0$ large enough, we have*

$$\langle L(u),u\rangle>0 \text{ for all } u\in X \text{ such that } \|u\|=R.$$

Proof. Let $u\in X$ be such that $\|u\|>1$. Hence, in view of Lemmas 2.3 and 3.2 and Proposition 3.5 and the Hölder inequality, we obtain

$$\begin{aligned}\langle L(u),u\rangle&=\int_{\mathbb{R}^N}\left[|\nabla_x u|^{G(x,y)}+a(x)\left|\nabla_y u\right|^{G(x,y)}\right]dx\,dy\\&-\int_{\mathbb{R}^N}\alpha_1 u\nabla_x r.\nabla_x u\,dx\,dy-\int_{\mathbb{R}^N}\alpha_2[a(x)]^{\frac{1}{G(x,y)}}u\nabla_y r.\nabla_y u\,dx\,dy\\&-\int_{\mathbb{R}^N}f((x,y),u)v\,dx\,dy\\&\ge\int_{\mathbb{R}^N}\left[|\nabla_x u|^{G(x,y)}+a(x)\left|\nabla_y u\right|^{G(x,y)}\right]dx\,dy-\int_{\mathbb{R}^N}\alpha_1 u\nabla_x r.\nabla_x u\,dx\,dy\\&-\int_{\mathbb{R}^N}\alpha_2[a(x)]^{\frac{1}{G(x,y)}}u\nabla_y r.\nabla_y u\,dx\,dy-\int_{\mathbb{R}^N}a(z)u\,dz,\end{aligned}$$

$$
\begin{aligned}
&\geq \|u\|^{G^-} - \|\nabla_x r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|\alpha_1^{\frac{G(\cdot)-1}{\beta(\cdot)G(\cdot)}} u\|_{\frac{\beta(\cdot)G(\cdot)}{G(\cdot)-1}} \|\nabla_x u\|_{G(\cdot)} \\
&- \|\nabla_y r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|\alpha_2^{\frac{G(\cdot)-1}{\beta(\cdot)G(\cdot)}} u\|_{\frac{\beta(\cdot)G(\cdot)}{G(\cdot)-1}} \|a^{\frac{1}{G(\cdot)}} \nabla_y u\|_{G(\cdot)} - \|a\|_{\frac{G(\cdot)}{G(\cdot)-1}} \|u\|_{G(\cdot)} \\
&\geq \|u\|^{G^-} - C\|\nabla_x r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|K^{\frac{G(\cdot)-1}{\beta(\cdot)G(\cdot)}} u\|_{\frac{\beta(\cdot)G(\cdot)}{G(\cdot)-1}} \|\nabla_x u\|_{G(\cdot)} \\
&- C\|\nabla_y r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|K^{\frac{G(\cdot)-1}{\beta(\cdot)G(\cdot)}} u\|_{\frac{\beta(\cdot)G(\cdot)}{G(\cdot)-1}} \|a^{\frac{1}{G(\cdot)}} \nabla_y u\|_{G(\cdot)} - \|a\|_{\frac{G(\cdot)}{G(\cdot)-1}} \|u\|_{G(\cdot)} \\
&\geq \|u\|^{G^-} - C\|\nabla_x r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|u\|^2 - C\|\nabla_y r\|_{\frac{G(\cdot)\beta(\cdot)}{(G(\cdot)-1)(\beta(\cdot)-1)}} \|u\|^2 \\
&- \|a\|_{\frac{G(\cdot)}{G(\cdot)-1}} \|u\|,
\end{aligned}
$$

where C is a positive constant. Choosing $\|u\| = R$ large enough, we deduce from the last inequality that

$$\langle L(u), u\rangle > 0 \text{ for all } u \in X \text{ such that } \|u\| = R.$$

This completes the proof of Lemma 4.5. □

Proof of Theorem 4.3 completed. It is clear that L is also demicontinuous and bounded. Then, in light of Lemmas 4.4 and 4.5 and using the topological degree theory for (S_+) type mappings, we conclude that

$$deg(L, B(0, R), 0) = 1,$$

where R is defined in Lemma 4.5. Therefore the equation $L(u) = 0$ has at least one solution $u \in B(0, R)$. From assumption (H_1), we can conclude that u is a nontrivial weak solution of equation (4.1). This completes the proof of Theorem 4.3. □

4.2. *Singular problem*

In this subsection, we work under conditions introduced in Proposition 3.5. Here, we are interested in weak solutions to nonlinear singular problems. Precisely, we study the following singular double phase equation

$$-\Delta_{G,a} u + |u|^{G(x,y)-2} u = \frac{b(x, y)}{u^{\sigma(x,y)}}, \quad (x, y) \in \mathbb{R}^N, \tag{4.10}$$

where $\sigma(\cdot) \in C^1(\mathbb{R}^N)$, $0 < \sigma(\cdot) < 1$. The assumption on function b is the following:
(A) $b > 0$ in $\mathbb{R}^N$, $b \in L^1(\mathbb{R}^N) \bigcap L^{G(\cdot)}(\mathbb{R}^N) \bigcap L^{\frac{G(\cdot)}{G(\cdot)-1}}(\mathbb{R}^N)$ and $\frac{b}{K} \in L^\infty(\mathbb{R}^N)$.

Definition 4.6. We say that $u \in X \setminus \{0\}$ is a weak solution of problem (4.10) if $u \geq 0, u \neq 0, u^{-\sigma(\cdot)} v \in L^1(\mathbb{R}^N)$ for all $v \in X \setminus \{0\}$ and

$$\int_{\mathbb{R}^N} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + a(x) \left|\nabla_y u\right|^{G(x,y)-2} \nabla_y u \nabla_y v \right] dx\, dy$$
$$= \int_{\mathbb{R}^N} \frac{b(z)}{u^{\sigma(z)}} v dz.$$

Our main result is the following existence theorem.

Theorem 4.7. *Let* (A), (G) *and* (K) *be satisfied. Then problem* (4.10) *admits at least one non-trivial positive weak solution.*

To prove the above theorem, we first consider a perturbation of (4.10) which removes the singularity. So, we consider the following approximation of problem (4.10):

$$\begin{aligned} &-\Delta_{G,a} u + |u|^{G(x,y)-2} u = \frac{b(x,y)}{(u+\epsilon)^{\sigma(x,y)}}, \qquad (x,y) \in \mathbb{R}^N, \\ &u > 0. \end{aligned} \tag{4.11}$$

The main way to deal with this problem is the topological approach. So, given $f \in L^{G(\cdot)}(\mathbb{R}^N)$, $f \geq 0$ and $\epsilon \in (0,1)$, we consider the following equation:

$$\begin{aligned} &-\Delta_{G,a} u + |u|^{G(x,y)-2} u = \frac{b(x,y)}{(f(x,y)+\epsilon)^{\sigma(x,y)}}, \qquad (x,y) \in \mathbb{R}^N, \\ &u > 0. \end{aligned} \tag{4.12}$$

For the above problem we have the following result.

Proposition 4.8. *Suppose that* (A), (G) *and* (K) *hold. Then problem* (4.12) *admits a unique positive solution* $u_\epsilon \in X$.

Proof. Let $B_G : L^{G(\cdot)}(\mathbb{R}^N) \to L^{G'(\cdot)}(\mathbb{R}^N)$ be the map defined by

$$B_G(u) = |u|^{G(\cdot)-2} u \quad \text{for all } \; u \in L^{G(\cdot)}(\mathbb{R}^N).$$

Using the Simon inequality (see [33]), B_G is bounded, continuous, strictly monotone. Then we consider the map $A_G : X \to X^*$ defined by

$$< A_G(u), v >= \int_{\mathbb{R}^N} \left[|\nabla_x u|^{G(x,y)-2} \nabla_x u \nabla_x v + a(x) \left|\nabla_y u\right|^{G(x,y)-2} \nabla_y u \nabla_y v \right] dx\, dy,$$

for all $u, v \in X$. Using the same argument, we can deduce that this operator is bounded continuous, strictly monotone. It follows that the operator $V_G = A_G + B_G$ is bounded continuous, strictly monotone (thus, maximal monotone, too). On the other hand, in light of Lemma 3.3, we have that V is coercive. We know that a maximal monotone coercive operator is surjective. Then, since $b(.)[f(.)+\epsilon]^{-\gamma(.)} \in L^{\frac{G(\cdot)}{G(\cdot)-1}}(\mathbb{R}^N)$, we can find $v_\epsilon \in X$ such that

$$\langle V(v_\epsilon), h\rangle = \langle b(.)[f(.)+\epsilon]^{-\gamma(.)}, h\rangle, \text{ for every } h \in X. \tag{4.13}$$

In (4.13) we choose $h = -v_\epsilon^-$ ($v_\epsilon^- = \max(-v_\epsilon, 0)$). Thus, using the fact that $(f(\cdot)+\epsilon) > 0$, we obtain that v_ϵ is a nonnegative and $v_\epsilon \neq 0$. Moreover, the strict monotonicity of $V(.)$ implies that this solution is unique. Finally, the anisotropic maximum principle of Zhang [34] implies that $v_\epsilon > 0$. This completes the proof of Proposition 4.8. □

Using Proposition 4.8, we can define the solution map $L_\epsilon : L^{G(\cdot)}(\mathbb{R}^N) \to L^{G(\cdot)}(\mathbb{R}^N)$ for problem (4.12) by

$$L_\epsilon(f) = v_\epsilon.$$

Proposition 4.9. *Suppose that assumptions of Proposition 4.8 are fulfilled. Then problem (4.11) admits a unique positive solution $u_\epsilon \in X$.*

Proof. In view of Proposition 4.8, we have

$$< A_G(v_\epsilon), h > + \int_{\mathbb{R}^N} |v_\epsilon|^{G(z)-2} v_\epsilon h dz = \int_{\mathbb{R}^N} b(z)[f(z)+\epsilon]^{-\gamma(z)} h dz, \text{ for all } h \in X. \tag{4.14}$$

In (4.14) we choose $h = v_\epsilon = L_\epsilon(f) \in X$ and we obtain

$$\rho(v_\epsilon) = \int_{\mathbb{R}^N} b(z)[f(z)+\epsilon]^{-\gamma(z)} v_\epsilon dz,$$

which implies that there exists a positive constant C such that

$$\min(\|L_\epsilon(f)\|^{G^-}, \|L_\epsilon(f)\|^{G^+}) \le C_\epsilon \|b\|_{\frac{G(\cdot)}{G(\cdot)-1}} \|L_\epsilon(f)\|$$

and

$$\|L_\epsilon(f)\| \le C_\epsilon, \quad \text{for all } f \in L^{G(\cdot)}(\mathbb{R}^N). \tag{4.15}$$

In what follows, we prove that $L_\epsilon(.)$ is continuous. To this end, let $f_n \to f$ in $L^{G(\cdot)}(\mathbb{R}^N)$. From (4.15) we have that $(L_\epsilon(f_n) = u_n)_{n\in\mathbb{N}}$ is bounded in X. So, we may assume that

$$u_n \rightharpoonup u \text{ in } X.$$

Thus, using conditions (B) and (K), we infer that

$$\begin{aligned}\int_{\mathbb{R}^N} b(z)[f(z)+\epsilon]^{-\gamma(z)}(u_n-u)dz &\le \frac{1}{\epsilon^{\sigma^+}} \int_{\mathbb{R}^N} b^{\frac{G(z)-1}{G(z)}}(z) b^{\frac{1}{G(z)}}(z)(u_n-u)dz \\ &\le \frac{C}{\epsilon^{\sigma^+}} \int_{\mathbb{R}^N} b^{\frac{G(z)-1}{G(z)}}(z) K^{\frac{1}{G(z)}}(z)(u_n-u)dz\end{aligned}$$

$$\leq \frac{C}{\epsilon^{\sigma^+}} \|b^{\frac{G(\cdot)-1}{G(\cdot)}}\|_{\frac{G(\cdot)}{G(\cdot)-1}} \|K^{\frac{1}{G(\cdot)}}(u_n - u)\|_{G(\cdot)}.$$

This leads to

$$\lim_{n\to+\infty} \int\limits_{\mathbb{R}^N} b(z)[f(z)+\epsilon]^{-\gamma(z)}(u_n - u)dz = 0. \tag{4.16}$$

Here we used Proposition 3.5. On the other hand, we have

$$\langle \rho'(u_n), h\rangle = \int\limits_{\mathbb{R}^N} b(z)[f_n(z)+\epsilon]^{-\gamma(z)} h dz, \text{ for all } h \in X \text{ and } n \in \mathbb{N}. \tag{4.17}$$

In (4.17) we choose $h = u_n - u \in X$, pass to the limit as $n \to +\infty$ and use (4.16). Then we obtain

$$\lim_{n\to+\infty} \langle \rho'(u_n), u_n - u\rangle = 0.$$

So, by Lemma 3.3,

$$u_n \to u \text{ in } X. \tag{4.18}$$

If in (4.17) we pass to the limit as $n \to +\infty$ and use (4.18), we obtain that

$$\langle \rho'(u), h\rangle = \int\limits_{\mathbb{R}^N} \frac{b(z)}{(f(z)+\epsilon)^{\gamma(z)}} h dz,$$

and

$$L_\epsilon(f) = u.$$

This proves that $L_\epsilon(.)$ is continuous. The continuity of $L_\epsilon(.)$, together with (4.15) and Proposition 3.5, permits the use of the Schauder-Tychonov fixed point theorem (see [24]) and we find $u_\epsilon \in X$ such that $L_\epsilon(u_\epsilon) = u_\epsilon$ and so, u_ϵ is a positive solution of (4.11).
Next we show the uniqueness of this solution. Suppose that $v_\epsilon \in X$ is another positive solution of (4.11). We have

$$\begin{aligned} 0 &\leq \langle \rho'(u_\epsilon) - \rho'(v_\epsilon), (u_\epsilon - v_\epsilon)^+\rangle \\ &= \int\limits_{\mathbb{R}^N} [\frac{b(z)}{(u_\epsilon+\epsilon)^{\gamma(z)}} - \frac{b(z)}{(v_\epsilon+\epsilon)^{\gamma(z)}}](u_\epsilon - v_\epsilon)^+ dz \leq 0, \end{aligned}$$

which implies that $u_\epsilon \leq v_\epsilon$. Interchanging the roles of u_ϵ and v_ϵ in the above argument, we also have that $v_\epsilon \leq u_\epsilon$, therefore $u_\epsilon = v_\epsilon$. This completes the proof of Proposition 4.9. □

Now, we prove the following monotonicity property of the map $\epsilon \to u_\epsilon$.

Proposition 4.10. *Assume that (B), (G) and (K) hold. Then the map $\epsilon \to u_\epsilon$ from $(0, 1]$ into X is nonincreasing.*

Proof. Let $0 < \epsilon' < \epsilon \leq 1$ and let $u_\epsilon, u_{\epsilon'} \in X$ be the corresponding unique positive solutions of problem (4.11).

We define the following function:

$$f_\epsilon(z,x) = \frac{b(z)}{[x^+ + \epsilon]^{\gamma(z)}}, \text{ if } x \leq u_{\epsilon'}(z) \text{ and } f_\epsilon(z,x) = \frac{b(z)}{[u_{\epsilon'}(z) + \epsilon]^{\gamma(z)}}, \text{ if } x > u_{\epsilon'}(z).$$

We set $F_\epsilon(z,x) = \int\limits_0^x f_\epsilon(z,s)ds$ and we introduce the functional $I_\epsilon : X \to \mathbb{R}$ defined by

$$I_\epsilon(u) = \int\limits_{\mathbb{R}^N} \frac{|\nabla_x u|^{G(x,y)}}{G(x,y)} dxdy + \int\limits_{\mathbb{R}^N} a(x) \frac{|\nabla_y u|^{G(x,y)}}{G(x,y)} dxdy + \int\limits_{\mathbb{R}^N} \frac{|u|^{G(x,y)}}{G(x,y)} dxdy$$
$$- \int\limits_{\mathbb{R}^N} F_\epsilon(z,u)dz.$$

Evidently I_ϵ is of class C^1. If $u \in X$ is large enough, we have

$$I_\epsilon(u) \geq \frac{\rho(u)}{G^-} - \frac{\|b\|_1}{\epsilon^{\gamma^+}} \geq \frac{\|u\|^{G^-}}{G^-} - \frac{\|b\|_{\frac{G(\cdot)}{G(\cdot)-1}}}{\epsilon^{\gamma^+}}.$$

Therefore, I_ϵ is coercive. On the other hand, by condition (B), we can prove that I_ϵ is weakly lower semicontinuous. Then, invoking the Weierstrass-Tonelli theorem, we can find $v_\epsilon \in X$ such that

$$I_\epsilon(v_\epsilon) = \inf_{u \in X} I_\epsilon(u).$$

This implies that

$$< \rho'(v_\epsilon), h >= \int\limits_{\mathbb{R}^N} f_\epsilon(z, v_\epsilon) h dz, \text{ for all } h \in X. \tag{4.19}$$

In (4.19) we choose $h = -v_\epsilon^- \in X$ and obtain

$$\rho(v_\epsilon^-) = -\int\limits_{\mathbb{R}^N} \frac{b(z)v_\epsilon^-}{\epsilon^{\gamma(z)}} dz \leq 0.$$

Hence,

$$v_\epsilon \geq 0, \quad v_\epsilon \neq 0.$$

Now, in (4.19) we choose $h = [v_\epsilon - u_{\epsilon'}]^+ \in X$. We get

$$\langle \rho'(v_\epsilon), (v_\epsilon - u_{\epsilon'})^+\rangle = \int_{\mathbb{R}^N} b(z)\frac{[v_\epsilon - u_{\epsilon'}\}^+}{[u_{\epsilon'} + \epsilon]^{\gamma(z)}}dz \leq \langle \rho'(u_{\epsilon'}), (v_\epsilon - u_{\epsilon'})^+\rangle,$$

and so

$$v_\epsilon \leq u_{\epsilon'}.$$

It follows, using the definition of $f_\epsilon(.,.)$ and Proposition 4.9, that $v_\epsilon = u_\epsilon$. Then, $u_\epsilon \leq u_{\epsilon'}$. This completes the proof of Proposition 4.10. □

Proof of Theorem 4.7 completed. Let $(\epsilon_n) \subseteq (0, 1]$ be a sequence such that $\epsilon_n \to 0^+$ as $n \to +\infty$ and u_n be as in Proposition 4.9. Then

$$\langle \rho'(u_n), h\rangle = \int_{\mathbb{R}^N} \frac{b(z)}{[u_n + \epsilon_n]^{\gamma(z)}}hdz, \quad \text{for all } \ h \in X, \quad \text{all } \ n \in \mathbb{N}. \tag{4.20}$$

In (4.20) we choose $h = u_n$ and use Proposition 4.10. Hence

$$\rho(u_n) \leq G^+ \int_{\mathbb{R}^N} \frac{b(z)}{u_1^{\gamma(z)}}u_n dz$$

which implies that (u_n) is bounded in X. Therefore, we can find $u \in X$ such that

$$u_n \rightharpoonup u \ \text{ in } \ X \ \text{ and } \ u_n \to u \ \text{ a.e. in } \ \mathbb{R}^N.$$

Consequently, combining Proposition 3.5 and the dominated convergence theorem, with the fact that $u_1 \leq u_n$ (see Proposition 4.10), we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \frac{b(z)}{[u_n + \epsilon_n]^{\gamma(z)}}hdz = \int_{\mathbb{R}^N} \frac{b(z)}{u^{\gamma(z)}}hdz, \ \text{for every } h \in X. \tag{4.21}$$

Also, it is easy to see that

$$\lim_{n\to+\infty} < \rho'(u_n), h > = < \rho'(u), h >, \ \text{for every } h \in X. \tag{4.22}$$

Then, by (4.21) and (4.22) and passing to the limit as $n \to +\infty$ in (4.20), we conclude that

$$< A_G(u), h > + \int_{\mathbb{R}^N} |u|^{G(z)-2}uhdz = \int_{\mathbb{R}^N} \frac{b(z)}{u^{\gamma(z)}}hdz \text{ for all } h \in X.$$

This proves that u is a weak solution of problem (4.10). Since $u_1 \leq u_n$ for all $n \in \mathbb{N}$, we have $u > 0$. Finally, we show the uniqueness of this positive solution. So, suppose that $v \in X$ is another positive solution of equation (4.10). As in the proof of Proposition 4.10, we can prove that $u = v$. The proof of Theorem 4.7 is now complete. □

Acknowledgments

The authors would like to thank Professor Nikolaos S. Papageorgiou for his numerous comments and suggestions on the initial version of this paper. The research of Vicenţiu D. Rădulescu and Dušan D. Repovš was supported by the Slovenian Research Agency program P1-0292. The research of Vicenţiu D. Rădulescu was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS/CCCDI–UEFISCDI, project number PCE 137/2021, within PNCDI III. Dušan D. Repovš also acknowledges the support of the Slovenian Research Agency grants N1-0083 and N1-0114.

References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (4) (1976/1977) 337–403.
[2] J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, Philos. Trans. R. Soc. Lond. Ser. A 306 (1496) (1982) 557–611.
[3] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, A weighted anisotropic variant of the Caffarelli-Kohn-Nirenberg inequality and applications, Nonlinearity 31 (4) (2018) 1516–1534.
[4] A. Bahrouni, V.D. Rădulescu, D.D. Repovš, Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves, Nonlinearity 32 (7) (2019) 2481–2495.
[5] A. Bahrouni, V.D. Rădulescu, P. Winkert, Double phase problems with variable growth and convection for the Baouendi-Grushin operator, Z. Angew. Math. Phys. 71 (6) (2020) 183.
[6] A. Bahrouni, V.D. Rădulescu, Singular double-phase systems with variable growth for the Baouendi-Grushin operator, Discrete Contin. Dyn. Syst. 41 (9) (2021) 4283–4296.
[7] A. Bahrouni, D.D. Repovš, Existence and nonexistence of solutions for $p(x)$-curl systems arising in electromagnetism, Complex Var. Elliptic Equ. 63 (2) (2018) 292–301.
[8] M.S. Baouendi, Sur une classe d'opérateurs elliptiques dégénérés, Bull. Soc. Math. Fr. 95 (1967) 45–87.
[9] L. Beck, G. Mingione, Lipschitz bounds and non-uniform ellipticity, Commun. Pure Appl. Math. 73 (5) (2020) 944–1034.
[10] M. Cencelj, V.D. Rădulescu, D.D. Repovš, Double phase problems with variable growth, Nonlinear Anal. 177 (2018) 270–287.
[11] D. Edmunds, J. Rákosnik, Sobolev embeddings with variable exponent, Stud. Math. 143 (3) (2000) 267–293.
[12] M. Eleuteri, P. Marcellini, E. Mascolo, Regularity for scalar integrals without structure conditions, Adv. Calc. Var. 13 (3) (2020) 279–300.
[13] X. Fan, J. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. 262 (2001) 749–760.
[14] B. Franchi, M.C. Tesi, A finite element approximation for a class of degenerate elliptic equations, Math. Comput. 69 (1999) 41–63.
[15] J. Giacomoni, S. Tiwari, G. Warnault, Quasilinear parabolic problem with $p(x)$-Laplacian: existence, uniqueness of weak solutions and stabilization, NoDEA Nonlinear Differ. Equ. Appl. 23 (2016).
[16] V.V. Grushin, On a class of hypoelliptic operators, Math. USSR Sb. 12 (1970) 458–476.
[17] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, V. Zizler, Biorthogonal Systems in Banach Spaces, Springer, New York, 2008.
[18] J. Liu, P. Pucci, H. Wu, Q. Zhang, Existence and blow-up rate of large solutions of $p(x)$-Laplacian equations with gradient terms, J. Math. Anal. Appl. 457 (1) (2018) 944–977.
[19] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 3 (1986) 391–409.
[20] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q–growth conditions, J. Differ. Equ. 90 (1991) 1–30.
[21] P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. 501 (1) (2021) 124408.

[22] G. Mingione, V.D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, J. Math. Anal. Appl. 501 (1) (2021) 125197.
[23] J. Musielak, Orlicz Spaces and Modular Spaces, Springer-Verlag, Berlin, 1983.
[24] N.S. Papageorgiou, P. Winkert, Applied Nonlinear Functional Analysis, De Gruyter, Berlin, 2018.
[25] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear singular problems with indefinite potential term, Anal. Math. Phys. 9 (4) (2019) 2237–2262.
[26] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Double-phase problems with reaction of arbitrary growth, Z. Angew. Math. Phys. 69 (4) (2018) 108.
[27] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Double-phase problems and a discontinuity property of the spectrum, Proc. Am. Math. Soc. 147 (2019) 2899–2910.
[28] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis – Theory and Methods, Springer Monographs in Mathematics, Springer, Cham, 2019.
[29] P. Pucci, L. Temperini, Existence for fractional (p, q)-systems with critical and Hardy terms in $\mathbb{R}^N$, Nonlinear Anal. 211 (2021) 112477.
[30] V.D. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, Nonlinear Anal. 121 (2015) 336–369.
[31] V.D. Rădulescu, Isotropic and anisotropic double-phase problems: old and new, Opusc. Math. 39 (2019) 259–279.
[32] V.D. Rădulescu, D.D. Repovš, Partial Differential Equations with Variable Exponents, CRC Press, Boca Raton, FL, 2015.
[33] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, in: Journées d'Analyse Non Linéaire, Proc. Conf., Besançon, 1977, in: Lecture Notes in Math., vol. 665, Springer, Berlin, 1978, pp. 205–227.
[34] Q. Zhang, A strong maximum principle for differential equations with nonstandard p(x)-growth conditions, J. Math. Anal. Appl. 312 (2005) 125–143.
[35] Q. Zhang, V.D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, J. Math. Pures Appl. (9) 118 (2018) 159–203.