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CLASSIFYING HOMOGENEOUS CELLULAR ORDINAL BALLEANS UP TO COARSE EQUIVALENCE

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Abstract. For every ballean X, we introduce two cardinal characteristics $\mathrm{cov}^\flat(X)$ and $\mathrm{cov}^\sharp(X)$ describing the capacity of balls in X. We observe that these characteristics are invariant under coarse equivalence and prove that two cellular ordinal balleans X, Y are coarsely equivalent if $\mathrm{cof}(X) = \mathrm{cof}(Y)$ and $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X) = \mathrm{cov}^\flat(Y) = \mathrm{cov}^\sharp(Y)$. This implies that a cellular ordinal ballean X is homogeneous if and only if $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X)$. Moreover, two homogeneous cellular ordinal balleans X, Y are coarsely equivalent if and only if $\mathrm{cof}(X) = \mathrm{cof}(Y)$ and $\mathrm{cov}^\sharp(X) = \mathrm{cov}^\sharp(Y)$ if and only if each of these balleans coarsely embeds into the other. This means that the coarse structure of a homogeneous cellular ordinal ballean X is fully determined by the values of $\mathrm{cof}(X)$ and $\mathrm{cov}^\sharp(X)$. For every limit ordinal γ, we define a ballean $2^{<\gamma}$ (called the Cantor macro-cube) that, in the class of cellular ordinal balleans of cofinality $\mathrm{cf}(\gamma)$, plays a role analogous to the role of the Cantor cube 2^κ in the class of zero-dimensional compact Hausdorff spaces. We also characterize balleans which are coarsely equivalent to $2^{<\gamma}$. This can be considered as an asymptotic analogue of Brouwer's characterization of the Cantor cube 2^ω.

Introduction. In this paper we study the structure of ordinal balleans, i.e., balleans that have well-ordered base of their coarse structure. Such balleans were introduced by Protasov [10]. Some basic facts about ordinal balleans are discussed in Section 1. The main result of the paper, presented in Section 2, is a criterion for recognizing coarsely equivalent cellular ordinal balleans. In Section 3, we shall use this criterion to classify homogeneous cellular ordinal balleans up to coarse equivalence. In Section 4, we apply this criterion to characterize balleans $2^{<\gamma}$ (called Cantor macro-cubes) that are universal objects in the class of cellular ordinal balleans. In Section 4, we also identify the natural coarse structure on additively indecomposable ordinals.

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1. Ordinal balleans. The notion of a ballean was introduced by Protasov [11] as a large scale counterpart of a uniform space, and it is a modification of the notion of a coarse space introduced by Roe [15]. Both notions are defined as sets endowed with certain families of entourages.

By an *entourage* on a set X we understand any reflexive symmetric relation $\varepsilon \subset X \times X$. This means that ε contains the diagonal $\Delta_X = \{(x,y) \in X \times X : x = y\}$ of $X \times X$, and it is symmetric in the sense that $\varepsilon = \varepsilon^{-1}$ where $\varepsilon^{-1} := \{(y,x) \in X \times X : (x,y) \in \varepsilon\}$. An entourage $\varepsilon \subset X \times X$ will be called *cellular* if it is transitive, i.e., it is an equivalence relation on X.

Each entourage $\varepsilon \subset X \times X$ determines a cover $\{B(x,\varepsilon) : x \in X\}$ of X by ε-balls $B(x,\varepsilon) := \{y \in X : (x,y) \in \varepsilon\}$. It follows that $\varepsilon = \bigcup_{x\in X}\{x\} \times B(x,\varepsilon) = \bigcup_{x\in X} B(x,\varepsilon) \times \{x\}$. For a subset $A \subset X$ we let $B(A,\varepsilon) := \bigcup_{a\in A} B(a,\varepsilon)$ denote the ε-neighborhood of A.

A *ballean* is a pair $(X, \mathcal{E}_X)$ consisting of a set X and a family $\mathcal{E}_X$ of entourages on X (called the *set of radii*) such that $\bigcup \mathcal{E}_X = X \times X$ and for any entourages $\varepsilon, \delta \in \mathcal{E}_X$ their composition

$$\varepsilon \circ \delta = \{(x,z) \in X \times X : \exists y \in X \ (x,y) \in \varepsilon,\ (y,z) \in \delta\}$$

is contained in some entourage $\eta \in \mathcal{E}_X$. A ballean $(X, \mathcal{E}_X)$ is called a *coarse space* if the family $\mathcal{E}_X$ is closed under taking subentourages, i.e., for any $\varepsilon \in \mathcal{E}_X$ any entourage $\delta \subset \varepsilon$ belongs to $\mathcal{E}_X$. In this case, the set of radii $\mathcal{E}_X$ is called a *coarse structure* on X. A subfamily $\mathcal{B} \subset \mathcal{E}_X$ is called a *base* of a coarse structure $\mathcal{E}_X$ if each entourage $\varepsilon \in \mathcal{E}_X$ is contained in some entourage $\delta \in \mathcal{B}$. It follows that the set of radii $\mathcal{E}_X$ of a ballean $(X, \mathcal{E}_X)$ is a base of a unique coarse structure ${\downarrow}\mathcal{E}_X$ (consisting of all possible subentourages $\delta \subset \varepsilon \in \mathcal{E}_X$). So, balleans can be considered as coarse spaces with a fixed base of their coarse structure. Coarse spaces and coarse structures were introduced by Roe [15].

Each subset $A \subset X$ of a ballean $(X, \mathcal{E}_X)$ carries the induced ballean structure $\mathcal{E}_A = \{\varepsilon \cap (A \times A) : \varepsilon \in \mathcal{E}_X\}$. The ballean $(A, \mathcal{E}_A)$ will be called a *subballean* of $(X, \mathcal{E}_X)$.

By definition, the *cofinality* $\mathrm{cof}(\mathbf{X})$ of a ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is equal to the smallest cardinality of a base of the coarse structure ${\downarrow}\mathcal{E}_X$. We identify cardinals with the smallest ordinals of the given cardinality.

Now we consider some examples of balleans.

Example 1.1. Every metric space (X,d) carries a canonical ballean structure $\mathcal{E}_X = \{\Delta_\varepsilon\}_{\varepsilon\in\mathbb{R}_+}$ consisting of the entourages

$$\Delta_\varepsilon = \{(x,y) \in X \times X : d(x,y) \le \varepsilon\}$$

parametrized by $\mathbb{R}_+ = [0,\infty)$. The ballean structure $\mathcal{E}_X = \{\Delta_\varepsilon\}_{\varepsilon\in\mathbb{R}_+}$ generates the coarse structure ${\downarrow}\mathcal{E}_X$ consisting of all subentourages of the entourages Δ_ε, $\varepsilon \in \mathbb{R}_+$.

A ballean $\mathbf{X}$ is called *metrizable* if its coarse structure is generated by a suitable metric. Metrizable balleans belong to the class of ordinal balleans. A ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is defined to be *ordinal* if its coarse structure $\downarrow\mathcal{E}_X$ has a well-ordered base $\mathcal{B} \subset \mathcal{E}_X$. The latter means that $\mathcal{B}$ can be enumerated as $\{\varepsilon_\alpha\}_{\alpha<\kappa}$ for some ordinal κ such that $\varepsilon_\alpha \subset \varepsilon_\beta$ for $\alpha < \beta < \kappa$. Passing to a cofinal subset of κ, we can always assume that κ is a regular cardinal, equal to the cofinality $\mathrm{cof}(\mathbf{X})$ of $\mathbf{X} = (X, \mathcal{E}_X)$ (i.e., the smallest cardinality of a base of the coarse structure $\downarrow\mathcal{E}_X$).

Ordinal balleans can be characterized as balleans $\mathbf{X} = (X, \mathcal{E}_X)$ whose cofinality equals the *additivity number*

$$\mathrm{add}(\mathbf{X}) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subset \downarrow\mathcal{E}_X,\ \bigcup\mathcal{A} \notin \downarrow\mathcal{E}_X \setminus \{X \times X\}\Big\}.$$

PROPOSITION 1.2. *A ballean* $\mathbf{X}$ *is ordinal if and only if* $\mathrm{cof}(\mathbf{X}) = \mathrm{add}(\mathbf{X})$.

Proof. Assuming that a ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is ordinal, fix a well-ordered base $\{\varepsilon_\alpha\}_{\alpha<\kappa}$ of the coarse structure $\downarrow\mathcal{E}_X$. Passing to a cofinal subsequence, we can assume that $\kappa = \mathrm{cf}(\kappa)$ is a regular cardinal. If $\kappa = 1$, then the ballean $(X, \mathcal{E}_X)$ is bounded, and hence for the entourage $X \times X \in \downarrow\mathcal{E}_X$ the family $\mathcal{A} = \{X \times X\}$ has cardinality $|\mathcal{A}| = 1$ and $\bigcup\mathcal{A} = X \times X \notin \mathcal{E}_X \setminus \{X \times X\}$. Therefore, $\mathrm{add}(\mathbf{X}) = 1 = \mathrm{cof}(\mathbf{X})$. So, we assume that κ is infinite, and hence $\varepsilon_\alpha \neq X \times X$ for all $\alpha < \kappa$. Since $\mathrm{add}(\mathbf{X}) \le \mathrm{cof}(\mathbf{X})$, it suffices to check that $\mathrm{cof}(\mathbf{X}) \le \mathrm{add}(\mathbf{X})$. The definition of $\mathrm{cof}(\mathbf{X})$ implies that $\mathrm{cof}(\mathbf{X}) \le \kappa$. The inequality $\mathrm{add}(\mathbf{X}) \ge \kappa \ge \mathrm{cof}(\mathbf{X})$ will follow as soon as we check that for any family $\mathcal{A} \subset \mathcal{E}_X$ with $|\mathcal{A}| < \kappa$ we have $\bigcup\mathcal{A} \in \downarrow\mathcal{E}_X \setminus \{X \times X\}$. For every $A \in \mathcal{A}$ find an ordinal $\alpha_A < \kappa$ such that $A \subset \varepsilon_{\alpha_A}$. By the regularity of κ, the cardinal $\beta = \sup\{\alpha_A : A \in \mathcal{A}\}$ is strictly smaller than κ. Consequently, $A \subset \varepsilon_{\alpha_A} \subset \varepsilon_\beta$ for every $A \in \mathcal{A}$, and hence $\bigcup\mathcal{A} \subset \varepsilon_\beta$ and $\bigcup\mathcal{A} \in \downarrow\mathcal{E}_X \setminus \{X \times X\}$. This completes the proof of $\mathrm{add}(\mathbf{X}) = \mathrm{cof}(\mathbf{X})$ for ordinal balleans.

Now we shall prove that a ballean $\mathbf{X} = (X, \mathcal{E}_X)$ is ordinal if $\mathrm{add}(\mathbf{X}) = \mathrm{cof}(\mathbf{X})$. Fix any base $\{\varepsilon_\alpha\}_{\alpha<\mathrm{cof}(\mathbf{X})} \subset \downarrow\mathcal{E}_X$ of the coarse structure $\downarrow\mathcal{E}_X$. By definition of $\mathrm{add}(\mathbf{X})$, for every $\alpha < \mathrm{cof}(\mathbf{X}) = \mathrm{add}(\mathbf{X})$, the union $\tilde{\varepsilon}_\alpha = \bigcup_{\beta\le\alpha}\varepsilon_\beta$ belongs to the coarse structure $\downarrow\mathcal{E}_X$. Then $\{\tilde{\varepsilon}_\alpha\}_{\alpha<\mathrm{cof}(\mathbf{X})}$ is a well-ordered base of $\downarrow\mathcal{E}_X$, which means that $\mathbf{X}$ is ordinal. ∎

An important property of ordinal balleans of uncountable cofinality is their cellularity. A ballean $(X, \mathcal{E}_X)$ is called *cellular* if its coarse structure $\downarrow\mathcal{E}_X$ has a base consisting of cellular entourages (i.e., equivalence relations). It can be shown that a ballean $(X, \mathcal{E}_X)$ is cellular if and only if for every $\varepsilon \in \mathcal{E}_X$ the cellular entourage $\varepsilon^{<\omega} = \bigcup_{n\in\omega}\varepsilon^n$ belongs to the coarse structure $\downarrow\mathcal{E}_X$. Here $\varepsilon^0 = \Delta_X$ and $\varepsilon^{n+1} = \varepsilon^n \circ \varepsilon$ for all $n \in \omega$. This characterization implies the following simple fact.

PROPOSITION 1.3. *Every ordinal ballean* $\mathbf{X} = (X, \mathcal{E}_X)$ *with* $\mathrm{cof}(\mathbf{X})$ *uncountable is cellular.*

REMARK 1.4. By [14, Theorem 3.1.3], a ballean X is cellular if and only if it has asymptotic dimension zero. A metrizable ballean X is cellular if and only if its coarse structure is generated by an *ultrametric* (i.e., a metric d satisfying $d(x,z) \le \max\{d(x,y), d(y,z)\}$ for all $x, y, z \in X$). More information on cellular balleans can be found in [14, Chapter 3]. For information on spaces of asymptotic dimension zero, see [4].

EXAMPLE 1.5. Let X be a set and γ be an ordinal identified with the set $[0,\gamma)$ of smaller ordinals. A function $d : X \times X \to [0,\gamma)$ is called a γ-*ultrametric* if

$$d(x,x) = 0, \quad d(x,y) = d(y,x) \quad \text{and} \quad d(x,z) \le \max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in X$. The γ-ultrametric d induces the cellular ballean structure $\mathcal{E}_d = \{\varDelta_\alpha\}_{\alpha<\gamma}$ consisting of the entourages $\varDelta_\alpha = \{(x,y) \in X \times X : d(x,y) \le \alpha\}$ for $\alpha < \gamma$.

EXAMPLE 1.6. Let λ, γ be ordinals and $f : [0,\lambda) \to [0,\gamma)$ be a function such that $\sup f([0,\alpha]) < \gamma$ for all ordinals $\alpha < \lambda$. The map f determines a γ-ultrametric d_f on $[0,\lambda)$ defined by $d_f(x,y) = d_f(y,x) = \sup f\big((x,y]\big)$ for all ordinals $x < y < \gamma$. The γ-ultrametric space $([0,\lambda), d_f)$ is called the γ-*comb* determined by f (see [12]).

EXAMPLE 1.7. Every infinite cardinal κ carries a natural ballean structure $\mathcal{E}_\kappa = \{\varepsilon_\alpha\}_{\alpha<\kappa}$ consisting of the entourages

$$\varepsilon_\alpha = \{(x,y) \in \kappa \times \kappa : x \le y + \alpha,\ y \le x + \alpha\}$$

parametrized by ordinals $\alpha < \kappa$. The resulting ordinal ballean $(\kappa, \mathcal{E}_\kappa)$ will be denoted by $\overset{\leftrightarrow}{\kappa}$. The cardinal balleans $\overset{\leftrightarrow}{\kappa}$ were introduced in [8]. By [8, Theorem 3], the ballean $\overset{\leftrightarrow}{\kappa}$ is cellular for any uncountable cardinal κ.

EXAMPLE 1.8. Given an ordinal γ and a transfinite sequence $(\kappa_\alpha)_{\alpha<\gamma}$ of non-zero cardinals, consider the ballean

$$\coprod_{\alpha\in\gamma} \kappa_\alpha = \Big\{ (x_\alpha)_{\alpha\in\gamma} \in \prod_{\alpha\in\gamma} \kappa_\alpha : |\{\alpha \in \gamma : x_\alpha \ne 0\}| < \omega \Big\}$$

endowed with the ballean structure $\{\varepsilon_\beta\}_{\beta<\gamma}$ consisting of the entourages

$$\varepsilon_\beta = \Big\{ ((x_\alpha)_{\alpha\in\gamma}, (y_\alpha)_{\alpha\in\gamma}) \in \Big(\coprod_{\alpha<\gamma} \kappa_\alpha\Big)^2 : \forall \alpha > \beta\ (x_\alpha = y_\alpha) \Big\} \quad \text{for } \beta < \gamma.$$

The ballean $\coprod_{\alpha\in\gamma} \kappa_\alpha$ is called the *asymptotic product* of the cardinals κ_α, $\alpha \in \gamma$. It is a cellular ordinal ballean whose cofinality equals $\mathrm{cf}(\gamma)$, the cofinality of the ordinal γ.

If all cardinals κ_α, $\alpha \in \gamma$, are equal to a fixed cardinal κ, then $\coprod_{\alpha\in\gamma} \kappa_\alpha$ will be denoted by $\kappa^{<\gamma}$. For a limit ordinal γ the ballean $2^{<\gamma}$ is called a *Cantor macro-cube*. The Cantor macro-cube $2^{<\omega}$ was characterized in [3]. This characterization will be extended to all Cantor macro-cubes in Theorem 4.3.

Balleans are objects of the (coarse) category whose morphisms are coarse maps. A map $f\colon X \to Y$ between two balleans $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ is called *coarse* if for each $\varepsilon \in \mathcal{E}_X$ there is $\delta \in \mathcal{E}_Y$ such that $\{(f(x), f(y)) : (x, y) \in \varepsilon\} \subset \delta$. A map $f\colon X \to Y$ is called a *coarse isomorphism* if f is bijective and both f and f^{-1} are coarse. In this case the balleans $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are called *coarsely isomorphic*. It follows that each ballean $(X, \mathcal{E}_X)$ is coarsely isomorphic to the coarse space $(X, {\downarrow}\mathcal{E}_X)$.

Coarse isomorphisms play the role of isomorphisms in the coarse category (whose objects are balleans and whose morphisms are coarse maps). Probably a more important notion is that of coarse equivalence of balleans. Two balleans $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ are *coarsely equivalent* if they contain coarsely isomorphic large subspaces $L_X \subset X$ and $L_Y \subset Y$. A subset L of a ballean $(X, \mathcal{E}_X)$ is called *large* if $X = B(L, \varepsilon)$ for some entourage $\varepsilon \in \mathcal{E}_X$.

Coarse equivalences can be alternatively defined using multi-valued maps. By a multi-valued map (briefly, a *multi-map*) $\Phi\colon X \multimap Y$ between two sets X, Y we understand any subset $\Phi \subset X \times Y$. For a subset $A \subset X$, the *image* of A under Φ is $\Phi(A) := \{y \in Y : \exists a \in A\ (a, y) \in \Phi\}$. Given $x \in X$ we write $\Phi(x)$ instead of $\Phi(\{x\})$.

The inverse $\Phi^{-1}\colon Y \multimap X$ of the multi-map $\Phi\colon X \multimap Y$ is the multi-map

$$\Phi^{-1} := \{(y, x) \in Y \times X : (x, y) \in \Phi\} \subset Y \times X$$

assigning to each $y \in Y$ the set $\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}$. For two multi-maps $\Phi\colon X \multimap Y$ and $\Psi\colon Y \multimap Z$ we define their composition $\Psi \circ \Phi\colon X \multimap Z$ as usual:

$$\Psi \circ \Phi = \{(x, z) \in X \times Z : \exists y \in Y\ [(x, y) \in \Phi \text{ and } (y, z) \in \Psi]\}.$$

A multi-map $\Phi\colon X \multimap Y$ between two balleans $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ is called *coarse* if for every $\varepsilon \in \mathcal{E}_X$ there is a $\delta \in \mathcal{E}_Y$ containing the set $\omega_\Phi(\varepsilon) := \bigcup_{(x,y)\in\varepsilon} \Phi(x) \times \Phi(y)$ called the *ε-oscillation* of Φ. More precisely, for a function $\varphi\colon \mathcal{E}_X \to \mathcal{E}_Y$, a multi-map $\Phi\colon X \multimap Y$ is defined to be *φ-coarse* if $\omega_\Phi(\varepsilon) \subset \varphi(\varepsilon)$ for every $\varepsilon \in \mathcal{E}_X$. So, $\Phi\colon X \multimap Y$ is coarse if and only if it is φ-coarse for some $\varphi\colon \mathcal{E}_X \to \mathcal{E}_Y$. It follows that a (single-valued) map $f\colon X \to Y$ is coarse if and only if it is coarse as a multi-map.

A multi-map $\Phi\colon X \multimap Y$ between two balleans is called a *coarse embedding* if $\Phi^{-1}(Y) = X$ and both Φ and Φ^{-1} are coarse. If in addition $\Phi(X) = Y$, then $\Phi\colon X \multimap Y$ is called a *coarse equivalence*. By analogy with [3, proof of Proposition 2.1], it can be shown that two balleans X, Y are coarsely equivalent if and only if there is a coarse equivalence $\Phi\colon X \multimap Y$.

The study of balleans (or coarse spaces) up to coarse equivalence is one of the principal tasks of Coarse Geometry [5], [6], [14], [15].

EXAMPLE 1.9. By [12], each ordinal cellular ballean $(X, \mathcal{E}_X)$ is coarsely isomorphic to some γ-comb $([0,\lambda), d_f)$.

EXAMPLE 1.10. Let G be a group. An ideal $\mathcal{I}$ in the Boolean algebra of all subsets of G is called a *group ideal* if $G = \bigcup \mathcal{I}$ and if for any $A, B \in \mathcal{I}$ we get $AB^{-1} \in \mathcal{I}$.

Let $\mathcal{I}$ be a group ideal $\mathcal{I}$ on a group G and X be a transitive G-space endowed with an action $G \times X \to X$ of the group G. The G-space X carries the ballean structure $\mathcal{E}_{X,G,\mathcal{I}} = \{\varepsilon_A\}_{A\in\mathcal{I}}$ consisting of the entourages $\varepsilon_A = \{(x,y) \in X : x \in (A \cup \{1_G\} \cup A^{-1}) \cdot y\}$ parametrized by sets $A \in \mathcal{I}$. Here by 1_G we denote the unit of the group G.

By [7, Theorems 1 and 3], every (cellular) ballean $(X, \mathcal{E}_X)$ is coarsely isomorphic to the ballean $(X, \mathcal{E}_{X,G,\mathcal{I}})$ for a suitable group G of permutations of X and a suitable group ideal $\mathcal{I}$ of G (having a base consisting of subgroups of G).

EXAMPLE 1.11. Let G be a group endowed with the ballean $\mathcal{E}_G$ consisting of the entourages $\varepsilon_F = \{(x,y) \in G \times G : xy^{-1} \in F\}$ parametrized by finite subsets $F = F^{-1} \subset G$ containing 1_G. By [11, 9.8] the ballean $(G, \mathcal{E}_G)$ is cellular if and only if the group G is locally finite (in the sense that each finite subset of G is contained in a finite subgroup of G). By [3], any two infinite countable locally finite groups G, H are coarsely equivalent. On the other hand, by [9], two countable locally finite groups G, H are coarsely isomorphic if and only if $\phi_G = \phi_H$. Here $\phi_G \colon \Pi \to \omega \cup \{\omega\}$ is the *factorizing function* of G. It is defined on the set Π of prime numbers and assigns to each $p \in \Pi$ the (finite or infinite) number

$$\phi_G(p) = \sup\{k \in \omega : G \text{ contains a subgroup of cardinality } p^k\}.$$

2. A criterion for coarse equivalence of cellular ordinal balleans. In this section we introduce two cardinal characteristics called covering numbers of a ballean, and using them we give a criterion for coarse equivalence of two cellular ordinal balleans.

Given $A \subset X$ and an entourage $\varepsilon \subset X \times X$ consider the cardinal

$$\mathrm{cov}_\varepsilon(A) := \min\{|C| : C \subset X,\ A \subset B(C,\varepsilon)\},$$

equal to the smallest number of ε-balls covering A.

For every ballean $(X, \mathcal{E}_X)$ consider the following cardinals:

- $\mathrm{cov}^\sharp(X, \mathcal{E}_X)$, the smallest cardinal κ for which there is an $\varepsilon \in \mathcal{E}_X$ such that $\sup_{x\in X} \mathrm{cov}_\varepsilon(B(x,\delta)) < \kappa$ for every $\delta \in \mathcal{E}_X$;
- $\mathrm{cov}^\flat(X, \mathcal{E}_X)$, the largest cardinal κ such that for any cardinal $\lambda < \kappa$ and $\varepsilon \in \mathcal{E}_X$ there is $\delta \in \mathcal{E}_X$ such that $\min_{x\in X} \mathrm{cov}_\varepsilon(B(x,\delta)) \geq \lambda$.

It follows that

$$\operatorname{cov}^\sharp(X) = \min_{\varepsilon\in\mathcal{E}_X} \sup_{\delta\in\mathcal{E}_X} \big(\sup_{x\in X} \operatorname{cov}_\varepsilon(B(x,\delta))\big)^+,$$
$$\operatorname{cov}^\flat(X) = \min_{\varepsilon\in\mathcal{E}_X} \sup_{\delta\in\mathcal{E}_X} \big(\min_{x\in X} \operatorname{cov}_\varepsilon(B(x,\delta))\big)^+,$$

where κ^+ denotes the smallest cardinal which is larger than κ.

The following proposition can be proved by analogy with [1, proof of Lemmas 3.1 and 3.2].

Proposition 2.1. *If a ballean X coarsely embeds into a ballean Y, then* $\operatorname{cov}^\sharp(X) \le \operatorname{cov}^\sharp(Y)$. *If balleans X, Y are coarsely equivalent, then* $\operatorname{cov}^\flat(X) = \operatorname{cov}^\flat(Y)$ *and* $\operatorname{cov}^\sharp(X) = \operatorname{cov}^\sharp(Y)$.

Observe that $\operatorname{cov}^\sharp(X) \le \omega$ means that the ballean X has bounded geometry, while $\operatorname{cov}^\flat(X) \ge \omega$ means that X has no isolated balls (see [2]). By [3], any two metrizable cellular balleans of bounded geometry and without isolated balls are coarsely equivalent. In [1] this result was extended to the following criterion: two metrizable cellular balleans X, Y are coarsely equivalent if $\operatorname{cov}^\flat(X) = \operatorname{cov}^\sharp(X) = \operatorname{cov}^\sharp(Y) = \operatorname{cov}^\flat(Y)$. In this paper we further extend this criterion to cellular ordinal balleans and prove the following main result.

Theorem 2.2. *Let X, Y be cellular ordinal balleans with* $\operatorname{cof}(X)=\operatorname{cof}(Y)$.

(1) *If* $\operatorname{cov}^\sharp(X) \le \operatorname{cov}^\flat(Y)$, *then X is coarsely equivalent to a subballean of Y.*
(2) *If* $\operatorname{cov}^\flat(X) = \operatorname{cov}^\sharp(X) = \operatorname{cov}^\sharp(Y) = \operatorname{cov}^\flat(Y)$, *then X and Y are coarsely equivalent.*

The proof will be presented in Section 6. First we shall discuss some applications of this theorem.

3. Classifying homogeneous cellular ordinal balleans. In this section we shall apply Theorem 2.2 to show that for a cellular ordinal ballean X the equality $\operatorname{cov}^\flat(X) = \operatorname{cov}^\sharp(X)$ is equivalent to the homogeneity of X, defined as follows.

A ballean $(X, \mathcal{E}_X)$ is called *homogeneous* if there is a function $\varphi\colon \mathcal{E}_X \to \mathcal{E}_X$ such that for any $x, y \in X$ there is a coarse equivalence $\Phi\colon X \multimap X$ such that $y \in \Phi(x)$ and both Φ and Φ^{-1} are φ-coarse. Recall that $\Phi\colon X \multimap X$ is φ-coarse if $\omega_\Phi(\varepsilon) := \bigcup_{(x,y)\in\varepsilon} \Phi(x) \times \Phi(x) \subset \varphi(\varepsilon)$ for every $\varepsilon \in \mathcal{E}_X$.

The following proposition shows that homogeneity is preserved by coarse equivalence.

Proposition 3.1. *A ballean X is homogeneous if and only if it is coarsely equivalent to a homogeneous ballean Y.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that a ballean $(X,\mathcal{E}_X)$ admits a coarse equivalence $\Phi\colon X \multimap Y$ with a homogeneous ballean $(Y,\mathcal{E}_Y)$. By the homogeneity of $(Y,\mathcal{E}_Y)$, there is a function $\varphi_Y : \mathcal{E}_Y \to \mathcal{E}_Y$ such that for any $y,y' \in Y$ there is a coarse equivalence $\Psi\colon Y \multimap Y$ such that $y' \in \Psi(y)$ and both Ψ and Ψ^{-1} are φ_Y-coarse. Since Φ is a coarse equivalence, there are functions $\varphi_{X,Y} : \mathcal{E}_X \to \mathcal{E}_Y$ and $\varphi_{Y,X} : \mathcal{E}_Y \to \mathcal{E}_X$ such that $\omega_\Phi(\varepsilon) \subset \varphi_{X,Y}(\varepsilon)$ and $\omega_{\Phi^{-1}}(\delta) \subset \varphi_{Y,X}(\delta)$ for all $\varepsilon \in \mathcal{E}_X$ and $\delta \in \mathcal{E}_Y$. We claim that the function

$$\varphi_X := \varphi_{Y,X} \circ \varphi_Y \circ \varphi_{X,Y} : \mathcal{E}_X \to \mathcal{E}_X$$

witnesses that X is homogeneous. Indeed, given any points x,x', we can choose $y \in \Phi(x)$, $y' \in \Phi(x')$ and find a coarse equivalence $\Psi_Y : Y \to Y$ such that $y' \in \Psi_Y(y)$ and both Ψ_Y and Ψ_Y^{-1} are φ_Y-coarse. It can be shown that the multi-map $\Psi_X := \Phi^{-1} \circ \Psi_Y \circ \Phi\colon X \multimap X$ has the desired properties: $x' \in \Phi^{-1}(y') \subset \Phi^{-1}(\Psi_Y(y)) \subset \Phi^{-1}(\Psi_Y(\Phi(x))) = \Phi_X(x)$ and $\omega_{\Phi_X}(\varepsilon) \cup \omega_{\Phi_X^{-1}}(\varepsilon) \subset \varphi_X(\varepsilon)$ for all $\varepsilon \in \mathcal{E}_X$. ∎

PROPOSITION 3.2. *If a ballean X is homogeneous, then* $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X)$.

Proof. Since $\mathrm{cov}^\flat(X) \le \mathrm{cov}^\sharp(X)$, it suffices to check that $\mathrm{cov}^\flat(X) \ge \mathrm{cov}^\sharp(X)$. This will follow as soon as for given $\varepsilon \in \mathcal{E}_X$ and $\kappa < \mathrm{cov}^\sharp(X)$, we find $\delta \in \mathcal{E}_X$ such that $\min_{x\in X} \mathrm{cov}_\varepsilon(B(x,\delta)) \ge \kappa$. By the homogeneity of X, there is a function $\varphi\colon \mathcal{E}_X \to \mathcal{E}_X$ such that for any $x,y \in X$ there is a coarse equivalence $\Phi\colon X \multimap X$ such that $y \in \Phi(x)$ and both Φ and Φ^{-1} are φ-coarse.

By the definition of $\mathrm{cov}^\sharp(X) > \kappa$, for $\varepsilon' = \varphi(\varepsilon)$, there are $\delta' \in \mathcal{E}_X$ and $x' \in X$ such that $\mathrm{cov}_{\varepsilon'}(B(x',\delta')) \ge \kappa$. We claim that $\delta = \varphi(\delta') \in \mathcal{E}_X$ has the required property: $\min_{x\in X} \mathrm{cov}_\varepsilon(B(x,\delta)) \ge \kappa$.

Assume conversely that $\mathrm{cov}_\varepsilon(B(x,\delta)) < \kappa$ for some $x \in X$. By the homogeneity of X and the choice of φ, there is a coarse equivalence $\Phi\colon X \multimap X$ such that $x' \in \Phi(x)$ and both Φ and Φ^{-1} are φ-coarse. Since $\mathrm{cov}_\varepsilon(B(x,\delta)) < \kappa$, there is a subset $C \subset X$ with $|C| < \kappa$ such that $B(x,\delta) \subset \bigcup_{c\in C} B(c,\varepsilon)$. For every $c \in C$ fix $y_c \in \Phi(c)$. Observe that for every $b \in B(c,\varepsilon)$ we have $(b,c) \in \varepsilon$. Therefore $\Phi(b) \times \Phi(c) \subset \omega_\Phi(\varepsilon) \subset \varphi(\varepsilon)$ and $\Phi(b) \subset B(y_c,\varphi(\varepsilon)) = B(y_c,\varepsilon')$, which implies $\Phi(B(c,\varepsilon)) \subset B(y_c,\varepsilon')$. Taking into account that $B(x,\delta) \subset \bigcup_{c\in C} B(c,\varepsilon)$, we get $\Phi(B(x,\delta)) \subset \bigcup_{c\in C} \Phi(B(c,\varepsilon)) \subset \bigcup_{c\in C} B(y_c,\varepsilon')$, which implies $\mathrm{cov}_{\varepsilon'}(\Phi(B(x,\delta))) \le |C| < \kappa$.

We claim that $B(x',\delta') \subset \Phi(B(x,\delta))$. Indeed, for any $y' \in B(x',\delta')$ we can fix $y \in \Phi^{-1}(x')$ and observe that $(y',x') \in \delta'$ implies $(y,x) \in \Phi^{-1}(y') \times \Phi^{-1}(x') \subset \omega_{\Phi^{-1}}(\delta') \subset \varphi(\delta') = \delta$ (since Φ^{-1} is φ-coarse). Then $y \in B(x,\delta)$ and $y' \in \Phi(y) \subset \Phi(B(x,\delta))$. Finally, we get $B(x',\delta') \subset \Phi(B(x,\delta))$ and $\mathrm{cov}_{\varepsilon'}(B(x',\delta')) \le \mathrm{cov}_{\varepsilon'}(\Phi(B(x,\delta))) \le |C| < \kappa$, which contradicts the choice of δ' and x'. ∎

THEOREM 3.3. *A cellular ordinal ballean X is homogeneous if and only if* $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X)$.

Proof. The "only if" part follows from Proposition 3.2. To prove the "if" part, assume X is a cellular ordinal ballean with $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X)$. Let $\gamma = \mathrm{cof}(X) = \mathrm{add}(X)$. The definition of $\kappa = \mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X)$ implies that there exists a non-decreasing transfinite sequence $(\kappa_\alpha)_{\alpha<\gamma}$ of cardinals such that $\kappa = \sup_{\alpha<\gamma} \kappa_\alpha^+$. Choose an increasing transfinite sequence $(G_\alpha)_{\alpha<\gamma}$ of groups such that $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for every limit ordinal $\alpha < \gamma$ and $|G_{\alpha+1}/G_\alpha| = \kappa_\alpha$ for every ordinal $\alpha < \gamma$.

Consider the group $G = \bigcup_{\alpha<\gamma} G_\alpha$ endowed with the ballean structure $\mathcal{E}_G = (\varepsilon_\alpha)_{\alpha<\gamma}$ consisting of the entourages

$$\varepsilon_\alpha := \{(x, y) \in G : x^{-1}y \in G_\alpha\} \quad \text{for } \alpha < \gamma.$$

It is clear that the left shifts are id-coarse isomorphisms of $(G, \mathcal{E}_G)$, which implies that the ballean $(G, \mathcal{E}_G)$ is homogeneous. It is clear that $\mathrm{add}(G, \mathcal{E}_G) = \mathrm{cof}(G, \mathcal{E}_G) = \gamma$ and

$$\mathrm{cov}^\flat(G, \mathcal{E}_G) = \mathrm{cov}^\sharp(G, \mathcal{E}_G) = \min_{\alpha<\gamma} \sup_{\alpha\le\beta<\gamma} |G_\beta/G_\alpha|^+ = \sup_{\alpha<\gamma} \kappa_\alpha^+ = \kappa.$$

Applying Theorem 2.2, we conclude that X is coarsely equivalent to the homogeneous ballean $(G, \mathcal{E}_G)$, and hence X is homogeneous according to Proposition 3.1. ∎

The following corollary of Theorems 2.2 and 3.3 shows that the cardinals $\mathrm{cof}(X)$ and $\mathrm{cov}^\sharp(X)$ fully determine the coarse structure of a homogeneous cellular ordinal ballean X.

THEOREM 3.4. *For any homogeneous cellular ordinal balleans X, Y the following conditions are equivalent:*

(1) *X and Y are coarsely equivalent;*
(2) *X is coarsely equivalent to a subspace of Y and vice versa;*
(3) $\mathrm{cof}(X) = \mathrm{cof}(Y)$ *and* $\mathrm{cov}^\sharp(X) = \mathrm{cov}^\sharp(Y)$.

Proof. The implication (1)⇒(2) is trivial, (2)⇒(3) follows by the invariance of cof and $\mathrm{cov}^\sharp$ under coarse equivalence and their monotonicity under taking subspaces, and (3)⇒(1) follows from Theorems 2.2 and 3.3. ∎

4. Recognizing the coarse structure of Cantor macro-cubes and cardinal balleans. It is easy to see that for any ordinal γ and transfinite sequence $(\kappa_\alpha)_{\alpha\in\gamma}$ of non-zero cardinals the asymptotic product $\coprod_{\alpha\in\gamma} \kappa_\alpha$ is a homogeneous cellular ordinal ballean whose cofinality equals $\mathrm{cf}(\gamma)$, the cofinality of the ordinal γ. In particular, the Cantor macro-cube $2^{<\gamma}$ is a homogeneous cellular ordinal ballean with $\mathrm{cof}(2^{<\gamma}) = \mathrm{cf}(\gamma)$.

To evaluate the covering numbers of $2^{<\gamma}$, for an ordinal γ, consider the ordinal

$$\lfloor\gamma\rfloor := \min\{\alpha : \gamma = \beta + \alpha \text{ for some } \beta < \gamma\}$$

called the *tail* of γ, and the cardinal

$$\lceil\gamma\rceil := \min\{\alpha : \gamma \le \beta + |\alpha| \text{ for some } \beta < \gamma\}$$

called the *cardinal tail* of γ. It is clear that $\lfloor\gamma\rfloor \le \lceil\gamma\rceil$. Moreover,

$$\lceil\gamma\rceil = \begin{cases} |\lfloor\gamma\rfloor| & \text{if } \lfloor\gamma\rfloor \text{ is a cardinal,} \\ |\lfloor\gamma\rfloor|^+ & \text{otherwise.} \end{cases}$$

The equality $\gamma = \lfloor\gamma\rfloor$ holds if and only if the ordinal γ is *additively indecomposable*, which means that $\alpha + \beta < \gamma$ for any ordinals $\alpha, \beta < \gamma$.

The following proposition can be derived from the definition of $2^{<\gamma}$.

PROPOSITION 4.1. *For every ordinal γ the Cantor macro-cube $2^{<\gamma}$ is a cellular ordinal ballean with*

$$\mathrm{add}(2^{<\gamma}) = \mathrm{cof}(2^{<\gamma}) = \mathrm{cf}(\gamma) \quad \textit{and} \quad \mathrm{cov}^\flat(2^{<\gamma}) = \mathrm{cov}^\sharp(2^{<\gamma}) = \lceil\gamma\rceil. \;\blacksquare$$

The following theorem (which can be derived from Proposition 4.1 and Theorem 2.2) shows that in the class of cellular ordinal balleans, the Cantor macro-cubes $2^{<\gamma}$ play a role analogous to the role of the Cantor cubes 2^κ in the class of zero-dimensional compact Hausdorff spaces.

THEOREM 4.2. *Let γ be any ordinal and X be any cellular ordinal ballean such that* $\mathrm{cof}(X) = \mathrm{cf}(\gamma)$.

(1) *If $\lceil\gamma\rceil \le \mathrm{cov}^\flat(X)$, then $2^{<\gamma}$ is coarsely equivalent to a subspace of X.*
(2) *If $\mathrm{cov}^\sharp(X) \le \lceil\gamma\rceil$, then X is coarsely equivalent to a subspace of $2^{<\gamma}$.*
(3) *If $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X) = \lceil\gamma\rceil$, then X is coarsely equivalent to $2^{<\gamma}$.*

Proposition 4.1 and Theorem 4.2 imply the following characterization of $2^{<\gamma}$ which extends the characterization of $2^{<\omega}$ proved in [3].

THEOREM 4.3. *For any ordinal γ and any ballean X the following conditions are equivalent:*

(1) *X is coarsely equivalent to $2^{<\gamma}$.*
(2) *X is cellular,* $\mathrm{add}(X) = \mathrm{cof}(X) = \mathrm{cf}(\gamma)$ *and* $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X) = \lceil\gamma\rceil$.

COROLLARY 4.4. *For any ordinals β, γ the Cantor macro-cubes $2^{<\beta}$ and $2^{<\gamma}$ are coarsely equivalent if and only if* $\mathrm{cf}(\beta) = \mathrm{cf}(\gamma)$ *and* $\lceil\beta\rceil = \lceil\gamma\rceil$.

Finally, we identify the coarse structure of the ballean $\overset{\leftrightarrow}{\gamma}$ supported by an additively indecomposable ordinal γ. Given any non-zero ordinal γ we consider the family $\{\varepsilon_\alpha\}_{\alpha<\gamma}$ of the entourages

$$\varepsilon_\alpha = \{(x,y) \in \gamma \times \gamma : x \le y + \alpha \text{ and } \; y \le x + \alpha\}$$

for $\alpha < \gamma$. It is easy to see that $\overset{\leftrightarrow}{\gamma} := (\gamma, \{\varepsilon_\alpha\})_{\alpha<\gamma}$ is a ballean if and only if the ordinal γ is additively indecomposable (which means that $\alpha + \beta < \gamma$ for any ordinals $\alpha, \beta < \gamma$).

The following theorem classifies the balleans $\overset{\leftrightarrow}{\gamma}$ up to coarse equivalence.

THEOREM 4.5. *For any additively indecomposable ordinal γ the ballean $\overset{\leftrightarrow}{\gamma}$ is coarsely equivalent to:*

- $\overset{\leftrightarrow}{\omega}$ *if and only if $\gamma = \beta \cdot \omega$ for some β;*
- $2^{<\gamma}$, *otherwise.*

Proof. If $\gamma = \beta \cdot \omega$ for some ordinal β, then $\overset{\leftrightarrow}{\gamma}$ is coarsely equivalent to $\overset{\leftrightarrow}{\omega}$ since $\overset{\leftrightarrow}{\gamma}$ contains the large subset $L = \{\beta \cdot n : n \in \omega\}$, which is coarsely isomorphic to $\overset{\leftrightarrow}{\omega}$.

Now assume that $\gamma \neq \beta \cdot \omega$ for any ordinal β. Since γ is additively indecomposable, this means that $\beta \cdot \omega < \gamma$ for any $\beta < \gamma$, which implies that $\overset{\leftrightarrow}{\gamma}$ is cellular. Since $\mathrm{add}(\overset{\leftrightarrow}{\gamma}) = \mathrm{cof}(\overset{\leftrightarrow}{\gamma}) = \mathrm{cf}(\gamma)$ and $\mathrm{cov}^\flat(\overset{\leftrightarrow}{\gamma}) = \mathrm{cov}^\sharp(\overset{\leftrightarrow}{\gamma}) = \lceil\gamma\rceil$, the cellular ordinal ballean $\overset{\leftrightarrow}{\gamma}$ is coarsely equivalent to $2^{<\gamma}$ according to Theorem 4.3. ■

REMARK 4.6. For any ordinal γ the balleans $2^{<\gamma}$ and $\overset{\leftrightarrow}{\omega}$ are not coarsely equivalent, since $2^{<\gamma}$ is cellular, whereas $\overset{\leftrightarrow}{\omega}$ is not.

5. Embedding cellular ordinal balleans into asymptotic products of cardinals. In this section, we construct coarse embeddings of cellular ordinal balleans into asymptotic products of cardinals. Such embeddings will play a crucial role in the proof of Theorem 2.2 presented in the next section.

Observe that for any transfinite sequence $(\kappa_\alpha)_{\alpha<\gamma}$ of cardinals, the asymptotic product $\coprod_{\alpha<\gamma} \kappa_\alpha$ carries an operation of coordinatewise addition of sequences induced by addition of ordinals. For $\beta < \gamma$ and $y \in \kappa_\alpha$ let $y \cdot \delta_\beta$ denote the sequence $(x_\alpha)_{\alpha<\gamma} \in \coprod_{\alpha<\gamma} \kappa_\alpha$ such that $x_\alpha = y$ if $\alpha = \beta$, and $x_\alpha = 0$ otherwise. It follows that each $(x_\alpha)_{\alpha<\gamma} \in \coprod_{\alpha<\gamma} \kappa_\alpha$ can be written as $\sum_{\alpha\in A} x_\alpha \cdot \delta_\alpha$ for the finite set $A = \{\alpha < \gamma : x_\alpha \neq 0\}$.

The following lemma exploits and develops the decomposition technique used in [9], [11, §10], and [13].

LEMMA 5.1. *Let X be an ordinal ballean of infinite cofinality γ, and $(\varepsilon_\alpha)_{\alpha<\gamma}$ be a well-ordered base of the coarse structure of X consisting of cellular entourages such that $\varepsilon_\beta = \bigcup_{\alpha<\beta} \varepsilon_\alpha$ for all limit ordinals $\beta < \gamma$. For every $\alpha < \gamma$ and $x \in X$ let $\kappa_\alpha(x) = \mathrm{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1}))$. Set $\kappa_\alpha = \min_{x\in X} \kappa_\alpha(x)$ and $\bar\kappa_\alpha = \sup_{x\in X} \kappa_\alpha(x)$. Then the ballean X is coarsely equivalent to a subballean $Y \subset \coprod_{\alpha<\gamma} \bar\kappa_\alpha$ containing $\coprod_{\alpha<\lambda} \kappa_\alpha$.*

Proof. For any $x,y\in X$, let
$$d(x,y):=\min\{\alpha<\gamma:(x,y)\in\varepsilon_\alpha\},$$
and observe that if $(x,y)\notin\varepsilon_0$, then the ordinal $d(x,y)$ is not limit (as $\varepsilon_\beta=\bigcup_{\alpha<\beta}\varepsilon_\alpha$ for any limit $\beta<\gamma$). Consequently, we can find an ordinal $d^-(x,y)$ such that $d(x,y)=d^-(x,y)+1$.

Fix any well-ordering $\preceq$ of X. Given a non-empty subset $B\subset X$, denote by $\min B$ the smallest point of B with respect to the well-order $\preceq$, and for every $\alpha<\gamma$ let $c_\alpha\colon X\to X$ be the map assigning to each $x\in X$ the smallest element $c_\alpha(x)=\min B(x,\varepsilon_\alpha)$ of the ball $B(x,\varepsilon_\alpha)$. Since ε_α is an equivalence relation, $B(x,\varepsilon_\alpha)=B(c_\alpha(x),\varepsilon_\alpha)$. To simplify the notation, we shall denote $B(x,\varepsilon_\alpha)$ by $B_\alpha(x)$.

Observe that for every $\alpha<\gamma$ and $B\in\{B_{\alpha+1}(x):x\in X\}$, the set $c_\alpha(X)\cap B$ has cardinality $\kappa_\alpha(\min B)$, so we can fix a map $n_{\alpha,B}:B\to\kappa_\alpha(\min B)$ such that $\{B_\alpha(x):x\in B\}=\{n^{-1}_{\alpha,B}(\beta):\beta\in\kappa_\alpha(\min B)\}$ and $n^{-1}_{\alpha,B}(0)=B_\alpha(\min B)$. Finally, define a map $n_\alpha\colon X\to\bar\kappa_\alpha$ by assigning to each $y\in X$ the number $n_\alpha(y):=n_{\alpha,B_{\alpha+1}(y)}(y)$ of the ε_α-ball containing y in the partition of the $\varepsilon_{\alpha+1}$-ball $B_{\alpha+1}(y)$. The definition of κ_α implies that $\kappa_\alpha\subset\kappa_\alpha(x)=n_\alpha(B_{\alpha+1}(x))$ for every $x\in X$.

For every $x\in X$, define $f_x\colon X\to\coprod_{\alpha<\gamma}\bar\kappa_\alpha$ by the recursive formula
$$f_x(y)=\begin{cases}0 & \text{if } d(x,y)=0,\\ f_{\min B_{d^-(x,y)}(y)}(y)+n_{d^-(x,y)}(y)\cdot\delta_{d^-(x,y)} & \text{otherwise.}\end{cases}$$
Since $d(\min B_{d^-(x,y)}(y),y)<d(x,y)$, the function f_x is well-defined.

It can be shown that for every $x\in X$, the function $f_x\colon X\to\coprod_{\alpha<\gamma}\bar\kappa_\alpha$ determines a coarse equivalence of X with the subspace $f(X)$ of $\coprod_{\alpha<\gamma}\bar\kappa_\alpha$ containing $\coprod_{\alpha<\gamma}\kappa_\alpha$. ■

6. Proof of Theorem 2.2. Assume that X,Y are cellular balleans with $\gamma=\mathrm{add}(X)=\mathrm{cof}(X)=\mathrm{cof}(Y)=\mathrm{add}(X)$ and $\kappa=\mathrm{cov}^\flat(X)=\mathrm{cov}^\sharp(X)=\mathrm{cov}^\sharp(Y)=\mathrm{cov}^\flat(Y)$ for some cardinals γ and κ. Let $\mathcal{E}_X$, $\mathcal{E}_Y$ denote the respective ballean structures.

We shall consider four cases.

1) $\gamma=0$. In this case X,Y are empty, and hence coarsely equivalent.

2) $\gamma=1$. Then X,Y are bounded, and hence coarsely equivalent.

3) $\gamma=\omega$. Since X is a cellular ballean with $\mathrm{cof}(X)=\gamma=\omega$, the coarse structure $\downarrow\mathcal{E}_X$ has a well-ordered base $\{\varepsilon_n\}_{n\in\omega}$ consisting of equivalence relations such that $\varepsilon_0=\Delta_X$. In this case the formula
$$d_X(x,x')=\min\{n\in\omega:(x,x')\in\varepsilon_n\}$$
defines an ultrametric $d_X\colon X\times X\to\omega$ generating the coarse structure of X. By analogy, we can define an ultrametric d_Y generating the coarse structure

of Y. Since $\mathrm{cov}^\flat(X) = \mathrm{cov}^\sharp(X) = \mathrm{cov}^\sharp(Y) = \mathrm{cov}^\flat(Y)$, we can apply [1, Theorem 1.2] (proved by the technique of towers created in [3]) to conclude that the ultrametric spaces X and Y are coarsely equivalent.

4) $\gamma > \omega$. Since X, Y are ordinal balleans with $\mathrm{cof}(X) = \mathrm{cof}(Y) = \gamma$, we can fix well-ordered bases $\{\tilde\varepsilon_\alpha\}_{\alpha<\gamma}$ and $\{\tilde\delta_\alpha\}_{\alpha<\gamma}$ of the coarse structures $\downarrow\mathcal{E}_X$ and $\downarrow\mathcal{E}_Y$, respectively. By induction on $\alpha < \gamma$ we shall construct well-ordered sequences $\{\varepsilon_\alpha\}_{\alpha<\gamma} \subset \downarrow\mathcal{E}_X$ and $\{\delta_\alpha\}_{\alpha<\gamma} \subset \downarrow\mathcal{E}_Y$ such that for every $\alpha < \gamma$ the following conditions will be satisfied:

(a) $\varepsilon_\alpha = \bigcup_{\beta<\alpha} \varepsilon_\beta$ and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ if the ordinal α is limit;
(b) ε_α and δ_α are cellular entourages;
(c) $\tilde\varepsilon_\alpha \subset \varepsilon_{\alpha+1}$ and $\tilde\delta_\alpha \subset \delta_{\alpha+1}$;
(d) $\min\limits_{x\in X} \mathrm{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1})) = \sup\limits_{x\in X} \mathrm{cov}_{\varepsilon_\alpha}(B(x, \varepsilon_{\alpha+1}))$
$= \min\limits_{y\in Y} \mathrm{cov}_{\delta_\alpha}(B(y, \delta_{\alpha+1})) = \sup\limits_{y\in Y} \mathrm{cov}_{\delta_\alpha}(B(y, \delta_{\alpha+1})) = \kappa_\alpha$
for some cardinal κ_α.

We start the inductive construction by choosing cellular entourages $\varepsilon_0 \in \mathcal{E}_X$ and $\delta_0 \in \mathcal{E}_Y$ such that

$$\sup_{x\in X} \mathrm{cov}_{\varepsilon_0}(B(x, \varepsilon)) < \kappa \quad \text{and} \quad \sup_{y\in Y} \mathrm{cov}_{\delta_0}(B(y, \delta)) < \kappa$$

for any $\varepsilon \in \downarrow\mathcal{E}_X$ and $\delta \in \downarrow\mathcal{E}_Y$. The existence of ε_0 and δ_0 follows from the cellularity of X, Y and the definition of $\mathrm{cov}^\sharp(X) = \mathrm{cov}^\sharp(Y) = \kappa$. Assume that for some ordinal $\alpha < \gamma$ and all ordinals $\beta < \alpha$, the cellular entourages ε_β and δ_β have already been constructed. If α is limit, then we set $\varepsilon_\alpha = \bigcup_{\beta<\alpha} \varepsilon_\beta$ and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$. Observe that the entourages ε_α and δ_β are cellular as unions of increasing chains of cellular entourages. Moreover, $\varepsilon_\alpha \in \downarrow\mathcal{E}_X$ and $\delta_\beta \in \downarrow\mathcal{E}_Y$ as $\alpha < \gamma = \mathrm{add}(X) = \mathrm{add}(Y)$.

Next, assume that α is not limit, and hence $\alpha = \beta + 1$ for some ordinal β. Taking into account the choice of ε_0, δ_0 and using the definitions of $\mathrm{cov}^\flat(X) = \mathrm{cov}^\flat(Y)$, we can construct two increasing sequences of cellular entourages, $\{\varepsilon'_n\}_{n\in\omega} \subset \downarrow\mathcal{E}_X$ and $\{\delta'_n\}_{n\in\omega} \subset \downarrow\mathcal{E}_Y$, such that

$$\sup_{x\in X} \mathrm{cov}_{\varepsilon'_n}(B(x, \varepsilon'_{n+1})) \le \min_{y\in Y} \mathrm{cov}_{\delta'_n}(B(y, \delta'_{n+1})),$$
$$\sup_{y\in Y} \mathrm{cov}_{\delta'_n}(B(y, \delta'_{n+1})) \le \min_{x\in X} \mathrm{cov}_{\varepsilon'_{n+1}}(B(x, \varepsilon'_{n+2})).$$

The entourages ε'_1 and δ'_1 can be chosen so that $\tilde\varepsilon_\alpha \subset \varepsilon'_1$ and $\tilde\delta_\alpha \subset \delta'_1$. Since $\mathrm{add}(X) = \mathrm{add}(Y) > \omega$, the entourages $\varepsilon_{\alpha+1} = \bigcup_{n\in\omega} \varepsilon'_n$ and $\delta_{\alpha+1} = \bigcup_{n\in\omega} \delta'_n$ belong to $\downarrow\mathcal{E}_X$ and $\downarrow\mathcal{E}_Y$, respectively, and have the properties (b)–(d), required in the inductive construction.

By Lemma 5.1, there are coarse equivalences $f_X\colon X \to \coprod_{\alpha<\gamma} \kappa_\alpha$ and $f_Y\colon Y \to \coprod_{\alpha<\gamma} \kappa_\alpha$. Then the multi-map $f_Y^{-1} \circ f_X\colon X \multimap Y$ is a coarse equivalence. ∎

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