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# New Techniques for Computing Geometric Index

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*Dedicated to Sibe Mardešić (1927–2016)*

**Abstract.** We introduce new general techniques for computing the geometric index of a link $L$ in the interior of a solid torus $T$. These techniques simplify and unify previous ad hoc methods used to compute the geometric index in specific examples and allow a simple computation of geometric index for new examples where the index was not previously known. The geometric index measures the minimum number of times any meridional disc of $T$ must intersect $L$. It is related to the algebraic index in the sense that adding up signed intersections of an interior simple closed curve $C$ in $T$ with a meridional disc gives $\pm$ the algebraic index of $C$ in $T$. One key idea is introducing the notion of geometric index for solid chambers of the form $B^2 \times I$ in $T$. We prove that if a solid torus can be divided into solid chambers by meridional discs in a specific (and often easy to obtain) way, then the geometric index can be easily computed.

**Mathematics Subject Classification.** Primary 57M25, 54E45, 54F65; Secondary 57N75, 57M30, 57N10.

**Keywords.** Algebraic index, Geometric index, Whitehead link, Bing link, McMillan link, Gabai link, Antoine link.

## 1. Introduction

The geometric index of a link $L$ in the interior of a solid torus $T$ measures the minimum number of times any meridional disc of $T$ must intersect $L$. The geometric index of $L$ in $T$ often seems intuitively obvious, but it is surprisingly difficult to prove that intuition corresponds to the actual index. See for example the links in Figs. 2, 3, and 4. Various ad hoc methods were used to compute the geometric index of the links in Fig. 4 in [2] and [7], but these methods do not apply to links in general. The method described in this

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paper encompasses the ad hoc arguments used in those papers and applies more generally to links in a solid torus.

In this paper, we introduce new techniques for computing the geometric index of $L$ in $T$. The strategy is to break up the solid torus into chambers $C_i$ of the form $B^2 \times I$ by a carefully selected collection of meridional discs, and to analyze the part of the link $L$ in each $C_i$. The chambers can in practice be chosen so that all but two of the segments in $C_i$ meet each end of $C_i$ in a single point. This can be done by isolating turning points or pairs of turning points in components of $L$ in separate chambers. This method holds for the links in the literature that we have examined. Each meridional disc of $C_i$ must intersect such a segment. This reduces the analysis in each such chamber to determining whether an arc with both endpoints in one end of the chamber links an arc with both endpoints in the other end. To make this precise, we define the geometric index of $L \cap C_i$ in $C_i$. If each such geometric index is $n$, and if each meridional disc in the chosen collection intersects $L$ transversely in $n$ points, then we show that the geometric index of $L$ in $T$ is $n$.

This is stated precisely in Theorem 4.4. The three corollaries that follow from this theorem give various methods for computing the geometric index from information about the chambers. One might think a simpler result could be obtained by not requiring that each meridional disc in the collection intersects $L$ in exactly $n$ points, and that one could simply take the minimum geometric index of the link in the collection of chambers. The example in Fig. 1 shows that this is not the case.

In each subfigure, Chamber A has geometric index 4 and Chambers B and C have geometric index 2. But the overall geometric indices of the two examples differ.

A specific case of this result for certain special links in the setting of only one meridional disc appears in [2]. Another specific simple case of this
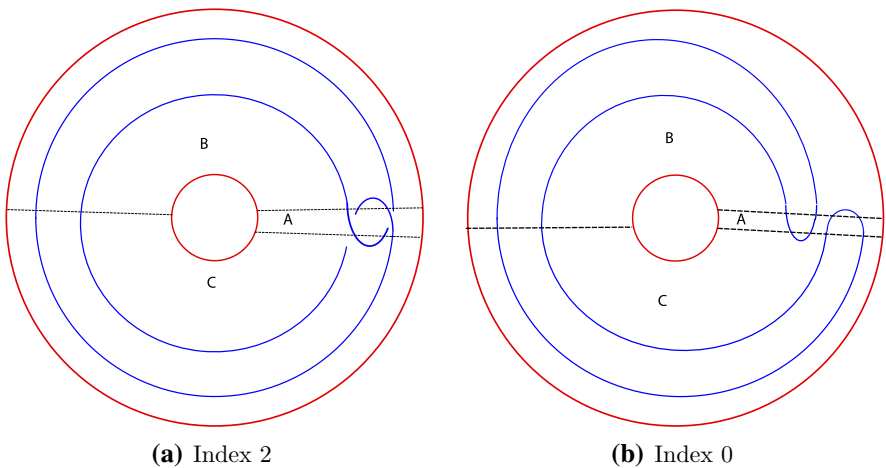


**(a)** Index 2                    **(b)** Index 0

Figure 1. Chambers with Same Indices

result for one specialized type of link and two meridional discs appears in [7]. Section 2 contains terminology, definitions and basic results about geometric and algebraic indices. This section also indicates how geometric index computations can be extremely useful in proving certain geometric results. Section 3 defines geometric index for chambers of the form $B^2 \times I$ and computes the geometric index in a few special cases. Section 4 contains the proof of Theorem 4.4 and three corollaries. Section 5 contains examples of applications of the main results.

## 2. Definitions and Results about Geometric and Algebraic Indices

If $S$ is a solid torus embedded in another solid torus $T$, the *algebraic index* of $S$ in $T$, $a(S,T)$, is defined to be $|\alpha|$, where $\alpha$ is the integer in $H_1(T)$ represented by the center line of $S$. Algebraic index is multiplicative, so that if $S \subset T \subset U$ are solid tori, the algebraic index of $S$ in $U$ is the product of the algebraic index of $S$ in $T$ with the algebraic index of $T$ in $U$. Note that the algebraic index of a Whitehead link in the torus containing it is 0, as is the algebraic index of each component of a Bing link. See Fig. 3.

Schubert introduced the notion of geometric index in [15]. If $K$ is a link in the interior of a solid torus $T$, then we denote the *geometric index* of $K$ in $T$ by $N(K,T)$. The geometric index is the minimum, over all meridional discs $D$ of $T$, of $|K \cap D|$. A *core* of a solid torus $T$ in 3-space is a simple closed curve $J$ such that $T$ is a regular neighborhood of $J$. Likewise, a core for a finite union of disjoint solid tori is a link consisting of one core of each of the solid tori. If $T$ is a solid torus and $M$ is a finite union of disjoint solid tori so that $M \subset Int\ T$, then the geometric index $N(M,T)$ of $M$ in $T$ is $N(K,T)$, where $K$ is a core of $M$.

Figure 2 indicates a collection of new examples. Each circled part of the inner link in Fig. 2(a) can be replaced by one of the patterns in Fig. 2(b) to produce many different examples. These examples do not fit into any previous class of examples such as the Gabai or McMillan links in [7]. More complicated links can also be obtained by additional varying of the linking or winding pattern of the strands as one progresses out from the center of the diagram.

It is clear that the algebraic index of the inner link in Fig. 2a is 0. Our techniques show that the geometric index is 8 for each example. See Sect. 5.

The geometric indices of the links in Fig. 3 are all two. The algebraic index of the Whitehead Link in Fig. 3a is zero as is the algebraic index of each component in Fig. 3b, c. The algebraic index of the link in Fig. 3d is two. The geometric indices of the links in Fig. 4 are indicated in the figure. More details will be provided in Sect. 5. See [7] for more discussion of the Gabai and McMillan links in Fig. 4.

*Remark* 2.1. It is well known that the algebraic index of a simple closed curve $J$ in a solid torus $T$ can be computed as follows. Choose a meridional disc $D$ that intersects $J$ transversely. Assign a positive and a negative side to $D$
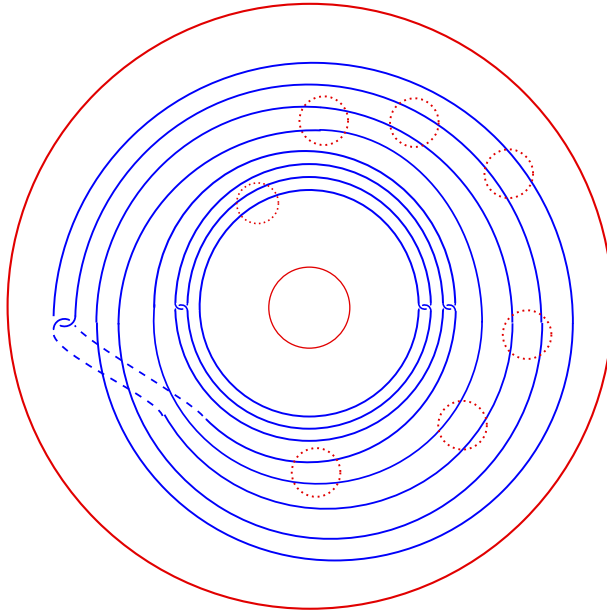
**(a)** Complicated Link(s)
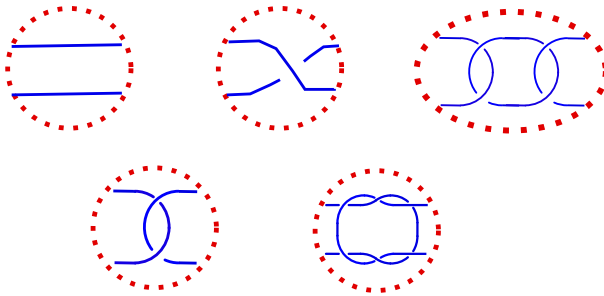


**(b)** Replacement Strands

Figure 2. Some Complicated Links

in $T$ so that a simple closed curve passing through $D$ once, going from the negative to the positive side, has algebraic index 1 in $T$. Assign a $\pm 1$ to each intersection of $J$ with $D$ corresponding to passing from the negative to the positive or from the positive to the negative side. Then, the algebraic index of $J$ in $T$ is the absolute value of the sum of the numbers assigned to each intersection of $J$ with $D$. In particular, the algebraic and geometric indices have the same parity.

*Remark* 2.2. Note that the previous remark immediately implies that the geometric index is greater than or equal to the algebraic index.

The next few results follow from Schubert [15] and regular neighborhood theory [14]. They indicate the usefulness of being able to compute geometric

(a) Whitehead Link

(b) Bing Link
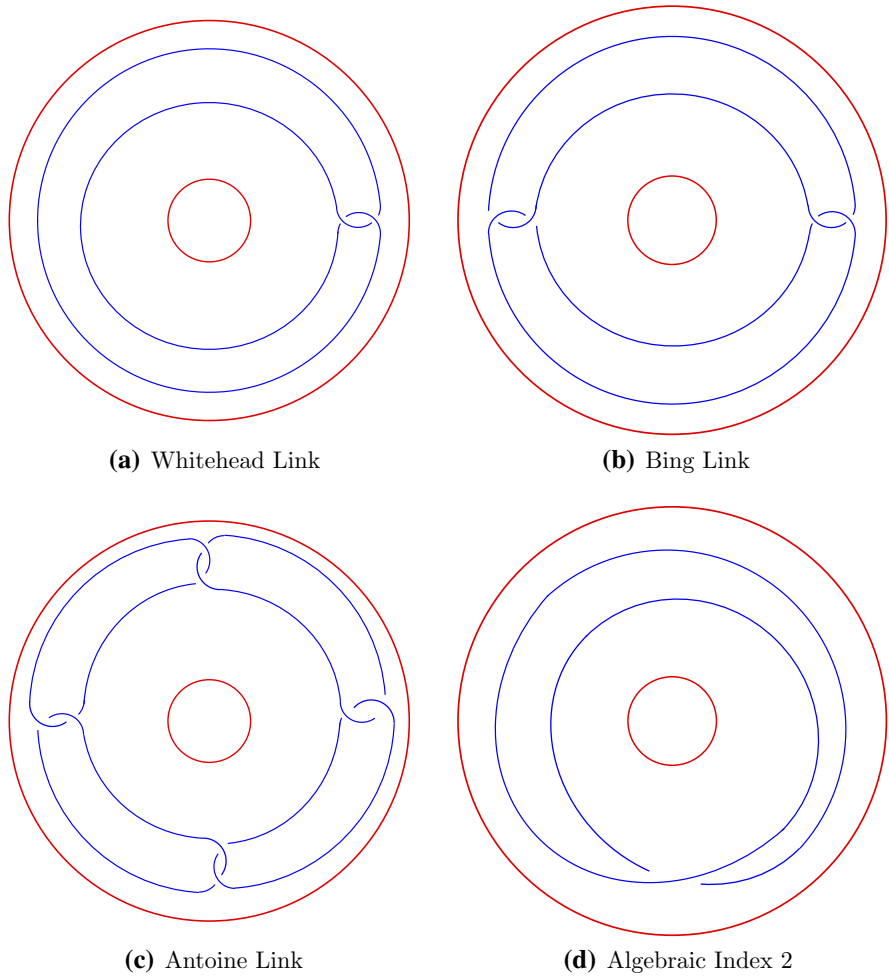
(c) Antoine Link

(d) Algebraic Index 2

Figure 3. Links with Geometric Index 2

index. For more details on the proofs of these lemmas, see Section 3 in [8]. We include a proof of the third lemma to give an indication of the techniques used.

**Lemma 2.3.** *Let $T_0$ and $T_1$ be unknotted solid tori in $S^3$ with $T_0 \subset IntT_1$ and $N(T_0, T_1) = 1$. Then, $\partial T_0$ and $\partial T_1$ are parallel; i.e., the manifold $T_1 - IntT_0$ is homeomorphic to $\partial T_0 \times I$ where $I$ is the closed unit interval $[0, 1]$.*

**Lemma 2.4.** *Let $T_0$ be a finite union of disjoint solid tori in $S^3$. Let $T_1$ and $T_2$ be solid tori so that $T_0 \subset IntT_1$ and $T_1 \subset IntT_2$. Then, $N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2)$.*
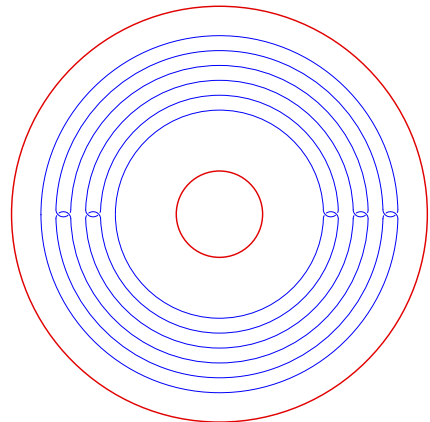
**Lemma 2.5.** *Let $T$ be a solid torus in $S^3$ and let $T_1, \dots T_k$ be disjoint unknotted solid tori in $T$, each of geometric index 0 in $T$. Then, the geometric index of $\cup_{i=1}^k T_i$ in $T$ is even.*
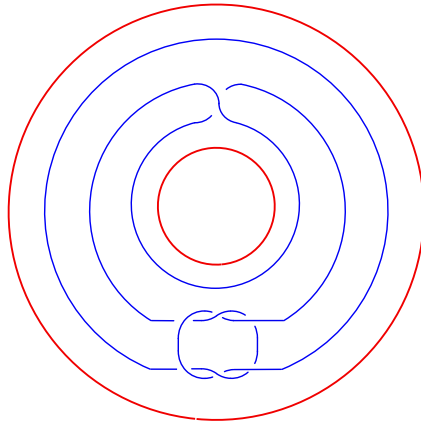
**(a)** Knotted Link - Index 3
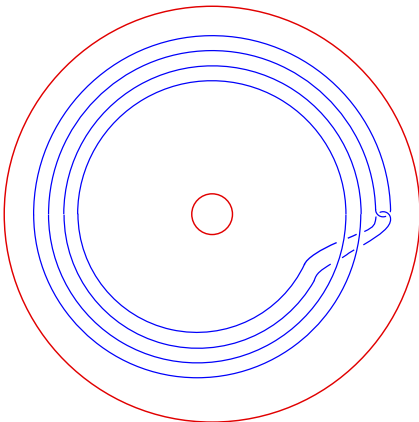


**(b)** McMillan Link - Index 4



**(c)** Gabai Link - Index 6

Figure 4. Additional links

*Proof.* If the geometric index were odd, then there would be a meridional disc $D$ of $T$ that would intersect the cores of $\cup_{i=1}^{k} T_i$ transversally an odd number of times. This would mean that $D$ must intersect the core of some $T_i$ an odd number of times. But if a meridional disc of $T$ intersects a simple closed curve $J$ transversally an odd number of times, the algebraic index of $J$ in $T$ is odd by Remark 2.1, and so $J$ is essential in $T$. However, for each $i$, the core of $T_i$ is inessential because it lies in a ball in $T$.                    □

The next two results make use of the material on geometric index to determine when the boundaries of certain tori are parallel as a further illustration of the use of geometric index. See [17] for details. We include the proof of the first theorem to further indicate the techniques used.

**Theorem 2.6.** *Let $W$ be torus in a solid torus $T$ in $S^3$ with the core of $W$ forming a Whitehead link (as in Fig. 3(a)) in $T$. If $T' \subset T$ is a solid unknotted torus whose boundary separates $\partial W$ from $\partial T$, then $\partial T'$ is parallel to either $\partial W$ or $\partial T$.*

*Proof.* Since $\partial T'$ separates $\partial W$ from $\partial T$, we have $W \subset IntT'$ and $T' \subset IntT$. Since $N(W,T')\cdot N(T',T) = N(W,T) = 2$, either $N(W,T') = 1$ or $N(T',T) = 1$. The conclusion now follows from Lemma 2.3.                                                    □

**Theorem 2.7.** *Let $F_1 \cup F_2$ be a pair of disjoint tori in a solid torus $T$ in $S^3$, with the cores of $F_1$ and $F_2$ forming a Bing link (as in Fig. 3(b)) in $T$. If $S$ is the boundary of a solid unknotted torus that separates $\partial F_1 \cup \partial F_2 \cup \partial T$, then $S$ is parallel to one of $\partial F_1$, $\partial F_2$, $\partial T$.*

## 3. Geometric Index for Chambers

In this section, we introduce the concept of geometric index for chambers of the form $B^2 \times I$, formed by pairwise disjoint meridional discs in a solid torus.

A *chamber* $C$ is a space homeomorphic to $B^2 \times I$. The *top* of the chamber, $C_t$, corresponds to $B^2 \times \{1\}$ and the *bottom* of the chamber, $C_b$, corresponds to $B^2 \times \{0\}$. We think of the chamber $C$ as the region between two meridional discs in a solid torus, with the meridional discs corresponding to $C_t$ and $C_b$. An *interior meridional disc* $M$ in such a chamber $C$ is a disc in $C - (C_t \cup C_b)$, where $\partial M$ is essential in $\partial C - (C_t \cup C_b)$. Let $L$ be a collection of arcs and simple closed curves in a chamber $C$ so that each arc has its endpoints in $C_t \cup C_b$ and so that each simple closed curve is in $C - \partial C$. The *geometric index* of $L$ in $C$, $N(L,C)$, is the minimum of $|L \cap M|$ over all interior meridional discs $M$ of $C$.

Consider a link $L = A \cup B \cup S$ where $A$ is an arc with endpoints in $C_t$, $B$ is an arc with endpoints in $C_b$, and $S$ is a possibly empty union of circles. The link $L$ forms a *clasp* in $C$ if the geometric index of $L$ in $C$ is two. A *spanning arc* A in $C$ is an arc with one endpoint in $C_t$, and the other endpoint in $C_b$. See Fig. 5 for some examples.

**Lemma 3.1.** *The Whitehead clasp, the Square Knot clasp, and the Antoine clasp (pictured in Fig. 5b, Fig. 5c, and Fig. 5e) have geometric index 2 in the indicated chambers. Each spanning arc in a chamber has geometric index 1 in that chamber.*

*Proof.* Any interior meridional disc must intersect a spanning arc, even if the arc is knotted in the chamber. To see this, note that any meridional disc divides the chamber into two components with $C_b$ in one component and $C_t$ in the other. If the meridional disc misses the arc completely, then the arc is contained in one of the two components which cannot happen.

For the Whitehead clasp, consider Fig. 6. Let $D$ be a disc in the chamber with boundary $A$ together with an arc in one end of the chamber that joins the two points of $A$ in that end. There is no loss of generality in only considering interior meridional discs that intersect $A\cup B$ transversely since non-transverse intersections can be removed by a small general position move which only

**(a)** Link in Chamber **(b)** Whitehead Clasp **(c)** Square Knot Clasp **(d)** Spanning Arcs
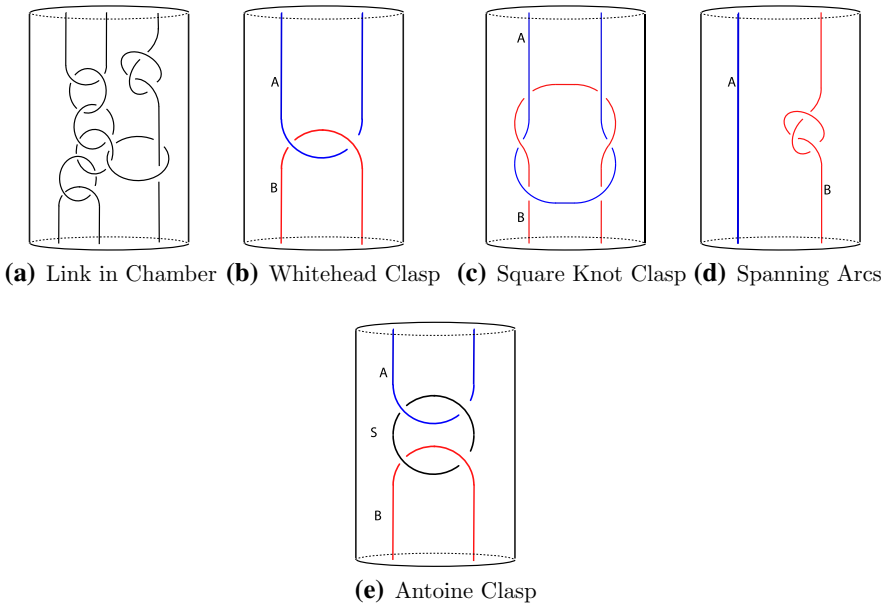


**(e)** Antoine Clasp
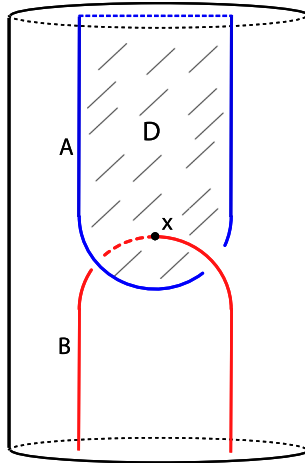
Figure 5. Clasps and Spanning Arcs



Figure 6. Whitehead Clasp with Spanning Disc

decreases the number of intersections. By an argument similar to that in the proof of Lemma 2.5, every such meridional disc must intersect each of $A$ and $B$ an even number of times. There is clearly an interior meridional disc that intersects $A \cup B$ twice, so we must only eliminate the possibility of no intersections.

Suppose $M$ is an interior meridional disc that misses $A \cup B$. By a general position adjustment away from $A \cup B$, we can assume that $M$ intersects the

disc $D$ in a finite number of simple closed curves. Consider an innermost such curve in $M$. If this curves bounds a disc in $D$ missing $x$, the disc in $M$ bounded by the curve together with the disc in $D$ bounded by the curve bound a 3 ball. This 3-ball can be used to push the disc in $M$ to the disc in $D$ which can then be pushed slightly off $D$. This process reduces the number of curves of intersection of $M$ with $D$.

If this innermost curve in $M$ bounds a disc in $D$ containing $x$, the disc in $M$ bounded by the curve together with the disc in $D$ bounded by the curve forms a 2-sphere that is pierced once by $B$. However, this is impossible. So eventually all curves of intersection of $M$ with $D$ can be eliminated. But then $M$ misses a spanning arc of the chamber which is impossible. So no interior meridional disc $M$ can miss $A \cup B$.

The proofs for the Square Knot clasp and the Antoine clasp are similar. Note that the Antoine clasp can be divided into two Whitehead clasps by adding a meridional disc in the middle of the cylinder in Fig. 5(e). Then, the techniques of Theorem 4.4 below can be used to show that the geometric index is two. □

## 4. Main Results

We now work towards proving the main result (Theorem 4.4) and some corollaries.

**Setup:** Throughout this section, let $L$ be a link in the interior of a solid torus $T$ and let $D_0, \ldots D_{m-1}$ be a cyclically ordered pairwise disjoint collection of meridional discs in $T$, each intersecting $L$ transversally in $n$ points. Let $C_i$ be the chamber bounded by $D_{i-1}$ and $D_{i \mod m}$ for $1 \leq i \leq m$.

*Remark* 4.1. Note that $N(L \cap C_i, C_i) \leq n$ for each $i$. This follows from the fact that an interior meridional disc parallel to and close enough to the ends of $C_i$ intersects $L \cap C_i$ in $n$ points by the setup.

**Lemma 4.2.** *If $N(L \cap C_i, C_i) \geq n$ for each $i$, and if $K$ is a meridional disc for $T$ that misses $\cup_{i=1}^{k} D_i$, then $|L \cap K| \geq n$.*

*Proof.* This follows since any such meridional disc is an interior meridional disc for one of the chambers. □

**Lemma 4.3.** *Assume $N(L \cap C_i, C_i) \geq n$ for each $i$. Let $D'$ be a disc that lies in the interior of $T$ with $D' \cap (\cup_{i=0}^{m-1} D_i) = \partial D'$. Then, $\partial D'$ is in some $D_i$. Let $D''$ be the disc bounded by $\partial D'$ in $D_i$. Then, $|L \cap D'| \geq |L \cap D''|$.*

*Proof.* Suppose, to the contrary, that $|L \cap D'| < |L \cap D''|$. Then, the meridional disc $(D_i - D'') \cup D'$ meets $L$ in fewer points than $D_i$. This disc can be pushed slightly off $D_i$ so that it still meets $L$ in fewer points than $D_i$. But this contradicts Lemma 4.2. □

**Theorem 4.4.** *Let $L$, $T$, and $D_0, \ldots D_{m-1}$ be as in the setup above. If any meridional disc in $T$ that misses $D = \cup_{i=0}^{i=m-1} D_i$ intersects $L$ in at least $n$ points, then $N(L, T) = n$.*

*Proof.* Since the setup implies $N(L,T) \leq n$, it suffices to show $N(L,T) \geq n$. Let $K$ be a meridional disc of $T$ so that $|K \cap L|$ is minimal. Without loss of generality, we may assume that $\partial K \cap (\bigcup_{i=0}^{m-1} \partial D_i) = \emptyset$, K is in general position with respect to D, and $L \cap K \cap D = \emptyset$.

Note that if $\partial K \cap (\bigcup_{i=0}^{m-1} \partial D_i) \neq \emptyset$, one can use standard general position techniques to make the intersection empty. For example, use the fact that there is an isotopy of $T$ taking $K$ to a close parallel copy of any particular $D_i$. This isotopy can be feathered or tapered off to the identity away from a small neighborhood of the boundary of $T$ without changing the intersection of $K$ with $L$.

We show by induction on the number of components of $K \cap D$ that $|K \cap L| \geq n$. If $K \cap D = \emptyset$, then by Lemma 4.2, $|L \cap K| \geq n$. Consider a simple closed curve component $c$ of $K \cap D$ that is innermost on K. Then, $c$ bounds a disc $k$ in $K$ and a disc $d$ in some $D_i$. Note that $c$ may not be innermost in $D$ among the simple closed curves of $K \cap D$. However, the (possibly) singular disc $(K - k) \cup d$ meets $L$ in no more points than $K$ by Lemma 4.3. The singularities, if any, of $(K - k) \cup d$ consist of disjoint double curves that can be cut apart to give a non-singular disc $K'$ that meets L in no more points than K does. See [9, Ch. 4] for details on cutting apart double curves. By pushing $K'$ slightly off $D_i$, we obtain a meridional disc $K''$ such that $|K'' \cap L|$ is still minimal and such that $K'' \cap D$ has fewer components than $K \cap D$. Therefore, by induction, $|K'' \cap L| \geq n$ and so $|K \cap L| \geq n$.  $\square$

**Corollary 4.5.** *Let $L$, $T$, and $D_0, \ldots D_{m-1}$ and $C_i$ be as in the setup above. If $N(L \cap C_i, C_i) = n$ for each $i$, then $N(L,T) = n$.*

*Proof.* Any meridional disc that misses $\cup_{i=0}^{i=m-1} D_i$ is an interior meridional disc for some chamber, and so intersects $L$ in at least $n$ points. The result now follows from Theorem 4.4.  $\square$

**Corollary 4.6.** *Let $L$, $T$, and $D_0, \ldots D_{m-1}$ and $C_i$ be as in the setup above. If, for each $i$, $C_i \cap L$ consists of $k_i$ clasps and $\ell_i$ spanning arcs where $2k_i + \ell = n$, then $N(L,T) = n$.*

*Proof.* The hypotheses show that any interior meridional disc in $C_i$ intersects $L$ in at least $2k_i + \ell = n$ points. (See Lemma 3.1). The result now follows from Corollary 4.5.  $\square$

**Corollary 4.7.** *Let $L$, $T$, and $D_0, \ldots D_{m-1}$ and $C_i$ be as in the setup above. Let $L'$ be a link in the interior of a solid torus $T'$ and let $D'_0, \ldots D'_{m-1}$ be a cyclically ordered collection of meridional discs in $T$, each intersecting $L'$ transversally in $n$ points. Let $C'_i$ be the region in $T'$ bounded by $D'_i$ and $D'_{(i+1) \bmod m}$. If $N(L',T') = n$, and if $(C_i, C_i \cap L) \cong (C'_i, C'_i \cap L')$ for each $i$, then $N(L,T) = n$.*

*Proof.* For each chamber $C'_i$, $N(L' \cap C'_i, C'_i)$ must be at least $n$. Otherwise, $N(L',T')$ would be less than $n$. So for each chamber $C_i$, $N(L \cap C_i, C_i)$ must be at least $n$. The result now follows from Corollary 4.5.  $\square$
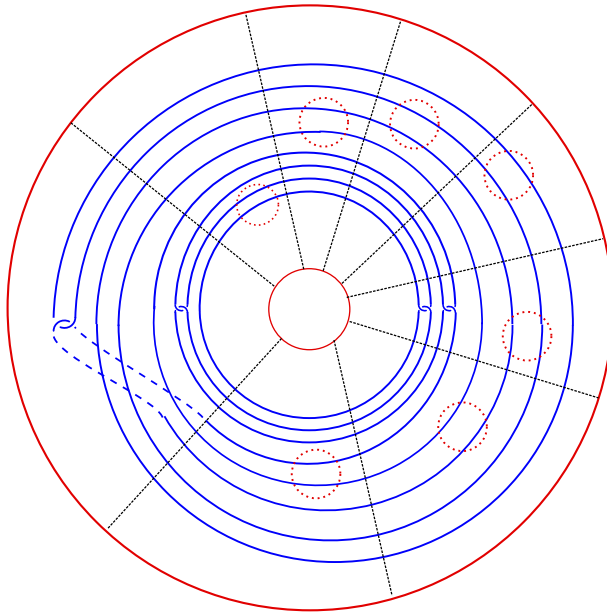
Figure 7. Chambers for Fig. 2

## 5. Examples and Applications

We apply the theorem and corollaries from the previous section to compute the geometric index of a number of new and old examples to illustrate the power of these results.

*Figure* 2 We begin with the new example(s) of Figure 2. Divide the outer torus into eight chambers as indicated in Fig. 7. Each chamber has a collection of spanning arcs, and possibly Whitehead clasps, Square knot clasps, or Antoine clasps. By Lemma 3.1, the geometric index of the inner link intersected with the chamber in the chamber is 8. The result now follows from Corollary 4.5 or from Corollary 4.6.

*Figures* 3a–c By Lemma 2.5, the geometric index of the link in each of these figures on the outer torus is 0 or 2. So it suffices to show the geometric index is not 0. The outer torus and link in Fig. 3c is a covering space of the outer tori and links in Fig. 3a, b. Any meridional disc in the outer torus in Fig. 3a, b that misses the interior link lifts to a meridional disc in the outer torus in Figure 3(c) that misses the interior link. So it suffices to show that the geometric index of the link in Fig. 3c is nonzero. There is a standard cut and paste argument that shows this. See for example [4, Section 9] or [13, Chapter 3-G].

An alternate approach using Corollary 4.6 is as follows. Divide the outer torus in four chambers as indicated in Fig. 8a. Each chamber has a single Whitehead clasp in it. By Lemma 3.1, the geometric index of the inner link intersected with the chamber in the chamber is 2. The result now follows from Corollary 4.5.
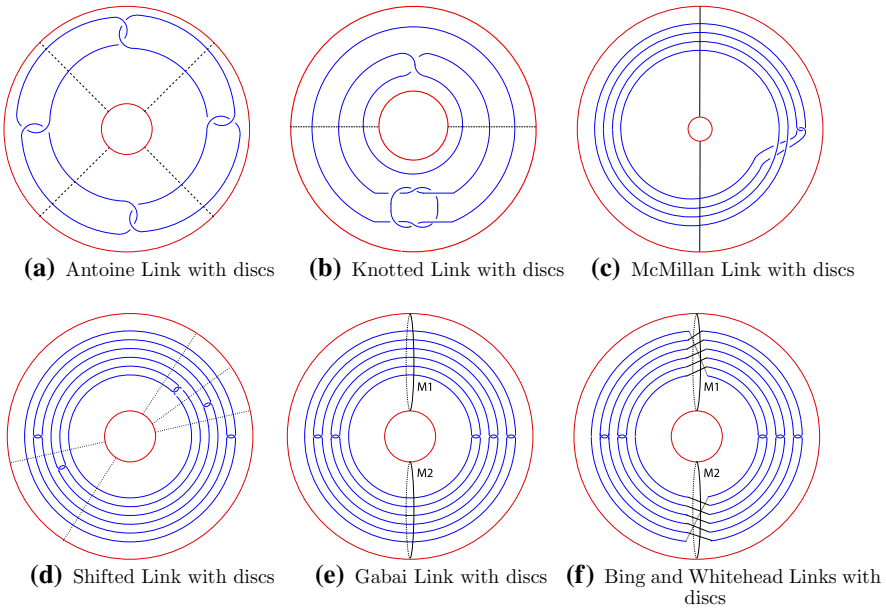
**(a)** Antoine Link with discs    **(b)** Knotted Link with discs    **(c)** McMillan Link with discs

**(d)** Shifted Link with discs    **(e)** Gabai Link with discs    **(f)** Bing and Whitehead Links with discs

Figure 8. Links with added meridional discs

*Figure* 3d The algebraic index is 2 since the inner curve has winding number 2 in the outer torus. Since the geometric index is greater than or equal to the algebraic index, and since there is clearly a meridional disc that intersects the inner curve twice, the geometric index is 2.

*Figure* 4b Divide the outer torus into two chambers as indicated in Fig. 8c. Each chamber has a Whitehead clasp and two spanning arcs or four spanning arcs. By Lemma 3.1, the geometric index of the inner link intersected with the chamber in the chamber is 4. The result now follows from Corollary 4.5 or from Corollary 4.6.

*Figure* 4a Divide the outer torus into two chambers as indicated in Fig. 8b. Each chamber has a Whitehead clasp and spanning arc or a Square Knot clasp and spanning arc. By Lemma 3.1, the geometric index of the inner link intersected with the chamber in the chamber is 3. The result now follows from Corollary 4.5 or from Corollary 4.6.

*Figure* 4c In Fig. 8d, we have slightly shifted the inner link by a homeomorphism and divided the outer torus into 5 chambers. The inner link, intersected with each chamber, consists of a Whitehead clasp and four spanning arcs. By Lemma 3.1, the geometric index of the inner link intersected with the chamber in the chamber is 6. The result now follows from Corollary 4.5 or from Corollary 4.6.

Alternately, consider Fig. 8e, f. Corollary 4.7 applies directly since the complements of the two indicated meridional discs are homeomorphic. Figure 8f consists of two Bing links and a Whitehead link, so the geometric index

of the inner link in the outer torus is 6. By Corollary 4.7, the geometric index of the inner link in the outer torus in Fig. 8e is also 6.

### Acknowledgements

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