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Multiplicity and concentration results for a (p, q)-Laplacian problem in $\mathbb{R}^N$

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Abstract. In this paper, we study the multiplicity and concentration of positive solutions for the following (p, q)-Laplacian problem:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $1 < p < q < N$, $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$, with $r \in \{p, q\}$, is the r-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the global Rabinowitz condition, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth. Using suitable variational arguments and Ljusternik–Schnirelmann category theory, we investigate the relation between the number of positive solutions and the topology of the set where V attains its minimum for small ε.

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1. Introduction

In this paper, we deal with the existence and multiplicity of solutions for the following $p\&q$-Laplacian problem:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{q-2}u\right) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter, $1 < p < q < N$, $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$, with $r \in \{p, q\}$, is the r-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth.

We recall that this class of problems arises from a general reaction-diffusion system

$$u_t = \mathrm{div}(D(u)\nabla u) + f(x, u) \quad x \in \mathbb{R}^N, t > 0,$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. As pointed out in [9], this equation appears in several applications such as biophysics, plasma physics and chemical reaction design. In these applications, u describes a concentration, $\mathrm{div}(D(u)\nabla u)$ corresponds to the diffusion with a diffusion coefficient $D(u)$, and the reaction term $f(x, u)$ relates to source and loss processes. Classical (p, q)-Laplacian problems in bounded or unbounded domains have been studied by several authors; see for instance [3,11–16,20] and references therein.

In order to precisely state our result, we introduce the assumptions on the potential V and the nonlinearity f. Along the paper, we assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the following condition introduced by Rabinowitz [21]:

$$0 < \inf_{x \in \mathbb{R}^N} V(x) = V_0 < \liminf_{|x| \to \infty} V(x) = V_\infty \in (0, \infty], \tag{V}$$

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and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ fulfills the following hypotheses:

(f_1) $f \in C^0(\mathbb{R}, \mathbb{R})$ and $f(t) = 0$ for all $t < 0$;

(f_2) $\displaystyle\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0$;

(f_3) there exists $r \in (q, q^*)$, with $q^* = \frac{Nq}{N-q}$, such that $\displaystyle\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{r-1}} = 0$;

(f_4) there exists $\vartheta \in (q, q^*)$ such that

$$0 < \vartheta F(t) = \vartheta \int_0^t f(\tau)\,\mathrm{d}\tau \le t f(t) \quad \text{for all } t > 0;$$

(f_5) the map $t \mapsto \dfrac{f(t)}{t^{q-1}}$ is increasing on $(0, \infty)$.

Since we deal with the multiplicity of solutions of (P_ε), we recall that if Y is a given closed subset of a topological space X, we denote by $cat_X(Y)$ the Ljusternik–Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y (see [25] for more details).

Let us denote by

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\} \quad \text{and} \quad M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \le \delta\}, \text{ for } \delta > 0.$$

Our main result can be stated as follows:

Theorem 1.1. *Assume that conditions (V) and (f_1)–(f_5) hold. Then for any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (P_ε) has at least $cat_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is a global maximum point of u_ε, then*

$$\lim_{\varepsilon\to 0} V(\varepsilon x_\varepsilon) = V_0,$$

and there exist $C_1, C_2 > 0$ such that

$$u_\varepsilon(x) \le C_1 e^{-C_2|x - x_\varepsilon|} \quad \textit{for all } x \in \mathbb{R}^N.$$

The proof of Theorem 1.1 will be obtained by using suitable variational techniques and category theory. We note that Theorem 1.1 improves Theorem 1.1 in [3], in which the authors assumed $f \in C^1$ and that there exist $C > 0$ and $\nu \in (p, q^*)$ such that

$$f'(t)t^2 - (q-1)f(t)t \ge Ct^\nu \quad \text{for all } t \ge 0.$$

Since we require that f is only continuous, the classical Nehari manifold arguments used in [3] do not work in our context, and in order to overcome the non-differentiability of the Nehari manifold, we take advantage of some variants of critical point theorems from [23]. Clearly, with respect to [3], a more accurate and delicate analysis will be needed to implement our variational machinery. To obtain multiple solutions, we use a technique introduced by Benci and Cerami in [7], which consists of making precise comparisons between the category of some sublevel sets of the energy functional $\mathcal{I}_\varepsilon$ associated with (P_ε) and the category of the set M. Since we aim to apply Ljusternik–Schnirelmann theory, we need to prove certain compactness property for the functional $\mathcal{I}_\varepsilon$. In particular, we will see that the levels of compactness are strongly related to the behavior of the potential V at infinity. A similar approach has been recently employed by the first author for fractional Schrödinger equations; see for example [5,6]. Finally, we prove the exponential decay of solutions by following some ideas from [13]. We would like to point out that our arguments are rather flexible and we believe that the ideas contained here can be applied in other situations to study problems driven by (p, q)-Laplacian operators, ϕ-Laplacian operator, or also fractional (p, q)-Laplacian problems, on the entire space.

The paper is organized as follows: in Sect. 2 we collect some facts about the involved Sobolev spaces and some useful lemmas. In Sect. 3, we provide some technical results which will be crucial to prove our main theorem. In Sect. 4, we deal with the autonomous problems associated to (P_ε). In Sect. 5, we obtain an existence result for (P_ε) for sufficiently small ε. Section 6 is devoted to the multiplicity result for (P_ε), and Sect. 7 to the concentration phenomenon.

2. Preliminaries

In this section, we recall some facts about the Sobolev spaces and we prove some technical lemmas which we will use later.

Let $p \in [1, \infty]$ and $A \subset \mathbb{R}^N$. We denote by $|u|_{L^p(A)}$ the $L^p(A)$-norm of a function $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^p(A)$. When $A = \mathbb{R}^N$, we simply write $|u|_p$ instead of $|u|_{L^p(\mathbb{R}^N)}$. For $p \in (1, \infty)$ and $N > p$, we define $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to

$$|\nabla u|_p^p = \int\limits_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x.$$

Let us denote by $W^{1,p}(\mathbb{R}^N)$ the set of functions $u \in L^p(\mathbb{R}^N)$ such that $|\nabla u|_p < \infty$, endowed with the natural norm

$$\|u\|_{1,p}^p = |\nabla u|_p^p + |u|_p^p.$$

We begin by recalling the following embedding theorem for Sobolev spaces.

Theorem 2.1. (see [1]) *Let $N > p$. Then there exists a constant $S_* > 0$ such that, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,*

$$|u|_{p^*}^p \leq S_*^{-1} |\nabla u|_p^p.$$

Moreover, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p_s^*]$ and compactly in $L^t_{loc}(\mathbb{R}^N)$ for any $t \in [1, p^*)$.

We recall the following Lions compactness lemma.

Lemma 2.1. (see [17]) *Let $N > p$ and $r \in [p, p^*)$. If $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and if*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int\limits_{\mathcal{B}_R(y)} |u_n|^r \mathrm{d}x = 0, \tag{2.1}$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p^*)$.

We also have the following useful lemma.

Lemma 2.2. (see [2,18]) *Let $\eta_n : \mathbb{R}^N \to \mathbb{R}^K$, $K \geq 1$, with $\eta_n \in L^t(\mathbb{R}^N) \times \cdots \times L^t(\mathbb{R}^N)$ $(t > 1)$, $\eta_n(x) \to 0$ a.e. in $\mathbb{R}^K$ and $A(y) = |y|^{t-2}y$, $y \in \mathbb{R}^K$. Then, if $|\eta_n|_t \leq C$ for all $n \in \mathbb{N}$, we have*

$$\int\limits_{\mathbb{R}^N} |A(\eta_n + w) - A(\eta_n) - A(w)|^{t'} \mathrm{d}x = o_n(1)$$

for each $w \in L^t(\mathbb{R}^N) \times \cdots \times L^t(\mathbb{R}^N)$ fixed, and $t' = \frac{t}{t-1}$ is the conjugate exponent of t.

For $\varepsilon > 0$, we define the space

$$\mathbb{X}_\varepsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(\varepsilon x) \left(|u|^p + |u|^q\right) \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon = \|u\|_{V,p} + \|u\|_{V,q},$$

where

$$\|u\|_{V,t}^t = |\nabla u|_t^t + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t \, dx \quad \text{for all } t > 1.$$

Then the following embedding lemma hold.

Lemma 2.3. (see [3]) *The space $\mathbb{X}_\varepsilon$ is continuously embedded into $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Therefore, $\mathbb{X}_\varepsilon$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, q^*]$ and compactly embedded in $L^t(B_R)$, for all $R > 0$ and any $t \in [1, q^*)$.*

Lemma 2.4. (see [3]) *If $V_\infty = \infty$, the embedding $\mathbb{X}_\varepsilon \subset L^m(\mathbb{R}^N)$ is compact for any $p \le m < q^*$.*

Finally we have the following splitting lemma which will be very useful in this work.

Lemma 2.5. *Let $\{u_n\} \subset \mathbb{X}_\varepsilon$ be a sequence such that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$. Set $v_n = u_n - u$. Then we have*

(*i*) $|\nabla v_n|_p^p + |\nabla v_n|_q^q = \left(|\nabla u_n|_p^p + |\nabla u_n|_q^q\right) - \left(|\nabla u|_p^p + |\nabla u|_q^q\right) + o_n(1)$,

(*ii*) $\displaystyle\int_{\mathbb{R}^N} V(\varepsilon x)\left(|v_n|^p + |v_n|^q\right) dx = \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u_n|^p + |u_n|^q\right) dx - \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u|^p + |u|^q\right) dx + o_n(1)$

(*iii*) $\displaystyle\int_{\mathbb{R}^N} \left(F(v_n) - F(u_n) + F(u)\right) dx = o_n(1)$,

(*iv*) $\displaystyle\sup_{\|w\|_\varepsilon \le 1} \int_{\mathbb{R}^N} |\left(f(v_n) - f(u_n) + f(u)\right) w| \, dx = o_n(1)$.

Proof. It is clear that (*i*) and (*ii*) are consequences of the well-known Brezis-Lieb lemma [8]. The proofs of (*iii*) and (*iv*) are given in [3] for $f \in C^1$. Since here we are assuming $f \in C^0$, we need to use different arguments. We start by proving (*iii*). Let us note that $u_n = v_n + u$ and

$$F(u_n) - F(v_n) = \int_0^1 \frac{d}{dt} F(v_n + tu) \, dt = \int_0^1 u f(v_n + tu) \, dt.$$

In view of (f_2) and (f_3), for any $\delta > 0$ there exists $c_\delta > 0$ such that

$$|f(t)| \le p\delta |t|^{p-1} + c_\delta |t|^{q^*-1} \quad \text{for all } t \in \mathbb{R}, \tag{2.2}$$

$$|F(t)| \le \delta |t|^p + c'_\delta |t|^{q^*} \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Using (2.2) with $\delta = 1$ and $(|a| + |b|)^r \le C(r)(|a|^r + |b|^r)$ for any $a, b \in \mathbb{R}$ and $r \ge 1$, we can see that

$$|F(u_n) - F(v_n)| \le C|v_n|^{p-1}|u| + C|u|^p + C|v_n|^{q^*-1}|u| + C|u|^{q^*}. \tag{2.4}$$

Fix $\eta > 0$. Applying the Young inequality $ab \le \eta a^r + C(\eta) b^{r'}$ for all $a, b > 0$, with $r, r' \in (1, \infty)$ such that $\frac{1}{r} + \frac{1}{r'} = 1$, to the first and the third term on the right hand side of (2.4), we deduce that

$$|F(u_n) - F(v_n)| \le \eta(|v_n|^p + |v_n|^{q^*}) + C_\eta(|u|^p + |u|^{q^*})$$

which together with (2.3) with $\delta = \eta$ implies that

$$|F(u_n) - F(v_n) - F(u)| \le \eta(|v_n|^p + |v_n|^{q^*}) + C'_\eta(|u|^p + |u|^{q^*}).$$

Let

$$G_{\eta,n}(x) = \max\left\{|F(u_n) - F(v_n) - F(u)| - \eta(|v_n|^p + |v_n|^{q^*}), 0\right\}.$$

Then $G_{\eta,n} \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ (recall that $v_n \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$), and $0 \leq G_{\eta,n} \leq C'_\eta(|u|^p + |u|^{q^*}) \in L^1(\mathbb{R}^N)$. As a consequence of the dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} G_{\eta,n}(x)\,\mathrm{d}x \to 0 \quad \text{ as } n \to \infty.$$

On the other hand, by the definition of $G_{\eta,n}$, it follows that

$$|F(v_n) - F(u_n) + F(u)| \leq \eta(|v_n|^p + |v_n|^{q^*}) + G_{\eta,n}$$

which together with the boundedness of (u_n) in $L^p(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$ yields

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |F(v_n) - F(u_n) + F(u)|\,\mathrm{d}x \leq C\eta.$$

By the arbitrariness of $\eta > 0$ we can deduce that (iii) holds. Finally, we prove (iv). For any fixed $\eta > 0$, by (f_2) we can choose $r_0 = r_0(\eta) \in (0, 1)$ such that

$$|f(t)| \leq \eta|t|^{p-1} \quad \text{ for } |t| \leq 2r_0. \tag{2.5}$$

On the other hand, by (f_3) we can pick $r_1 = r_1(\eta) > 2$ such that

$$|f(t)| \leq \eta|t|^{q^*-1} \quad \text{ for } |t| \geq r_1 - 1. \tag{2.6}$$

By the continuity of f, there exists $\delta = \delta(\eta) \in (0, r_0)$ satisfying

$$|f(t_1) - f(t_2)| \leq r_0^{p-1}\eta \quad \text{ for } |t_1 - t_2| \leq \delta,\ |t_1|, |t_2| \leq r_1 + 1. \tag{2.7}$$

Moreover, by (f_3) there exists a positive constant $c = c(\eta)$ such that

$$|f(t)| \leq c(\eta)|t|^{p-1} + \eta|t|^{q^*-1} \quad \text{ for all } t \in \mathbb{R}. \tag{2.8}$$

In what follows, we shall estimate the following term:

$$\int_{\mathbb{R}^N\setminus B_R(0)} |f(u_n - u) - f(u_n) - f(u)||w|\,\mathrm{d}x.$$

Using (2.8) and $u \in L^p(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)$, we can find $R = R(\eta) > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R(0)} |f(u)w|\,\mathrm{d}x &\leq c\left(\int_{\mathbb{R}^N\setminus B_R(0)} |u|^{q^*}\,\mathrm{d}x\right)^{\frac{q^*-1}{q^*}} |w|_{q^*} + c\left(\int_{\mathbb{R}^N\setminus B_R(0)} |u|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} |w|_p \\ &\leq c\eta\|w\|_{1,q} + c\eta\|w\|_{1,p} \leq c\eta\|w\|_\varepsilon.\end{aligned}$$

Set $A_n = \{x \in \mathbb{R}^N \setminus B_R(0) : |u_n(x)| \leq r_0\}$. Invoking (2.5) and applying the Hölder inequality, we get

$$\int_{A_n\cap\{|u|\leq\delta\}} |f(u_n) - f(u_n - u)||w|\,\mathrm{d}x \leq \eta(|u_n|_p^{p-1} + |u_n - u|_p^{p-1})|w|_p \leq c\eta\|w\|_\varepsilon. \tag{2.9}$$

Let $B_n = \{x \in \mathbb{R}^N \setminus B_R(0) : |u_n(x)| \geq r_1\}$. Then (2.6) and the Hölder inequality yield

$$\int_{B_n\cap\{|u|\leq\delta\}} |f(u_n) - f(u_n - u)||w|\,\mathrm{d}x \leq \eta(|u_n|_{q^*}^{q^*-1} + |u_n - u|_{q^*}^{q^*-1})|w|_{q^*} \leq c\eta\|w\|_\varepsilon. \tag{2.10}$$

Finally, define $C_n = \{x \in \mathbb{R}^N \setminus B_R(0) : r_0 \leq |u_n(x)| \leq r_1\}$. Since $u_n \in W^{1,p}(\mathbb{R}^N)$, it follows that $|C_n| < \infty$. Now (2.7) gives

$$\int\limits_{C_n \cap \{|u| \leq \delta\}} |f(u_n) - f(u_n - u)||w| \, \mathrm{d}x \leq r_0^{p-1} \eta |w|_p |C_n|^{\frac{p-1}{p}} \leq \eta |u_n|_p |w|_p \leq c\eta \|w\|_\varepsilon. \tag{2.11}$$

Putting together (2.9), (2.10) and (2.11), we obtain that

$$\int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \leq \delta\}} |f(u_n) - f(u_n - u)||w| \, \mathrm{d}x \leq c\eta \|w\|_\varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.12}$$

Next, we note that (2.8) implies

$$|f(u_n) - f(u_n - u)| \leq \eta (|u_n|^{q^*-1} + |u_n - u|^{q^*-1}) + c(\eta)(|u_n|^{p-1} + |u_n - u|^{p-1}),$$

so we can see that

$$\begin{aligned} &\int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}} |f(u_n) - f(u_n - u)||w| \, \mathrm{d}x \\ &\leq \int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}} \Big[\eta (|u_n|^{q^*-1} + |u_n - u|^{q^*-1})|w| + c(\eta)(|u_n|^{p-1} + |u_n - u|^{p-1})|w| \Big] \, \mathrm{d}x \\ &\leq c\eta \|w\|_\varepsilon + \int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}} c(\eta)(|u_n|^{p-1} + |u_n - u|^{p-1})|w| \, \mathrm{d}x. \end{aligned}$$

Since $u \in W^{1,p}(\mathbb{R}^N)$, we get $|(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}| \to 0$ as $R \to \infty$. Then choosing $R = R(\eta)$ large enough we can infer

$$\begin{aligned} &\int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}} c(\eta)(|u_n|^{p-1} + |u_n - u|^{p-1})|w| \, \mathrm{d}x \\ &\qquad \leq c(\eta)(|u_n|_{q^*}^{p-1} + |u_n - u|_{q^*}^{p-1}) \, |w|_{q^*} \, |(\mathbb{R}^N \setminus B_R(0)) \cap \{u \geq \delta\}|^{\frac{q^*-p}{p}} \leq \eta \|w\|_\varepsilon, \end{aligned}$$

where we have used the generalized Hölder inequality. Therefore,

$$\int\limits_{(\mathbb{R}^N \setminus B_R(0)) \cap \{|u| \geq \delta\}} |f(u_n) - f(u_n - u)||w| \, \mathrm{d}x \leq c\eta \|w\|_\varepsilon \quad \text{for all } n \in \mathbb{N},$$

which combined with (2.12) yields

$$\int\limits_{\mathbb{R}^N \setminus B_R(0)} |f(u_n) - f(u) - f(u_n - u)||w| \, \mathrm{d}x \leq c\eta \|w\|_\varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

Now, recalling that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, we may assume that, up to a subsequence, $u_n \to u$ strongly converges in $L^p(B_R(0))$ and there exists $h \in L^p(B_R(0))$ such that $|u_n(x)|, |u(x)| \leq |h(x)|$ for a. e. $x \in B_R(0)$.

It is clear that

$$\int\limits_{B_R(0)} |f(u_n - u)||w| \, \mathrm{d}x \leq c\eta \|w\|_\varepsilon \tag{2.14}$$

provided that n is big enough. Let us define $D_n = \{x \in B_R(0) : |u_n(x) - u(x)| \geq 1\}$. Thus,

$$\int\limits_{D_n} |f(u_n) - f(u)||w| \, \mathrm{d}x \leq \int\limits_{D_n} \Big(c(\eta)(|u|^{p-1} + |u_n|^{p-1}) + \eta (|u_n|^{q^*-1} + |u|^{q^*-1}) \Big) |w| \, \mathrm{d}x$$

$$\leq c\eta\|w\|_\varepsilon + 2c(\eta)\int\limits_{D_n} |h|^{p-1}|w|\,\mathrm{d}x$$
$$\leq c\eta\|w\|_\varepsilon + 2c(\eta)\left(\int\limits_{D_n} |h|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} |w|_p.$$

Observing that $|D_n| \to 0$ as $n \to \infty$, we can deduce that

$$\int\limits_{D_n} |f(u_n) - f(u)||w|\,\mathrm{d}x \leq c\eta\|w\|_\varepsilon. \tag{2.15}$$

Since $u \in W^{1,p}(\mathbb{R}^N)$, we know that $|\{|u| \geq L\}| \to 0$ as $L \to \infty$, so there exists $L = L(\eta) > 0$ such that for all n

$$\begin{aligned}
&\int\limits_{(B_R(0)\setminus D_n)\cap\{|u|\geq L\}} |f(u_n) - f(u)||w|\,\mathrm{d}x \\
&\leq \int\limits_{(B_R(0)\setminus D_n)\cap\{|u|\geq L\}} \left[\eta(|u_n|^{q^*-1} + |u|^{q^*-1})|w| + c(\eta)(|u_n|^{p-1} + |u|^{p-1})|w|\right]\mathrm{d}x \\
&\leq c\eta\|w\|_\varepsilon + c(\eta)(|u_n|_{q^*}^{p-1} + |u|_{q^*}^{p-1})\,|w|_{q^*}\,|(B_R(0)\setminus D_n)\cap\{|u|\geq L\}|^{\frac{q^*-p}{p}} \\
&\leq c\eta\|w\|_\varepsilon.
\end{aligned} \tag{2.16}$$

On the other hand, by the dominated convergence theorem we can infer

$$\int\limits_{(B_R(0)\setminus D_n)\cap\{|u|\leq L\}} |f(u_n) - f(u)|^p\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\int\limits_{(B_R(0)\setminus D_n)\cap\{|u|\leq L\}} |f(u_n) - f(u)||w|\,\mathrm{d}x \leq c\eta\|w\|_\varepsilon \tag{2.17}$$

for n large enough. Putting together (2.15), (2.16) and (2.17), we have

$$\int\limits_{B_R(0)} |f(u_n) - f(u)||w|\,\mathrm{d}x \leq c\eta\|w\|_\varepsilon.$$

This and (2.14) yield

$$\int\limits_{B_R(0)} |f(u_n) - f(u) - f(u_n - u)||w|\,\mathrm{d}x \leq c\eta\|w\|_\varepsilon. \tag{2.18}$$

Taking into account (2.13) and (2.18), we can conclude that for n large enough

$$\int\limits_{\mathbb{R}^N} |f(u_n) - f(u) - f(u_n - u)||w|\,\mathrm{d}x \leq c\eta\|w\|_\varepsilon.$$

This completes the proof of lemma. $\square$

3. Functional setting

In this section, we consider the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)\left(|u|^{p-2}u + |u|^{p-2}u\right) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{P_ε}$$

In order to study (P_ε), we look for critical points of the functional $\mathcal{I}_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(u) = \frac{1}{p}|\nabla u|_p^p + \frac{1}{q}|\nabla u|_q^q + \int_{\mathbb{R}^N} V(\varepsilon x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

It is easy to see that $\mathcal{I}_\varepsilon \in C^1(\mathbb{X}_\varepsilon, \mathbb{R})$ and its differential is given by

$$\begin{aligned} \langle \mathcal{I}_\varepsilon'(u), \varphi\rangle &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u)\,\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} f(u)\varphi\,\mathrm{d}x \end{aligned}$$

for any $u, \varphi \in \mathbb{X}_\varepsilon$. Now, let us introduce the Nehari manifold associated to $\mathcal{I}_\varepsilon$, that is

$$\mathcal{N}_\varepsilon = \left\{u \in \mathbb{X}_\varepsilon \setminus \{0\} : \langle \mathcal{I}_\varepsilon'(u), u\rangle = 0\right\},$$

and define

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u).$$

Let us note that $\mathcal{I}_\varepsilon$ possesses a mountain pass geometry [4].

Lemma 3.1. *The functional $\mathcal{I}_\varepsilon$ satisfies the following conditions:*
(i) there exist $\alpha, \rho > 0$ such that $\mathcal{I}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;
(ii) there exists $e \in \mathbb{X}_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $\mathcal{I}_\varepsilon(e) < 0$.

Proof. (i) Using (f_2) and (f_3), for any given $\xi > 0$ there exists $C_\xi > 0$ such that

$$|f(t)| \leq \xi|t|^{p-1} + C_\xi|t|^{r-1} \quad \text{for any } t \in \mathbb{R}, \tag{3.1}$$
$$|F(t)| \leq \frac{\xi}{p}|t|^p + \frac{C_\xi}{r}|t|^r \quad \text{for any } t \in \mathbb{R}. \tag{3.2}$$

Hence, taking $\xi \in (0, V_0)$, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(u) &\geq \frac{1}{p}\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - \frac{\xi}{p}|u|_p^p - \frac{C_\xi}{r}|u|_r^r \\ &\geq C_1\|u\|_{V,p}^p + \frac{1}{q}\|u\|_{V,q}^q - C_\xi'\|u\|_\varepsilon^r. \end{aligned}$$

Choosing $\|u\|_\varepsilon = \rho \in (0,1)$ and using $1 < p < q$, we have $\|u\|_{V,p} < 1$ and therefore $\|u\|_{V,p}^p \geq \|u\|_{V,p}^q$ which combined with $a^t + b^t \geq C_t(a+b)^t$ for any $a, b \geq 0$ and $t > 1$, yields

$$\mathcal{I}_\varepsilon(u) \geq C\|u\|_\varepsilon^q - C_\xi'\|u\|_\varepsilon^r.$$

Since $r > q$ we can find $\alpha > 0$ such that $\mathcal{I}_\varepsilon(u) \geq \alpha > 0$ for $\|u\|_\varepsilon = \rho$.
(ii) By (f_4), we can infer

$$F(t) \geq C_1|t|^\vartheta - C_2 \quad \text{for any } t \geq 0,$$

for some $C_1, C_2 > 0$. Taking $v \in C_c^\infty(\mathbb{R}^N)$ such that $v \geq 0$, $v \not\equiv 0$, we have

$$\mathcal{I}_\varepsilon(tv) \leq \frac{t^p}{p}\|v\|_\varepsilon^p + \frac{t^q}{q}\|v\|_\varepsilon^q - t^\vartheta C_1 \int\limits_{\operatorname{supp} v} v^\vartheta \mathrm{d}x + C_2|\operatorname{supp} v| \to -\infty \text{ as } t \to \infty.$$

$\square$

Now, in view of Lemma 3.1, we can use a version of mountain pass theorem without the Palais-Smale condition [25] to deduce the existence of a (PS)-sequence $\{u_n\}$ at level c'_ε, namely

$$\mathcal{I}_\varepsilon(u_n) \to c'_\varepsilon \quad \text{ and } \quad \mathcal{I}'_\varepsilon(u_n) \to 0,$$

where c'_ε is the mountain pass level of $\mathcal{I}_\varepsilon$ defined as

$$c'_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_\varepsilon(\gamma(t)),$$

and $\Gamma = \{\gamma \in C^0([0,1], \mathbb{X}_\varepsilon) : \gamma(0) = 0, \mathcal{I}_\varepsilon(\gamma(1)) < 0\}$.

Lemma 3.2. *The following holds*

$$c'_\varepsilon = c_\varepsilon = \inf_{u \in \mathbb{X}_\varepsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\varepsilon(tu).$$

Proof. For each $u \in \mathbb{X}_\varepsilon \setminus \{0\}$ and $t > 0$, let us introduce the function $h(t) = \mathcal{I}_\varepsilon(tu)$. Following the same arguments as in the proof of Lemma 3.1, we deduce that $h(0) = 0$, $h(t) < 0$ for t sufficiently large and $h(t) > 0$ for t sufficiently small. Hence, $\max_{t \geq 0} h(t)$ is achieved at $t = t_u > 0$ satisfying $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

Note that, if $u \in \mathcal{N}_\varepsilon$ then $u^+ \neq 0$. Indeed, from (f_1), we can deduce that

$$\|u\|_{V,p}^p + \|u\|_{V,q}^q = \int\limits_{\mathbb{R}^N} f(u)u \,\mathrm{d}x = \int\limits_{\mathbb{R}^N} f(u^+)u^+ \,\mathrm{d}x.$$

Now, if $u^+ \equiv 0$, then $\|u\|_{V,p}^p + \|u\|_{V,q}^q = 0$, that is $u \equiv 0$, and this is a contradiction in view of $u \in \mathcal{N}_\varepsilon$.

Next, we prove that t_u is the unique critical point of h. Assume by contradiction that there exist t_1 and t_2 such that $t_1 u, t_2 u \in \mathcal{N}_\varepsilon$, that is

$$t_1^{p-q}|\nabla u|_p^p + |\nabla u|_q^q + t_1^{p-q} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,\mathrm{d}x + \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^q \,\mathrm{d}x = \int\limits_{\{u>0\}} \frac{f(t_1 u)}{(t_1 u)^{q-1}} u^q \,\mathrm{d}x$$

and

$$t_2^{p-q}|\nabla u|_p^p + |\nabla u|_q^q + t_2^{p-q} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,\mathrm{d}x + \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^q \,\mathrm{d}x = \int\limits_{\{u>0\}} \frac{f(t_2 u)}{(t_2 u)^{q-1}} u^q \,\mathrm{d}x.$$

Subtracting term by term in the above equalities, we get

$$(t_1^{p-q} - t_2^{p-q})|\nabla u|_p^p + (t_1^{p-q} - t_2^{p-q}) \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,\mathrm{d}x = \int\limits_{\{u>0\}} \left[\frac{f(t_1 u)}{(t_1 u)^{q-1}} - \frac{f(t_2 u)}{(t_2 u)^{q-1}}\right] u^q \mathrm{d}x.$$

Now, if $t_1 < t_2$, from (f_5) and recalling that $p < q$, we can infer

$$0 < (t_1^{p-q} - t_2^{p-q})|\nabla u|_p^p + (t_1^{p-q} - t_2^{p-q}) \int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^p \,\mathrm{d}x = \int\limits_{\{u>0\}} \left[\frac{f(t_1 u)}{(t_1 u)^{q-1}} - \frac{f(t_2 u)}{(t_2 u)^{q-1}}\right] u^q \mathrm{d}x < 0,$$

which gives a contradiction. Now we can argue as in [25] to complete the proof. $\square$

Next, we prove the following useful result.

Lemma 3.3. *Let $\{u_n\}$ be a Palais-Smale sequence of $\mathcal{I}_\varepsilon$ at level c. Then*

(i) $\{u_n\}$ *is bounded in* $\mathbb{X}_\varepsilon$*.*
(ii) $u_n^- \to 0$ *in* $\mathbb{X}_\varepsilon$ *and we may assume that* $u_n \geq 0$ *for any* $n \in \mathbb{N}$*.*

Proof. (i) From (f_4), we have

$$\begin{aligned} C(1+\|u_n\|_\varepsilon) &\geq \mathcal{I}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{I}'_\varepsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p} + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^q_{V,q} + \frac{1}{\vartheta}\int\limits_{\mathbb{R}^N} (f(u_n)u_n - \vartheta F(u_n))\, \mathrm{d}x \\ &\geq \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p} + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^q_{V,q} \\ &\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)(\|u_n\|^p_{V,p} + \|u_n\|^q_{V,q}). \end{aligned}$$

Now, assume by contradiction that $\|u_n\|_\varepsilon \to \infty$. We shall distinguish among the following cases:
Case 1. $\|u_n\|_{V,p} \to \infty$ and $\|u_n\|_{V,q} \to \infty$.
Since $p < q$, we have, for n sufficiently large, that $\|u_n\|^{q-p}_{V,q} \geq 1$, that is $\|u_n\|^q_{V,q} \geq \|u_n\|^p_{V,q}$, and thus

$$\begin{aligned} C(1+\|u_n\|_\varepsilon) &\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\left(\|u_n\|^p_{V,p} + \|u_n\|^p_{V,q}\right) \\ &\geq C_1\left(\|u_n\|_{V,p} + \|u_n\|_{V,q}\right)^p = C_1\|u_n\|^p_\varepsilon, \end{aligned}$$

which gives a contradiction.
Case 2. $\|u_n\|_{V,p} \to \infty$ and $\|u_n\|_{V,q}$ is bounded.
We can see that

$$C\left(1+\|u_n\|_{V,p} + \|u_n\|_{V,q}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V,p}$$

implies

$$C\left(\frac{1}{\|u_n\|^p_{V,p}} + \frac{1}{\|u_n\|^{p-1}_{V,p}} + \frac{\|u_n\|_{V,q}}{\|u_n\|^p_{V,p}}\right) \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right),$$

and letting $n \to \infty$, we get $0 \geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right) > 0$, which yields a contradiction.
$\|u_n\|_{V,p}$ is bounded and $\|u_n\|_{V,q} \to \infty$.
Case 3. We can proceed similarly as in the case (2).

Hence, $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$ and we may assume that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.
(ii) Since $\langle \mathcal{I}'_\varepsilon(u_n), u_n^-\rangle = o_n(1)$, where $u_n^- = \min\{u_n, 0\}$, and $f(t) = 0$ for $t \leq 0$, we have that

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n^-\, \mathrm{d}x + \int\limits_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla u_n^-\, \mathrm{d}x \\ &\quad + \int\limits_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n)\, u_n^-\, \mathrm{d}x = o_n(1), \end{aligned}$$

from which it follows

$$\|u_n^-\|^p_{V,p} + \|u_n^-\|^q_{V,q} = o_n(1),$$

that is $u_n^- \to 0$ in $\mathbb{X}_\varepsilon$. Moreover, $\{u_n^+\}$ is bounded in $\mathbb{X}_\varepsilon$. Now, we prove that $\mathcal{I}_\varepsilon(u_n^+) \to c$ and $\mathcal{I}'_\varepsilon(u_n^+) = o_n(1)$. Clearly, $\|u_n\|_{V,t} = \|u_n^+\|_{V,t} + o_n(1)$ for $t \in \{p, q\}$. On the other hand, by (3.2), the mean value theorem, and since $u_n = u_n^+ + u_n^-$, we have

$$\left|\int\limits_{\mathbb{R}^N} F(u_n)\,\mathrm{d}x - \int\limits_{\mathbb{R}^N} F(u_n^+)\,\mathrm{d}x\right| \le C\int\limits_{\mathbb{R}^N} (|u_n|^{p-1} + |u_n|^{r-1})|u_n^-|\,\mathrm{d}x$$
$$\le C|u_n^-|_p + C|u_n^-|_r \le C\|u_n^-\|_{V,p} + C\|u_n^-\|_{V,q} \le C\|u_n^-\|_\varepsilon = o_n(1).$$

This shows that $\mathcal{I}_\varepsilon(u_n^+) \to c$. Next, we claim that $\mathcal{I}'_\varepsilon(u_n^+) = o_n(1)$. Fix $\varphi \in \mathbb{X}_\varepsilon$ such that $\|\varphi\|_\varepsilon \le 1$. Then we have

$$\begin{aligned}
&\big|\langle \mathcal{I}'_\varepsilon(u_n), \varphi\rangle - \langle \mathcal{I}'_\varepsilon(u_n^+), \varphi\rangle\big| \\
&\quad = \Big|\int\limits_{\mathbb{R}^N} [|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_n^+|^{p-2}\nabla u_n^+]\nabla\varphi\,\mathrm{d}x + \int\limits_{\mathbb{R}^N} [|\nabla u_n|^{q-2}\nabla u_n - |\nabla u_n^+|^{q-2}\nabla u_n^+]\nabla\varphi\,\mathrm{d}x \\
&\qquad + \int\limits_{\mathbb{R}^N} V(\varepsilon x)[(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n) - (|u_n^+|^{p-2}u_n^+ + |u_n^+|^{q-2}u_n^+)]\,\varphi\,\mathrm{d}x \\
&\qquad - \int\limits_{\mathbb{R}^N} [f(u_n) - f(u_n^+)]\varphi\,\mathrm{d}x\Big|.
\end{aligned}$$

Now, recalling that for all $\xi > 0$ there exists $C_\xi > 0$ such that

$$||a+b|^{t-2}(a+b) - |a|^{t-2}a| \le \xi|a|^{t-1} + C_\xi|b|^{t-1} \quad \text{for all } a, b \in \mathbb{R}^N \text{ and } t > 1,$$

we see that for $t \in \{p, q\}$ the following holds

$$\begin{aligned}
&\Big|\int\limits_{\mathbb{R}^N} [|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_n^+|^{t-2}\nabla u_n^+]\nabla\varphi\,\mathrm{d}x\Big| \\
&\quad \le \xi|\nabla u_n^+|_t^{t-1}|\nabla\varphi|_t + C_\xi|\nabla u_n^-|_t^{t-1}|\nabla\varphi|_t \\
&\quad \le \xi C + C'_\xi\|u_n^-\|_\varepsilon^{t-1}.
\end{aligned}$$

Consequently,

$$\limsup_{n\to\infty}\Big|\int\limits_{\mathbb{R}^N} [|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_n^+|^{t-2}\nabla u_n^+]\nabla\varphi\,\mathrm{d}x\Big| \le \xi C$$

and by the arbitrariness of $\xi > 0$ we get

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} [|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_n^+|^{t-2}\nabla u_n^+]\nabla\varphi\,\mathrm{d}x = 0.$$

A similar argument shows that

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} V(\varepsilon x)[(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n) - (|u_n^+|^{p-2}u_n^+ + |u_n^+|^{q-2}u_n^+)]\,\varphi\,\mathrm{d}x = 0.$$

Observing that

$$\begin{aligned}
\Big|\int\limits_{\mathbb{R}^N} [f(u_n) - f(u_n^+)]\varphi\,\mathrm{d}x\Big| &= \Big|\int\limits_{\mathbb{R}^N} f(u_n^-)\varphi\,\mathrm{d}x\Big| \\
&\le C\int\limits_{\mathbb{R}^N} (|u_n^-|^{p-1} + |u_n^-|^{r-1})|\varphi|\,\mathrm{d}x \\
&\le C(|u_n^-|_p^{p-1}|\varphi|_p + |u_n^-|_r^{r-1}|\varphi|_r)
\end{aligned}$$

$$\leq C(\|u_n^-\|_\varepsilon^{p-1} + \|u_n^-\|_\varepsilon^{r-1}) = o_n(1),$$

we can deduce that $|\langle \mathcal{I}_\varepsilon'(u_n), \varphi\rangle - \langle \mathcal{I}_\varepsilon'(u_n^+), \varphi\rangle| = o_n(1)$. Since $\langle \mathcal{I}_\varepsilon'(u_n), \varphi\rangle = o_n(1)$, we conclude that $\mathcal{I}_\varepsilon'(u_n^+) = o_n(1)$. □

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$. We begin by proving some properties of the functional $\mathcal{I}_\varepsilon$.

Lemma 3.4. *Under assumptions (V) and (f_1)–(f_5), for any $\varepsilon > 0$ we have:*

(i) $\mathcal{I}_\varepsilon'$ maps bounded sets of $\mathbb{X}_\varepsilon$ into bounded sets of $\mathbb{X}_\varepsilon$.
(ii) $\mathcal{I}_\varepsilon'$ is weakly sequentially continuous in $\mathbb{X}_\varepsilon$.
(iii) $\mathcal{I}_\varepsilon(t_n u_n) \to -\infty$ as $t_n \to \infty$, where $u_n \in K$ and $K \subset \mathbb{X}_\varepsilon \setminus \{0\}$ is a compact subset.

Proof. (i) Let $\{u_n\}$ be a bounded sequence in $\mathbb{X}_\varepsilon$ and $v \in \mathbb{X}_\varepsilon$. Then from assumptions (f_2) and (f_3) we can deduce that

$$\langle \mathcal{I}_\varepsilon'(u_n), v\rangle \leq C_1\|u_n\|_\varepsilon^{p-1}\|v\|_\varepsilon + C_2\|u_n\|_\varepsilon^{q-1}\|v\|_\varepsilon + C_3\|u_n\|_\varepsilon^{r-1}\|v\|_\varepsilon \leq C.$$

(ii) Let $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$. By Lemma 2.3, we have that $u_n \to u$ in $L^t_{loc}(\mathbb{R}^N)$ for all $t \in [1, q_s^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Then, for all $v \in C_c^\infty(\mathbb{R}^N)$, it follows from (3.1) and the dominated convergence theorem that

$$\langle \mathcal{I}_\varepsilon'(u_n), v\rangle \to \langle \mathcal{I}_\varepsilon'(u), v\rangle. \tag{3.3}$$

Since $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathbb{X}_\varepsilon$, we can take $\{v_j\} \subset C_c^\infty(\mathbb{R}^N)$ such that $\|v_j - v\|_\varepsilon \to 0$ as $j \to \infty$. Note that (3.1) and Lemma 2.3 yield

$$\begin{aligned} |\langle \mathcal{I}_\varepsilon'(u_n), v\rangle - \langle \mathcal{I}_\varepsilon'(u), v\rangle| &\leq |\langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), v_j\rangle| + |\langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), v - v_j\rangle| \\ &\leq |\langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), v_j\rangle| + C\int_{\mathbb{R}^N} (|u_n|^{p-1} + |u|^{p-1} + |u_n|^{r-1} + |u|^{r-1})|v - v_j| \,\mathrm{d}x \\ &\leq |\langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), v_j\rangle| + C\|v_j - v\|_\varepsilon. \end{aligned}$$

For any $\zeta > 0$, fix $j_0 \in \mathbb{N}$ such that $\|v_{j_0} - v\|_\varepsilon < \frac{\zeta}{2C}$. By (3.3), there is $n_0 \in \mathbb{N}$ such that

$$|\langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), v_{j_0}\rangle| < \frac{\zeta}{2} \quad \text{for all } n \geq n_0.$$

Thus,

$$|\langle \mathcal{I}_\varepsilon'(u_n), v\rangle - \langle \mathcal{I}_\varepsilon'(u), v\rangle| < \zeta \quad \text{for all } n \geq n_0$$

and this shows that $\mathcal{I}_\varepsilon'$ is weakly sequentially continuous in $\mathbb{X}_\varepsilon$.

(iii) Without loss of generality, we may assume that $\|u\|_\varepsilon \leq 1$ for each $u \in K$. For $u_n \in K$, after passing to a subsequence, we obtain that $u_n \to u \in \mathbb{S}_\varepsilon$. Then, using (f_4) and Fatou's lemma, we can see that

$$\begin{aligned} \mathcal{I}_\varepsilon(t_n u_n) &= \frac{t_n^p}{p}\|u_n\|_\varepsilon^p + \frac{t_n^q}{q}\|u_n\|_\varepsilon^q - \int_{\mathbb{R}^N} F(t_n u_n)\,\mathrm{d}x \\ &\leq t_n^\vartheta \left(\frac{\|u_n\|_\varepsilon^p}{t_n^{\vartheta-p}} + \frac{\|u_n\|_\varepsilon^q}{t_n^{\vartheta-q}} - \int_{\mathbb{R}^N} \frac{F(t_n u_n)}{t_n^\vartheta}\,\mathrm{d}x \right) \to -\infty \text{ as } n \to \infty. \end{aligned}$$

□

Lemma 3.5. *Under the assumptions of Lemma 3.4, for $\varepsilon > 0$ we have:*

(i) for all $u \in \mathbb{S}_\varepsilon$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon(u) = t_u u$ is the unique maximum of $\mathcal{I}_\varepsilon$ on $\mathbb{X}_\varepsilon$, where $\mathbb{S}_\varepsilon = \{u \in \mathbb{X}_\varepsilon : \|u\|_\varepsilon = 1\}$.
(ii) The set $\mathcal{N}_\varepsilon$ is bounded away from 0. Furthermore, $\mathcal{N}_\varepsilon$ is closed in $\mathbb{X}_\varepsilon$.

(*iii*) *There exists* $\alpha > 0$ *such that* $t_u \geq \alpha$ *for each* $u \in \mathbb{S}_\varepsilon$ *and, for each compact subset* $W \subset \mathbb{S}_\varepsilon$*, there exists* $C_W > 0$ *such that* $t_u \leq C_W$ *for all* $u \in W$.

(*iv*) *For each* $u \in \mathcal{N}_\varepsilon$, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in \mathcal{N}_\varepsilon$. *In particular,* $\mathcal{N}_\varepsilon$ *is a regular manifold diffeomorphic to the sphere in* $\mathbb{X}_\varepsilon$.

(*v*) $c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon \geq \rho > 0$ *and* $\mathcal{I}_\varepsilon$ *is bounded below on* $\mathcal{N}_\varepsilon$*, where* ρ *is independent of* ε.

Proof. (*i*) The proof follows the same lines as the proof of Lemma 3.2.
(*ii*) Using (3.1) and Lemma 2.3, for any $u \in \mathcal{N}_\varepsilon$ we have

$$\|u\|_{V,p}^p + \|u\|_{V,q}^q = \int_{\mathbb{R}^N} f(u)u \, \mathrm{d}x \leq \frac{\xi}{V_0}\|u\|_{V,p}^p + C_\xi \|u\|_\varepsilon^r.$$

Taking $\xi > 0$ sufficiently small, we can deduce that

$$C_1\|u\|_{V,p}^p + \|u\|_{V,q}^q \leq C\|u\|_\varepsilon^r.$$

Now, if $\|u\|_\varepsilon \geq 1$, we are done. If $\|u\|_\varepsilon < 1$, then $\|u\|_{V,p}^p \geq \|u\|_{V,p}^q$ so we get

$$C\|u\|_\varepsilon^r \geq C_1\|u\|_{V,p}^p + \|u\|_{V,q}^q \geq C_1\|u\|_{V,p}^q + \|u\|_{V,q}^q \geq C_2\|u\|_\varepsilon^q,$$

which implies that $\|u\|_\varepsilon \geq \kappa$ for some $\kappa > 0$.
Next, we prove that $\mathcal{N}_\varepsilon$ is closed in $\mathbb{X}_\varepsilon$. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. From Lemma 3.4, we infer that $\mathcal{I}_\varepsilon'(u_n)$ is bounded, so

$$\langle \mathcal{I}_\varepsilon'(u_n), u_n\rangle - \langle \mathcal{I}_\varepsilon'(u), u\rangle = \langle \mathcal{I}_\varepsilon'(u_n) - \mathcal{I}_\varepsilon'(u), u\rangle + \langle \mathcal{I}_\varepsilon'(u_n), u_n - u\rangle \to 0,$$

that is $\langle \mathcal{I}_\varepsilon'(u), u\rangle = 0$, which combined with $\|u\|_\varepsilon \geq \kappa$ implies that

$$\|u\|_\varepsilon = \lim_{n\to\infty} \|u_n\|_\varepsilon \geq \kappa > 0,$$

hence $u \in \mathcal{N}_\varepsilon$.
(*iii*) For each $u \in \mathbb{S}_\varepsilon$, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Then, using $\|u\|_\varepsilon \geq \kappa$, we also have $t_u = \|t_u u\|_\varepsilon \geq \kappa$. It remains we prove that $t_u \leq C_W$ for all $u \in W \subset \mathbb{S}_\varepsilon$. We argue by contradiction: we suppose that there exists a sequence $\{u_n\} \subset W \subset \mathbb{S}_\varepsilon$ such that $t_{u_n} \to \infty$. Since W is compact, we can find $u \in W$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

Now, using (f_4) we have

$$\begin{aligned}
\mathcal{I}_\varepsilon(u) &= \mathcal{I}_\varepsilon(u) - \frac{1}{q}\langle \mathcal{I}_\varepsilon'(u), u\rangle \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)|\nabla u|_p^p + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u|^p \mathrm{d}x - \int_{\mathbb{R}^N}\left(F(u) - \frac{1}{q}f(u)u\right)\mathrm{d}x \\
&= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|_{V,p}^p - \int_{\mathbb{R}^N}\left(F(u) - \frac{1}{q}f(u)u\right)\mathrm{d}x \geq 0,
\end{aligned}$$

and this is in contrast with Lemma 3.4-(*iii*) by which $\mathcal{I}_\varepsilon(t_{u_n}u_n) \to -\infty$ as $n \to \infty$.
(*iv*) Let us define the maps $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon \setminus \{0\} \to \mathcal{N}_\varepsilon$ and $m_\varepsilon : \mathbb{S}_\varepsilon \to \mathcal{N}_\varepsilon$ by setting

$$\hat{m}_\varepsilon(u) = t_u u \quad \text{and} \quad m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon}. \tag{3.4}$$

In view of (*i*)–(*iii*) and Proposition 3.1 in [23], we can deduce that m_ε is a homeomorphism between $\mathbb{S}_\varepsilon$ and $\mathcal{N}_\varepsilon$ and the inverse of m_ε is given by $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$. Therefore, $\mathcal{N}_\varepsilon$ is a regular manifold diffeomorphic to $\mathbb{S}_\varepsilon$.
(*v*) For $\varepsilon > 0$, $t > 0$ and $u \in \mathbb{X}_\varepsilon \setminus \{0\}$, we can see that (3.2) yields

$$\mathcal{I}_\varepsilon(tu) \geq \frac{t^p}{p}|\nabla u|_p^p + \frac{t^q}{q}|\nabla u|_q^q + \int_{\mathbb{R}^N} V(\varepsilon x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)\mathrm{d}x - \frac{\xi t^p}{V_0}\int_{\mathbb{R}^N} V_0|u|^p \, \mathrm{d}x - C_\xi t^r \int_{\mathbb{R}^N} |u|^r \, \mathrm{d}x$$

$$\geq \frac{t^p}{p}\left(1-\frac{\xi}{V_0}\right)\|u\|_{V,p}^p+\frac{t^q}{q}\|u\|_{V,q}^q-C_\xi t^r\|u\|_\varepsilon^r$$

so we can find $\rho>0$ such that $\mathcal{I}_\varepsilon(tu)\geq\rho>0$ for $t>0$ small enough. On the other hand, by using (i)–(iii), we get (see [23]) that

$$c_\varepsilon=\inf_{u\in\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon(u)=\inf_{u\in\mathbb{X}_\varepsilon\setminus\{0\}}\max_{t>0}\mathcal{I}_\varepsilon(tu)=\inf_{u\in\mathbb{S}_\varepsilon}\max_{t>0}\mathcal{I}_\varepsilon(tu) \tag{3.5}$$

which implies $c_\varepsilon\geq\rho$ and $\mathcal{I}_\varepsilon|_{\mathcal{N}_\varepsilon}\geq\rho$. □

Now we introduce the following functionals $\hat{\Psi}_\varepsilon:\mathbb{X}_\varepsilon\setminus\{0\}\to\mathbb{R}$ and $\Psi_\varepsilon:\mathbb{S}_\varepsilon\to\mathbb{R}$ defined by

$$\hat{\Psi}_\varepsilon=\mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))\quad\text{ and }\quad\Psi_\varepsilon=\hat{\Psi}_\varepsilon|_{\mathbb{S}_\varepsilon},$$

where $\hat{m}_\varepsilon(u)=t_u u$ is given in (3.4). As in [23], we have the following result:

Lemma 3.6. *Under the assumptions of Lemma 3.4, we have that for $\varepsilon>0$:*

(i) $\Psi_\varepsilon\in C^1(\mathbb{S}_\varepsilon,\mathbb{R})$, *and*

$$\langle\Psi'_\varepsilon(w),v\rangle=\|m_\varepsilon(w)\|_\varepsilon\langle\mathcal{I}'_\varepsilon(m_\varepsilon(w)),v\rangle\quad\text{ for } v\in T_w(\mathbb{S}_\varepsilon).$$

(ii) *$\{w_n\}$ is a Palais-Smale sequence for Ψ_ε if and only if $\{m_\varepsilon(w_n)\}$ is a Palais-Smale sequence for $\mathcal{I}_\varepsilon$. If $\{u_n\}\subset\mathcal{N}_\varepsilon$ is a bounded Palais-Smale sequence for $\mathcal{I}_\varepsilon$, then $\{m_\varepsilon^{-1}(u_n)\}$ is a Palais-Smale sequence for Ψ_ε.*

(iii) *$u\in\mathbb{S}_\varepsilon$ is a critical point of Ψ_ε if and only if $m_\varepsilon(u)$ is a critical point of $\mathcal{I}_\varepsilon$. Moreover, the corresponding critical values coincide and*

$$\inf_{\mathbb{S}_\varepsilon}\Psi_\varepsilon=\inf_{\mathcal{N}_\varepsilon}\mathcal{I}_\varepsilon=c_\varepsilon.$$

4. The autonomous problem

In this section, we deal with the autonomous problem associated with (P_ε), that is

$$\begin{cases}-\Delta_p u-\Delta_q u+\mu(|u|^{p-2}u+|u|^{q-2}u)=f(u) & \text{in } \mathbb{R}^N\\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \mu>0.\end{cases} \tag{AP_μ}$$

The functional associated with (AP_μ) is given by

$$\mathcal{J}_\mu(u)=\frac{1}{p}|\nabla u|_p^p+\frac{1}{q}|\nabla u|_q^q+\mu\left[\frac{1}{p}|u|_p^p+\frac{1}{q}|u|_q^q\right]-\int_{\mathbb{R}^N}F(u)\,\mathrm{d}x$$

which is well-defined on the space $\mathbb{Y}_\mu=W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\mu=\|u\|_{\mu,p}+\|u\|_{\mu,q},$$

where

$$\|u\|_{\mu,t}^t=|\nabla u|_t^t+\mu|u|_t^t\quad\text{ for all } t>1.$$

It is easy to check that $\mathcal{J}_\mu\in C^1(\mathbb{Y}_\mu,\mathbb{R})$ and its differential is given by

$$\langle\mathcal{J}'_\mu(u),\varphi\rangle=\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x$$
$$+\mu\left[\int_{\mathbb{R}^N}|u|^{p-2}u\,\varphi\,\mathrm{d}x+\int_{\mathbb{R}^N}|u|^{q-2}u\,\varphi\,\mathrm{d}x\right]-\int_{\mathbb{R}^N}f(u)\varphi\,\mathrm{d}x$$

for any $u, \varphi \in \mathbb{Y}_\mu$. Let us define the Nehari manifold associated with $\mathcal{J}_\mu$

$$\mathcal{M}_\mu = \{u \in \mathbb{Y}_\mu \setminus \{0\} : \langle \mathcal{J}'_\mu(u), u\rangle = 0\}.$$

We note that (f_4) yields

$$\begin{aligned} \mathcal{J}_\mu(u) &= \mathcal{J}_\mu(u) - \frac{1}{q}\langle \mathcal{J}'_\mu(u), u\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p_{\mu,p} - \int_{\mathbb{R}^N} \left(F(u) - \frac{1}{q}f(u)u\right) \mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right)\|u\|^p_{\mu,p} \quad \text{ for all } u \in \mathcal{M}_\mu. \end{aligned} \tag{4.1}$$

Arguing as in the previous section and using (4.1), it is easy to prove the following lemma.

Lemma 4.1. *Under the assumptions of Lemma 3.4, for $\mu > 0$ we have:*

(i) *for all $u \in \mathbb{S}_\mu$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{M}_\mu$. Moreover, $m_\mu(u) = t_u u$ is the unique maximum of $\mathcal{J}_\mu$ on $\mathbb{Y}_\mu$, where $\mathbb{S}_\mu = \{u \in \mathbb{Y}_\mu : \|u\|_\mu = 1\}$.*
(ii) *The set $\mathcal{M}_\mu$ is bounded away from 0. Furthermore, $\mathcal{M}_\mu$ is closed in $\mathbb{Y}_\mu$.*
(iii) *There exists $\alpha > 0$ such that $t_u \geq \alpha$ for each $u \in \mathbb{S}_\mu$ and, for each compact subset $W \subset \mathbb{S}_\mu$, there exists $C_W > 0$ such that $t_u \leq C_W$ for all $u \in W$.*
(iv) *$\mathcal{M}_\mu$ is a regular manifold diffeomorphic to the sphere in $\mathbb{Y}_\mu$.*
(v) *$d_\mu = \inf_{\mathcal{M}_\mu} \mathcal{J}_\mu > 0$ and $\mathcal{J}_\mu$ is bounded below on $\mathcal{M}_\mu$ by some positive constant.*
(vi) *$\mathcal{J}_\mu$ is coercive on $\mathcal{M}_\mu$.*

Now we define the following functionals $\hat{\Psi}_\mu : \mathbb{Y}_\mu \setminus \{0\} \to \mathbb{R}$ and $\Psi_\mu : \mathbb{S}_\mu \to \mathbb{R}$ by setting

$$\hat{\Psi}_\mu = \mathcal{J}_\mu(\hat{m}_\mu(u)) \quad \text{ and } \quad \Psi_\mu = \hat{\Psi}_\mu|_{\mathbb{S}_\mu}.$$

Then we obtain the following result:

Lemma 4.2. *Under the assumptions of Lemma 3.4, we have that for $\mu > 0$:*

(i) *$\Psi_\mu \in C^1(\mathbb{S}_\mu, \mathbb{R})$, and*

$$\langle \Psi'_\mu(w), v\rangle = \|m_\mu(w)\|_\mu \langle \mathcal{J}'_\mu(m_\mu(w)), v\rangle \quad \text{ for } v \in T_w(\mathbb{S}_\mu).$$

(ii) *$\{w_n\}$ is a Palais-Smale sequence for Ψ_μ if and only if $\{m_\mu(w_n)\}$ is a Palais-Smale sequence for $\mathcal{J}_\mu$. If $\{u_n\} \subset \mathcal{M}_\mu$ is a bounded Palais-Smale sequence for $\mathcal{J}_\mu$, then $\{m_\mu^{-1}(u_n)\}$ is a Palais-Smale sequence for Ψ_μ.*
(iii) *$u \in \mathbb{S}_\mu$ is a critical point of Ψ_μ if and only if $m_\mu(u)$ is a critical point of $\mathcal{J}_\mu$. Moreover, the corresponding critical values coincide and*

$$\inf_{\mathbb{S}_\mu} \Psi_\mu = \inf_{\mathcal{M}_\mu} \mathcal{J}_\mu = d_\mu.$$

Remark 4.1. As in (3.5), invoking (i)–(iii) of Lemma 4.1, we can see that d_μ admits the following minimax characterization

$$d_\mu = \inf_{u \in \mathcal{M}_\mu} \mathcal{J}_\mu(u) = \inf_{u \in \mathbb{Y}_\mu \setminus \{0\}} \max_{t>0} \mathcal{J}_\mu(tu) = \inf_{u \in \mathbb{S}_\mu} \max_{t>0} \mathcal{J}_\mu(tu). \tag{4.2}$$

Lemma 4.3. *Let $\{u_n\} \subset \mathcal{M}_\mu$ be a minimizing sequence for $\mathcal{J}_\mu$. Then $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$ and there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q \mathrm{d}x \geq \beta > 0.$$

Proof. Arguing as in the proof of Lemma 3.3, we can see that $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$. Now, in order to prove the other assertion of this lemma, we argue by contradiction. Assume that for any $R > 0$ it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{\mathcal{B}_R(y)} |u_n|^q \mathrm{d}x = 0.$$

Since $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$, it follows by Lemma 2.1 that

$$u_n \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{ for any } t \in (q, q^*). \tag{4.3}$$

Fix $\xi \in (0, \mu)$. Then, taking into account that $\{u_n\} \subset \mathcal{M}_\mu$ and (3.1), we have

$$\begin{aligned} 0 &= \langle \mathcal{J}_\mu'(u_n), u_n\rangle \\ &\geq |\nabla u_n|_p^p + |\nabla u_n|_q^q + \mu\left[|u_n|_p^p + |u_n|_q^q\right] - \xi |u_n|_p^p - C_\xi |u_n|_r^r \\ &\geq C_1 \|u_n\|_{s,p}^p + C_2\|u_n\|_{s,q}^q - C_3|u_n|_r^r, \end{aligned}$$

and in view of (4.3), we have that $\|u_n\|_\mu \to 0$. □

Next, we prove the following useful compactness result for the autonomous problem. For completeness, we recall that a critical point $u \neq 0$ of $\mathcal{J}_\mu$ satisfying $\mathcal{J}_\mu(u) = \inf_{\mathcal{M}_\mu} \mathcal{J}_\mu = d_\mu$ is called a ground state solution to (AP_μ); see chapter 4 in [25] for more details.

Lemma 4.4. *The problem (AP_μ) has a positive ground-state solution.*

Proof. By virtue of (v) of Lemma 4.1, we know that $d_\mu > 0$ for each $\mu > 0$. Moreover, if $u \in \mathcal{M}_\mu$ satisfies $\mathcal{J}_\mu(u) = d_\mu$, then $m_\mu^{-1}(u)$ is a minimizer of Ψ_μ and it is a critical point of Ψ_μ. In view of Lemma 4.2, we can see that u is a critical point of $\mathcal{J}_\mu$. Now we show that there exists a minimizer of $\mathcal{J}_\mu|_{\mathcal{M}_\mu}$. By Ekeland's variational principle [25] there exists a sequence $\{\nu_n\} \subset \mathbb{S}_\mu$ such that $\Psi_\mu(\nu_n) \to d_\mu$ and $\Psi_\mu'(\nu_n) \to 0$ as $n \to \infty$. Let $u_n = m_\mu(\nu_n) \in \mathcal{M}_\mu$. Then, thanks to Lemma 4.2, $\mathcal{J}_\mu(u_n) \to d_\mu$ and $\mathcal{J}_\mu'(u_n) \to 0$ as $n \to \infty$. Therefore, arguing as in the proof of Lemma 3.3, $\{u_n\}$ is bounded in $\mathbb{Y}_\mu$ which is a reflexive space, so we may assume that $u_n \rightharpoonup u$ in $\mathbb{Y}_\mu$ for some $u \in \mathbb{Y}_\mu$.

It is clear that $\mathcal{J}_\mu'(u) = 0$. Indeed, for all $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla\phi \,\mathrm{d}x \to \int\limits_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla \phi\,\mathrm{d}x, \quad \text{ for } t \in \{p, q\}, \\ &\int\limits_{\mathbb{R}^N} |u_n|^{t-2} u_n \phi\,\mathrm{d}x \to \int\limits_{\mathbb{R}^N} |u|^{t-2}u\phi\,\mathrm{d}x, \quad \text{ for } t \in \{p, q\}, \\ &\int\limits_{\mathbb{R}^N} f(u_n)\phi\,\mathrm{d}x \to \int\limits_{\mathbb{R}^N} f(u)\phi\,\mathrm{d}x, \end{aligned}$$

and using the fact that $\langle \mathcal{J}_\mu'(u_n), \phi\rangle = o_n(1)$, we can deduce that $\langle \mathcal{J}_\mu'(u), \phi\rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^N)$. By the density of $\phi \in C_c^\infty(\mathbb{R}^N)$ in $\mathbb{Y}_\mu$, we obtain that u is a critical point of $\mathcal{J}_\mu$.

Now, if $u \neq 0$, then u is a nontrivial solution to (AP_μ). Assume that $u = 0$. Then $\|u_n\|_\mu \not\to 0$ in $\mathbb{Y}_\mu$. Hence, arguing as in the proof of Lemma 4.3 we can find a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int\limits_{\mathcal{B}_R(y_n)} |u_n|^q\mathrm{d}x \geq \beta > 0. \tag{4.4}$$

Now, let us define

$$\tilde{v}_n(x) = u_n(x + y_n).$$

Due to the invariance by translations of $\mathbb{R}^N$, it is clear that $\|\tilde{v}_n\|_{\mu,t} = \|u_n\|_{\mu,t}$, with $t \in \{p,q\}$, so $\{\tilde{v}_n\}$ is bounded in $\mathbb{Y}_\mu$ and there exists $\tilde{v}$ such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{Y}_\mu$, $\tilde{v}_n \to \tilde{v}$ in $L^m_{loc}(\mathbb{R}^N)$ for any $m \in [1, q^*)$ and $\tilde{v} \neq 0$ in view of (4.4). Moreover, $\mathcal{J}_\mu(\tilde{v}_n) = \mathcal{J}_\mu(u_n)$ and $\mathcal{J}'_\mu(\tilde{v}_n) = o_n(1)$, and arguing as before it is easy to check that $\mathcal{J}'_\mu(\tilde{v}) = 0$.

Now, say u be the solution obtained before, and we prove that u is a ground-state solution. It is clear that $d_\mu \leq \mathcal{J}_\mu(u)$. On the other hand, by Fatou's lemma we can see that

$$\mathcal{J}_\mu(u) = \mathcal{J}_\mu(u) - \frac{1}{q}\langle \mathcal{J}'_\mu(u), u\rangle \leq \liminf_{n\to\infty} \left[\mathcal{J}_\mu(u_n) - \frac{1}{q}\langle \mathcal{J}'_\mu(u_n), u_n\rangle\right] = d_\mu,$$

which implies that $d_\mu = \mathcal{J}_\mu(u)$.

Finally, we prove that the ground state obtained earlier is positive. Indeed, taking $u^- = \min\{u, 0\}$ as test function in (AP_μ), and applying (f_1) and invoking the following inequality

$$|x-y|^{t-2}(x-y)(x^- - y^-) \geq |x^- - y^-|^t \quad \forall t > 1,$$

we can see that

$$\begin{aligned}\|u^-\|^p_{\mu,p} + \|u^-\|^q_{\mu,q} \leq & \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \mu|u|^{p-2}uu^- \,\mathrm{d}x \\ & + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla u^- \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \mu|u|^{q-2}uu^- \,\mathrm{d}x \\ = & \int_{\mathbb{R}^N} f(u)u^- \,\mathrm{d}x = 0,\end{aligned}$$

which implies that $u^- = 0$, that is $u \geq 0$ in $\mathbb{R}^N$. By the regularity results in [13], we have that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$ (in the exponential way). Applying the Harnack inequality in [24], we can see that $u > 0$ in $\mathbb{R}^N$. This completes the proof of the lemma. □

5. A first existence result for (P_ε)

In this section, we focus on the existence of a solution to (P_ε) provided that ε is sufficiently small. Let us start with the following useful lemma.

Lemma 5.1. *Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $\mathcal{I}_\varepsilon(u_n) \to c$ and $u_n \rightharpoonup 0$ in $\mathbb{X}_\varepsilon$. Then one of the following alternatives occurs:*

(a) $u_n \to 0$ in $\mathbb{X}_\varepsilon$;
(b) there are a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q \mathrm{d}x \geq \beta > 0.$$

Proof. Assume that (b) does not hold. Then, for any $R > 0$, the following holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathcal{B}_R(y)} |u_n|^q \mathrm{d}x = 0.$$

Since $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$, it follows by Lemma 2.1 that

$$u_n \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{ for any } t \in (q, q^*). \tag{5.1}$$

Now, we can argue as in the proof of Lemma 4.3 and deduce that $\|u_n\|_\varepsilon \to 0$ as $n \to \infty$. □

In order to get a compactness result for $\mathcal{I}_\varepsilon$, we need to prove the following auxiliary lemma.

Lemma 5.2. *Assume that $V_\infty < \infty$ and let $\{v_n\} \subset \mathcal{N}_\varepsilon$ be a sequence such that $\mathcal{I}_\varepsilon(v_n) \to d$ with $v_n \rightharpoonup 0$ in $\mathbb{X}_\varepsilon$. If $v_n \not\to 0$ in $\mathbb{X}_\varepsilon$, then $d \geq d_{V_\infty}$, where d_{V_∞} is the infimum of $\mathcal{J}_{V_\infty}$ over $\mathcal{M}_{V_\infty}$.*

Proof. Let $\{t_n\} \subset (0,\infty)$ be such that $\{t_n v_n\} \subset \mathcal{M}_{V_\infty}$. Our aim is to show that $\limsup_{n\to\infty} t_n \leq 1$. Assume by contradiction that there exist $\delta > 0$ and a subsequence, denoted again by $\{t_n\}$, such that

$$t_n \geq 1+\delta \quad \text{ for any } n \in \mathbb{N}. \tag{5.2}$$

Since $\{v_n\} \subset \mathbb{X}_\varepsilon$ is a bounded (PS) sequence for $\mathcal{I}_\varepsilon$, we have that $\langle \mathcal{I}'_\varepsilon(v_n), v_n\rangle = o_n(1)$, or equivalently

$$|\nabla v_n|_p^p + |\nabla v_n|_q^q + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^p \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} f(v_n)v_n \,\mathrm{d}x = o_n(1). \tag{5.3}$$

Since $t_n v_n \in \mathcal{M}_{V_\infty}$, we also have that

$$t_n^{p-q}|\nabla v_n|_p^p + |\nabla v_n|_q^q + t_n^{p-q}V_\infty \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x + V_\infty \int_{\mathbb{R}^N} |v_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} \frac{f(t_n v_n)}{(t_n v_n)^{q-1}} v_n^q \,\mathrm{d}x = 0. \tag{5.4}$$

Putting together (5.3) and (5.4), we get

$$\int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{(v_n)^{q-1}} \right) v_n^q \,\mathrm{d}x \leq \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|v_n|^q \mathrm{d}x. \tag{5.5}$$

Now, using assumption (V) we can see that, given $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that

$$V(\varepsilon x) \geq V_\infty - \zeta \quad \text{ for any } |x| \geq R. \tag{5.6}$$

From this, taking into account that $v_n \to 0$ in $L^q(B_R)$ and the boundedness of $\{v_n\}$ in $\mathbb{X}_\varepsilon$, we can infer

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))|v_n|^q \mathrm{d}x &= \int_{B_R(0)} (V_\infty - V(\varepsilon x))|v_n|^q \mathrm{d}x + \int_{\mathbb{R}^N\setminus B_R(0)} (V_\infty - V(\varepsilon x))|v_n|^q \mathrm{d}x \\ &\leq V_\infty \int_{B_R(0)} |v_n|^q \mathrm{d}x + \zeta \int_{\mathbb{R}^N\setminus B_R(0)} |v_n|^q \mathrm{d}x \\ &\leq o_n(1) + \zeta C. \end{aligned} \tag{5.7}$$

Combining (5.5) and (5.7), we have

$$\int_{\mathbb{R}^N} \left(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{(v_n)^{q-1}} \right) v_n^q \,\mathrm{d}x \leq o_n(1) + \zeta C. \tag{5.8}$$

Since $v_n \not\to 0$ in $\mathbb{X}_\varepsilon$, we can apply Lemma 5.1 to deduce the existence of a sequence $\{y_n\} \subset \mathbb{R}^N$ and two positive numbers $\bar{R}, \beta$ such that

$$\int_{\mathcal{B}_{\bar{R}}(y_n)} |v_n|^q \,\mathrm{d}x \geq \beta > 0. \tag{5.9}$$

Let us consider $\tilde{v}_n = v_n(x+y_n)$. Then we may assume that, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{X}_\varepsilon$. By (5.9), there exists $\Omega \subset \mathbb{R}^N$ with positive measure and such that $\tilde{v} > 0$ in Ω. From (5.2), (f_4) and (5.8), we can infer that

$$0 < \int_{\Omega} \left(\frac{f((1+\delta)\tilde{v}_n)}{((1+\delta)\tilde{v}_n)^{q-1}} - \frac{f(\tilde{v}_n)}{(\tilde{v}_n)^{q-1}} \right) \tilde{v}_n^q \,\mathrm{d}x \leq o_n(1) + \zeta C.$$

Taking the limit as $n \to \infty$ and applying Fatou's lemma, we obtain

$$0 < \int_{\Omega} \left(\frac{f((1+\delta)\tilde{v})}{((1+\delta)\tilde{v})^{q-1}} - \frac{f(\tilde{v})}{(\tilde{v})^{q-1}} \right) \tilde{v}^q \, \mathrm{d}x \leq \zeta C \quad \text{for any } \zeta > 0,$$

which is a contradiction.

Now we consider the following cases:

CASE 1: Assume that $\limsup_{n\to\infty} t_n = 1$. Thus there exists $\{t_n\}$ such that $t_n \to 1$. Taking into account that $\mathcal{I}_\varepsilon(v_n) \to c$, we have

$$\begin{aligned} c + o_n(1) &= \mathcal{I}_\varepsilon(v_n) \\ &= \mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + \mathcal{J}_{V_\infty}(t_n v_n) \\ &\geq \mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + d_{V_\infty}. \end{aligned} \tag{5.10}$$

Now, let us point out that

$$\begin{aligned} &\mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) \\ &= \frac{(1-t_n^p)}{p} |\nabla v_n|_p^p + \frac{(1-t_n^q)}{q} |\nabla v_n|_q^q + \frac{1}{p} \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p \mathrm{d}x \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^q V_\infty) |v_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)) \, \mathrm{d}x. \end{aligned} \tag{5.11}$$

Using condition (V), $v_n \to 0$ in $L^p(B_R(0))$, $t_n \to 1$, (5.6), and the fact that

$$V(\varepsilon x) - t_n^p V_\infty = (V(\varepsilon x) - V_\infty) + (1-t_n^p) V_\infty \geq -\zeta + (1-t_n^p) V_\infty \text{ for any } |x| \geq R,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p \mathrm{d}x \\ &= \int_{B_R(0)} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p \mathrm{d}x + \int_{\mathbb{R}^N \setminus B_R(0)} (V(\varepsilon x) - t_n^p V_\infty) |v_n|^p \mathrm{d}x \\ &\geq (V_0 - t_n^p V_\infty) \int_{B_R(0)} |v_n|^p \mathrm{d}x - \zeta \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^p \mathrm{d}x + V_\infty (1-t_n^p) \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^p \mathrm{d}x \\ &\geq o_n(1) - \zeta C. \end{aligned} \tag{5.12}$$

In a similar fashion, we can prove that

$$\int_{\mathbb{R}^N} (V(\varepsilon x) - t_n^q V_\infty) |v_n|^q \mathrm{d}x \geq o_n(1) - \zeta C. \tag{5.13}$$

Since $\{v_n\}$ is bounded in $\mathbb{X}_\varepsilon$, we can conclude that

$$\frac{(1-t_n^p)}{p} |\nabla v_n|_p^p = o_n(1) \quad \text{and} \quad \frac{(1-t_n^q)}{q} |\nabla v_n|_q^q = o_n(1). \tag{5.14}$$

Thus, putting together (5.11), (5.12), (5.13) and (5.14), we obtain

$$\mathcal{I}_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) \geq \int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n)) \, \mathrm{d}x + o_n(1) - \zeta C. \tag{5.15}$$

At this point, we aim to show that

$$\int_{\mathbb{R}^N} (F(t_n v_n) - F(v_n))\, \mathrm{d}x = o_n(1). \tag{5.16}$$

Applying the mean value theorem and (3.1), we can deduce that

$$\int_{\mathbb{R}^N} |F(t_n v_n) - F(v_n)|\, \mathrm{d}x \leq C|t_n - 1| \int_{\mathbb{R}^N} |v_n|^p \mathrm{d}x + C|t_n - 1| \int_{\mathbb{R}^N} |v_n|^r \mathrm{d}x.$$

Exploiting the boundedness of $\{v_n\}$, we get the assertion. Gathering (5.10), (5.15) and (5.16), we can infer that

$$c + o_n(1) \geq o_n(1) - \zeta C + d_{V_\infty},$$

and taking the limit as $\zeta \to 0$ we get $c \geq d_{V_\infty}$.

Case 2: Assume that $\limsup_{n\to\infty} t_n = t_0 < 1$. Then there is a subsequence, still denoted by $\{t_n\}$, such that $t_n \to t_0 (< 1)$ and $t_n < 1$ for any $n \in \mathbb{N}$. Let us observe that

$$\begin{aligned} c + o_n(1) &= \mathcal{I}_\varepsilon(v_n) - \frac{1}{q}\langle \mathcal{I}'_\varepsilon(v_n), v_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \|v_n\|^p_{V,p} + \int_{\mathbb{R}^N} \left(\frac{1}{q} f(v_n)v_n - F(v_n)\right) \mathrm{d}x. \end{aligned} \tag{5.17}$$

Recalling that $t_n v_n \in \mathcal{M}_{V_\infty}$, and using (f_5) and (5.17), we obtain

$$\begin{aligned} d_{V_\infty} &\leq \mathcal{J}_{V_\infty}(t_n v_n) \\ &= \mathcal{J}_{V_\infty}(t_n v_n) - \frac{1}{q}\langle \mathcal{J}'_{V_\infty}(t_n v_n), t_n v_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \|t_n v_n\|^p_{V,p} + \int_{\mathbb{R}^N} \left(\frac{1}{q} f(t_n v_n)t_n v_n - F(t_n v_n)\right) \mathrm{d}x \\ &\leq \left(\frac{1}{p} - \frac{1}{q}\right) \|v_n\|^p_{V,p} + \int_{\mathbb{R}^N} \left(\frac{1}{q} f(v_n)v_n - F(v_n)\right) \mathrm{d}x \\ &= c + o_n(1). \end{aligned}$$

Taking the limit as $n \to \infty$, we get $c \geq d_{V_\infty}$. □

At this point, we are able to prove the following compactness result.

Proposition 5.1. *Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be such that $\mathcal{I}_\varepsilon(u_n) \to c$, where $c < d_{V_\infty}$ if $V_\infty < \infty$ and $c \in \mathbb{R}$ if $V_\infty = \infty$. Then $\{u_n\}$ has a convergent subsequence in $\mathbb{X}_\varepsilon$.*

Proof. It is easy to see that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$. Then, up to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } \mathbb{X}_\varepsilon, \\ &u_n \to u \text{ in } L^m_{loc}(\mathbb{R}^N) \quad \text{for any } m \in [1, q^*), \\ &u_n \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{5.18}$$

By using assumptions (f_2)–(f_3), (5.18) and the fact that $\mathcal{C}^\infty_c(\mathbb{R}^N)$ is dense in $\mathbb{X}_\varepsilon$, it is easy to check that $\mathcal{I}'_\varepsilon(u) = 0$.

Now, let $v_n = u_n - u$. By Lemma 2.5, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(v_n) &= \mathcal{I}_\varepsilon(u_n) - \mathcal{I}_\varepsilon(u) + o_n(1) \\ &= c - \mathcal{I}_\varepsilon(u) + o_n(1) = d + o_n(1). \end{aligned} \tag{5.19}$$

Now, we prove that $\mathcal{I}'_\varepsilon(v_n) = o_n(1)$. For $t \in \{p, q\}$, by using Lemma 2.2 with $\eta_n = v_n$ and $w = u$, we get

$$\iint_{\mathbb{R}^{2N}} |A(u_n) - A(v_n) - A(u)|^{t'} \mathrm{d}x = o_n(1), \tag{5.20}$$

and arguing as in the proof of Lemma 3.3 in [18], we can see that

$$\int_{\mathbb{R}^N} V(\varepsilon x)||v_n|^{t-2}v_n - |u_n|^{t-2}u_n + |u|^{t-2}u|^{t'} \mathrm{d}x = o_n(1). \tag{5.21}$$

Hence, by using the Hölder inequality, for any $\varphi \in \mathbb{X}_\varepsilon$ such that $\|\varphi\|_\varepsilon \leq 1$, we get

$$\begin{aligned}
&|\langle \mathcal{I}'_\varepsilon(v_n) - \mathcal{I}'_\varepsilon(u_n) + \mathcal{I}'_\varepsilon(u), \varphi\rangle| \\
&\leq \left(\iint_{\mathbb{R}^{2N}} |A(u_n) - A(v_n) - A(u)|^{p'} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p'}} [\varphi]_{s,p} \\
&\quad + \left(\iint_{\mathbb{R}^{2N}} |A(u_n) - A(v_n) - A(u)|^{q'} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{q'}} [\varphi]_{s,q} \\
&\quad + \left(\int_{\mathbb{R}^N} V(\varepsilon x)||v_n|^{p-2}v_n - |u_n|^{p-2}u_n + |u|^{p-2}u|^{p'} \mathrm{d}x \right)^{p'} \left(\int_{\mathbb{R}^N} V(\varepsilon x)|\varphi|^p \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad + \left(\int_{\mathbb{R}^N} V(\varepsilon x)||v_n|^{q-2}v_n - |u_n|^{q-2}u_n + |u|^{q-2}u|^{q'} \mathrm{d}x \right)^{q'} \left(\int_{\mathbb{R}^N} V(\varepsilon x)|\varphi|^q \mathrm{d}x \right)^{\frac{1}{q}} \\
&\quad + \int_{\mathbb{R}^N} |(f(v_n) - f(u_n) + f(u))\varphi| \mathrm{d}x,
\end{aligned}$$

and in view of (iv) of Lemma 2.5, (5.20), (5.21), $\mathcal{I}'_\varepsilon(u_n) = 0$ and $\mathcal{I}'_\varepsilon(u) = 0$ we obtain the assertion.

Now, we note that by using (f_4) we can see that

$$\mathcal{I}_\varepsilon(u) = \mathcal{I}_\varepsilon(u) - \frac{1}{q}\langle \mathcal{I}'_\varepsilon(u), u\rangle \geq 0. \tag{5.22}$$

Assume $V_\infty < \infty$. It follows from (5.19) and (5.22) that

$$d \leq c < d_{V_\infty}$$

which together Lemma 5.2 gives $v_n \to 0$ in $\mathbb{X}_\varepsilon$, that is $u_n \to u$ in $\mathbb{X}_\varepsilon$.
Let us consider the case $V_\infty = \infty$. Then, we can use Lemma 2.4 to deduce that $v_n \to 0$ in $L^m(\mathbb{R}^N)$ for all $m \in [p, q^*)$. This, combined with assumptions (f_2) and (f_3), implies that

$$\int_{\mathbb{R}^N} f(v_n)v_n \mathrm{d}x = o_n(1). \tag{5.23}$$

Since $\langle \mathcal{I}'_\varepsilon(v_n), v_n\rangle = o_n(1)$, and applying (5.23) we can infer that

$$\|v_n\|^p_\varepsilon = o_n(1),$$

which yields $u_n \to u$ in $\mathbb{X}_\varepsilon$. $\square$

We conclude this section by giving the proof of the existence of a ground-state solution to (P_ε) (that is a nontrivial critical point u of $\mathcal{I}_\varepsilon$ such that $\mathcal{I}_\varepsilon(u) = \inf_{\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon = c_\varepsilon$) whenever $\varepsilon > 0$ is small enough.

Theorem 5.1. *Assume that (V) and (f_1)–(f_5) hold. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, problem (P_ε) admits a ground-state solution.*

Proof. By (v) of Lemma 3.5, we know that $c_\varepsilon \geq \rho > 0$ for each $\varepsilon > 0$. Moreover, if $u_\varepsilon \in \mathcal{N}_\varepsilon$ satisfies $\mathcal{I}_\varepsilon(u_\varepsilon) = c_\varepsilon$, then $m_\varepsilon^{-1}(u_\varepsilon)$ is a minimizer of Ψ_ε and it is a critical point of Ψ_ε. By virtue of Lemma 3.6, we can see that u_ε is a critical point of $\mathcal{I}_\varepsilon$. It remains to show that there exists a minimizer of $\mathcal{I}_\varepsilon|_{\mathcal{N}_\varepsilon}$. By Ekeland's variational principle [25], there exists a sequence $\{v_n\} \subset \mathbb{S}_\varepsilon$ such that $\Psi_\varepsilon(v_n) \to c_\varepsilon$ and $\Psi'_\varepsilon(v_n) \to 0$ as $n \to \infty$. Let $u_n = m_\varepsilon(v_n) \in \mathcal{N}_\varepsilon$. Then, by Lemma 3.6, we deduce that $\mathcal{I}_\varepsilon(u_n) \to c_\varepsilon$, $\langle \mathcal{I}'_\varepsilon(u_n), u_n \rangle = 0$ and $\mathcal{I}'_\varepsilon(u_n) \to 0$ as $n \to \infty$. Therefore, $\{u_n\}$ is a Palais-Smale sequence for $\mathcal{I}_\varepsilon$ at level c_ε. It is easy to check that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$ and we denote by u its weak limit. It is also easy to verify that $\mathcal{I}'_\varepsilon(u) = 0$.

When $V_\infty = \infty$, by using Lemma 2.4, we have $\mathcal{I}_\varepsilon(u) = c_\varepsilon$ and $\mathcal{I}'_\varepsilon(u) = 0$.

Now, we deal with the case $V_\infty < \infty$. In view of Proposition 5.1, it is enough to show that $c_\varepsilon < d_{V_\infty}$ for small ε. Without loss of generality, we may suppose that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x).$$

Let $\mu \in \mathbb{R}$ be such that $\mu \in (V_0, V_\infty)$. Clearly, $d_{V_0} < d_\mu < d_{V_\infty}$. Let us prove that there exists a function $w \in \mathbb{Y}_\mu$ with compact support such that

$$\mathcal{J}_\mu(w) = \max_{t \geq 0} \mathcal{J}_\mu(tw) \quad \text{and} \quad \mathcal{J}_\mu(w) < d_{V_\infty}. \tag{5.24}$$

Let $\psi \in C_c^\infty(\mathbb{R}^N, [0,1])$ be such that $\psi = 1$ in $B_1(0)$ and $\psi = 2$ in $\mathbb{R}^N \setminus B_2(0)$. For any $R > 0$, we set $\psi_R(x) = \psi(\frac{x}{R})$. We consider the function $w_R(x) = \psi_R(x) w^\mu(x)$, where w^μ is a ground-state solution to (AP_μ). By the dominated convergence theorem, we can see that

$$\lim_{R \to \infty} \|w_R - w^\mu\|_{1,p} + \|w_R - w^\mu\|_{1,q} = 0. \tag{5.25}$$

Let $t_R > 0$ be such that $\mathcal{J}_\mu(t_R w_R) = \max_{t \geq 0} \mathcal{J}_\mu(t w_R)$. Then, $t_R w_R \in \mathcal{M}_\mu$. Now there exists $\bar{r} > 0$ such that $\mathcal{J}_\mu(t_{\bar{r}} w_{\bar{r}}) < d_{V_\infty}$. Indeed, if $\mathcal{J}_\mu(t_R w_R) \geq d_{V_\infty}$ for any $R > 0$, using $t_R w_R \in \mathcal{M}_\mu$, (5.25) and w^μ is a ground state, we can deduce that $t_R \to 1$ and

$$d_{V_\infty} \leq \liminf_{R \to \infty} \mathcal{J}_\mu(t_R w_R) = \mathcal{J}_\mu(w^\mu) = d_\mu < d_{V_\infty},$$

which gives a contradiction. Then, taking $w = \psi_{\bar{r}} w^\mu$, we can conclude that (5.24) holds.

Now, by (V), we obtain that for some $\bar{\varepsilon} > 0$

$$V(\varepsilon x) \leq \mu \quad \text{for all } x \in \operatorname{supp} w \text{ and } \varepsilon \in (0, \bar{\varepsilon}). \tag{5.26}$$

Then, in the light of (5.24) and (5.26), we have for all $\varepsilon \in (0, \bar{\varepsilon})$

$$\max_{t>0} \mathcal{I}_\varepsilon(tw) \leq \max_{t>0} \mathcal{J}_\mu(tw) = \mathcal{J}_\mu(w) < d_{V_\infty}.$$

It follows from (3.5) that $c_\varepsilon < d_{V_\infty}$ for all $\varepsilon \in (0, \bar{\varepsilon})$. □

6. Multiple solutions for (P_ε)

This section is devoted to the study of the multiplicity of solutions to (P_ε). We begin by proving the following result which will be needed to implement the barycenter machinery.

Proposition 6.1. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that the translated sequence*

$$v_n(x) = u_n(x + \tilde{y}_n)$$

has a subsequence which converges in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\} = \{\varepsilon_n \tilde{y}_n\}$ is such that $y_n \to y \in M$.

Proof. Since $\langle \mathcal{I}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$, we know that $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$. Since $d_{V_0} > 0$, we can infer that $\|u_n\|_{\varepsilon_n} \not\to 0$. Therefore, as in the proof of Lemma 5.1, we can find a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int\limits_{B_R(\tilde{y}_n)} |u_n|^q \,\mathrm{d}x \geq \beta. \tag{6.1}$$

Let us define

$$v_n(x) = u_n(x + \tilde{y}_n).$$

In view of the boundedness of $\{u_n\}$ and (6.1), we may assume that $v_n \rightharpoonup v$ in $\mathbb{Y}_{V_0}$ for some $v \neq 0$. Let $\{t_n\} \subset (0,\infty)$ be such that $w_n = t_n v_n \in \mathcal{M}_{V_0}$, and we set $y_n = \varepsilon_n \tilde{y}_n$.
Thus, by using the change of variables $z \mapsto x + \tilde{y}_n$, $V(x) \geq V_0$ and the invariance by translation, we can see that

$$d_{V_0} \leq \mathcal{J}_{V_0}(w_n) \leq \mathcal{I}_{\varepsilon_n}(t_n v_n) \leq \mathcal{I}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1).$$

Hence, we can infer $\mathcal{J}_{V_0}(w_n) \to d_{V_0}$. This fact and $\{w_n\} \subset \mathcal{M}_{V_0}$ imply that there exists $K > 0$ such that $\|w_n\|_{V_0} \leq K$ for all $n \in \mathbb{N}$. Moreover, we can prove that the sequence $\{t_n\}$ is bounded in $\mathbb{R}$. In fact, $v_n \not\to 0$ in $\mathbb{Y}_{V_0}$, so there exists $\alpha > 0$ such that $\|v_n\|_{V_0} \geq \alpha$. Consequently, for all $n \in \mathbb{N}$, we have

$$|t_n|\alpha \leq \|t_n v_n\|_{V_0} = \|w_n\|_{V_0} \leq K,$$

which yields $|t_n| \leq \frac{K}{\alpha}$ for all $n \in \mathbb{N}$. Therefore, up to a subsequence, we may suppose that $t_n \to t_0 \geq 0$. Let us show that $t_0 > 0$. Otherwise, if $t_0 = 0$, by the boundedness of $\{v_n\}$, we get $w_n = t_n v_n \to 0$ in $\mathbb{Y}_{V_0}$, that is $\mathcal{J}_{V_0}(w_n) \to 0$ which is in contrast with the fact $d_{V_0} > 0$. Thus, $t_0 > 0$ and, up to a subsequence, we may assume that $w_n \rightharpoonup w = t_0 v \neq 0$ in $\mathbb{Y}_{V_0}$.
Therefore,

$$\mathcal{J}_{V_0}(w_n) \to d_{V_0} \quad \text{ and } \quad w_n \rightharpoonup w \neq 0 \text{ in } \mathbb{Y}_{V_0}.$$

From Lemma 4.4, we can deduce that $w_n \to w$ in $\mathbb{Y}_{V_0}$, that is $v_n \to v$ in $\mathbb{Y}_{V_0}$.
Now, we show that $\{y_n\}$ has a subsequence satisfying $y_n \to y \in M$. First, we prove that $\{y_n\}$ is bounded in $\mathbb{R}^N$. Assume by contradiction that $\{y_n\}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \to \infty$.
First, we deal with the case $V_\infty = \infty$. By using $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ and by changing the variable, we can see that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)\mathrm{d}x \\
&\quad \leq |\nabla v_n|_p^p + |\nabla v_n|_q^q + \int\limits_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)\mathrm{d}x \\
&\quad = \int\limits_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x = \int\limits_{\mathbb{R}^N} f(v_n)v_n \,\mathrm{d}x.
\end{aligned}$$

By applying Fatou's lemma and $v_n \to v$ in $\mathbb{Y}_{V_0}$, we deduce that

$$\infty = \liminf_{n\to\infty} \int\limits_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^p + |v_n|^q)\mathrm{d}x \leq \liminf_{n\to\infty} \int\limits_{\mathbb{R}^N} f(v_n)v_n \mathrm{d}x = \int\limits_{\mathbb{R}^N} f(v)v \,\mathrm{d}x < \infty,$$

which gives a contradiction.
Let us consider the case $V_\infty < \infty$. Taking into account that $w_n \to w$ strongly converges in $\mathbb{Y}_{V_0}$, condition (V) and using the change of variable $z = x + \tilde{y}_n$, we have

$$d_{V_0} = \mathcal{J}_{V_0}(w) < \mathcal{J}_{V_\infty}(w)$$

$$
\begin{aligned}
&\le \liminf_{n\to\infty}\left[\frac{1}{p}|\nabla w_n|_p^p+\frac{1}{q}|\nabla w_n|_q^q+\int_{\mathbb{R}^N} V(\varepsilon_n x+y_n)\left(\frac{1}{p}|w_n|^p+\frac{1}{q}|w_n|^q\right)\mathrm{d}x-\int_{\mathbb{R}^N} F(w_n)\,\mathrm{d}x\right]\\
&=\liminf_{n\to\infty}\left[\frac{t_n^p}{p}|\nabla u_n|_p^p+\frac{t_n^q}{q}|\nabla u_n|_q^q+\int_{\mathbb{R}^N} V(\varepsilon_n z)\left(\frac{t_n^p}{p}|u_n|^p+\frac{t_n^q}{q}|u_n|^q\right)\mathrm{d}z-\int_{\mathbb{R}^N} F(t_n u_n)\,\mathrm{d}z\right]\\
&=\liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(t_n u_n)\le \liminf_{n\to\infty}\mathcal{I}_{\varepsilon_n}(u_n)=d_{V_0} \qquad (6.2)
\end{aligned}
$$

which is a contradiction. Thus, $\{y_n\}$ is bounded and, up to a subsequence, we may assume that $y_n\to y$. If $y\notin M$, then $V_0<V(y)$ and we can argue as in (6.2) to get a contradiction. Therefore, we can conclude that $y\in M$. □

Let $\delta>0$ be fixed and let $\psi\in C^\infty([0,\infty),[0,1])$ be a nonincreasing function such that $\psi=1$ in $[0,\frac{\delta}{2}]$, $\psi=0$ in $[\delta,\infty)$ and $|\psi'|\le C$ for some $C>0$. For any $y\in M$, we define

$$
\Upsilon_{\varepsilon,y}(x)=\psi(|\varepsilon x-y|)\omega\left(\frac{\varepsilon x-y}{\varepsilon}\right),
$$

where $\omega\in\mathbb{X}_{V_0}$ is a ground-state solution to (AP_{V_0}) which exists by virtue of Lemma 4.4.

Let $t_\varepsilon>0$ be the unique positive number such that

$$
\mathcal{I}_\varepsilon(t_\varepsilon\Upsilon_{\varepsilon,y})=\max_{t\ge0}\mathcal{I}_\varepsilon(t\Upsilon_{\varepsilon,y}).
$$

Define the map $\Phi_\varepsilon: M\to\mathcal{N}_\varepsilon$ by setting $\Phi_\varepsilon(y):=t_\varepsilon\Upsilon_{\varepsilon,y}$. Then we can prove that

Lemma 6.1. *The functional Φ_ε satisfies the following limit*

$$
\lim_{\varepsilon\to0}\mathcal{I}_\varepsilon(\Phi_\varepsilon(y))=d_{V_0} \text{ uniformly in } y\in M. \tag{6.3}
$$

Proof. Assume by contradiction that there exist $\delta_0>0$, $\{y_n\}\subset M$ and $\varepsilon_n\to0$ such that

$$
|\mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-d_{V_0}|\ge\delta_0. \tag{6.4}
$$

Let us observe that the dominated convergence theorem implies

$$
|\nabla\Upsilon_{\varepsilon_n,y_n}|_p^p+\int_{\mathbb{R}^N}V(\varepsilon_n x)|\Upsilon_{\varepsilon_n,y_n}|^p\,\mathrm{d}x\to|\nabla\omega|_p^p+\int_{\mathbb{R}^N}V_0|\omega|^p\,\mathrm{d}x \tag{6.5}
$$

and

$$
|\nabla\Upsilon_{\varepsilon_n,y_n}|_q^q+\int_{\mathbb{R}^N}V(\varepsilon_n x)|\Upsilon_{\varepsilon_n,y_n}|^q\,\mathrm{d}x\to|\nabla\omega|_q^q+\int_{\mathbb{R}^N}V_0|\omega|^q\,\mathrm{d}x. \tag{6.6}
$$

Since $\langle\mathcal{I}'_{\varepsilon_n}(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}),t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}\rangle=0$, we can use the change of variable $z=\frac{\varepsilon_n x-y_n}{\varepsilon_n}$ to see that

$$
\begin{aligned}
&t_{\varepsilon_n}^p|\nabla\Upsilon_{\varepsilon_n,y_n}|_p^p+t_{\varepsilon_n}^q|\nabla\Upsilon_{\varepsilon_n,y_n}|_q^q+\int_{\mathbb{R}^N}V(\varepsilon_n x)\left(|t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}|^p+|t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n}|^q\right)\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}f(t_{\varepsilon_n}\Upsilon_{\varepsilon_n})t_{\varepsilon_n}\Upsilon_{\varepsilon_n}\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}f(t_{\varepsilon_n}\psi(|\varepsilon_n z|)\omega(z))t_{\varepsilon_n}\psi(|\varepsilon_n z|)\omega(z)\,\mathrm{d}z. \qquad (6.7)
\end{aligned}
$$

Now, we prove that $t_{\varepsilon_n} \to 1$. First we show that $t_{\varepsilon_n} \to t_0 < \infty$. Assume by contradiction that $|t_{\varepsilon_n}| \to \infty$. Then, using the fact that $\psi(|x|) = 1$ for $x \in B_{\frac{\delta}{2}}(0)$ and that $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{2\varepsilon_n}}(0)$ for n sufficiently large, we can see that (6.7) and (f_5) imply

$$\begin{aligned} t_{\varepsilon_n}^{p-q}|\nabla \Upsilon_{\varepsilon_n,y_n}|_p^p + |\nabla \Upsilon_{\varepsilon_n,y_n}|_q^q + \int\limits_{\mathbb{R}^N} V(\varepsilon_n x)\left(t_{\varepsilon_n}^{p-q}|\Upsilon_{\varepsilon_n,y_n}|^p + |\Upsilon_{\varepsilon_n,y_n}|^q\right) \mathrm{d}x \\ \geq \int\limits_{B_{\frac{\delta}{2}}(0)} \frac{f(t_{\varepsilon_n}\omega(z))}{(t_{\varepsilon_n}\omega(z))^{q-1}}(\omega(z))^q \mathrm{d}z \geq \frac{f(t_{\varepsilon_n}\omega(\bar{z}))}{(t_{\varepsilon_n}\omega(\bar{z}))^{q-1}} \int\limits_{B_{\frac{\delta}{2}}(0)} (\omega(z))^q \mathrm{d}z \end{aligned} \tag{6.8}$$

where $\bar{z} \in \mathbb{R}^N$ is such that $\omega(\bar{z}) = \min\{\omega(z) : |z| \leq \frac{\delta}{2}\} > 0$ (note that $\omega \in C(\mathbb{R}^N)$ and $\omega > 0$ in $\mathbb{R}^N$). Putting together (f_4), $p < q$, $t_{\varepsilon_n} \to \infty$, (6.5) and (6.6), we can see that (6.8) implies that $\|\Upsilon_{\varepsilon_n,y_n}\|_{V,q}^q \to \infty$, which gives a contradiction. Therefore, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0 \geq 0$. If $t_0 = 0$, we can use (6.5), (6.6), (6.7), $p < q$ and (f_2), to get

$$\|\Upsilon_{\varepsilon_n,y_n}\|_{V,p}^p \to 0,$$

which is a contradiction. Hence, $t_0 > 0$. Now, we show that $t_0 = 1$. Letting $n \to \infty$ in (6.7), we can see that

$$t_0^{p-q}|\nabla \omega|_p^p + |\nabla \omega|_q^q + \int\limits_{\mathbb{R}^N} V_0(t_0^{p-q}\omega^p \mathrm{d}x + \omega^q)\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} \frac{f(t_0\omega)}{(t_0\omega)^{q-1}}\omega^q\,\mathrm{d}x. \tag{6.9}$$

Since $\omega \in \mathcal{M}_{V_0}$, we have

$$|\nabla \omega|_p^p + |\nabla \omega|_q^q + \int\limits_{\mathbb{R}^N} V_0(\omega^p \mathrm{d}x + \omega^q)\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} f(\omega)\omega\,\mathrm{d}x. \tag{6.10}$$

Putting together (6.11) and (6.10), we find

$$(t_0^{p-q} - 1)|\nabla \omega|_p^p + (t_0^{p-q} - 1)\int\limits_{\mathbb{R}^N} V_0\omega^p\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} \left(\frac{f(t_0\omega)}{(t_0\omega)^{q-1}} - \frac{f(\omega)}{\omega^{q-1}}\right)\omega^q\,\mathrm{d}x. \tag{6.11}$$

By (f_5), we can deduce that $t_0 = 1$. This fact and the dominated convergence theorem yield

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} F(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n})\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} F(\omega)\,\mathrm{d}x. \tag{6.12}$$

Hence, taking the limit as $n \to \infty$ in

$$\begin{aligned} \mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p}|\nabla \Upsilon_{\varepsilon_n,y_n}|_p^p + \frac{t_{\varepsilon_n}^q}{q}|\nabla \Upsilon_{\varepsilon_n,y_n}|_q^q \\ &\quad + \int\limits_{\mathbb{R}^N} V(\varepsilon_n x)\left(\frac{t_{\varepsilon_n}^p}{p}|\Upsilon_{\varepsilon_n,y_n}|^p + \frac{t_{\varepsilon_n}^q}{q}|\Upsilon_{\varepsilon_n,y_n}|^q\right)\mathrm{d}x \\ &\quad - \int\limits_{\mathbb{R}^N} F(t_{\varepsilon_n}\Upsilon_{\varepsilon_n,y_n})\,\mathrm{d}x \end{aligned}$$

and exploiting (6.5), (6.6) and (6.12), we can deduce that

$$\lim_{n\to\infty} \mathcal{I}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \mathcal{J}_{V_0}(\omega) = d_{V_0}$$

which is impossible in view of (6.4). $\square$

Now, we are in the position to introduce the barycenter map. We take $\rho > 0$ such that $M_\delta \subset B_\rho(0)$, and we set $\chi : \mathbb{R}^N \to \mathbb{R}^N$ as follows

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)\,(|u|^p + |u|^q)\,\mathrm{d}x}{\int_{\mathbb{R}^N} (|u|^p + |u|^q)\,\mathrm{d}x}.$$

Lemma 6.2. *The functional Φ_ε verifies the following limit*

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M. \tag{6.13}$$

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0. \tag{6.14}$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n}, ψ and the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n](|\psi(|\varepsilon_n z|)\omega(z)|^p + |\psi(|\varepsilon_n z|)\omega(z)|^q)\,\mathrm{d}z}{\int_{\mathbb{R}^N} (|\psi(|\varepsilon_n z|)\omega(z)|^p + |\psi(|\varepsilon_n z|)\omega(z)|^q)\,\mathrm{d}z}.$$

Taking into account $\{y_n\} \subset M \subset B_\rho(0)$ and applying the dominated convergence theorem, we can infer that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which contradicts (6.14). $\square$

At this point, we introduce a subset $\widetilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by taking a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$, and setting

$$\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \mathcal{I}_\varepsilon(u) \leq d_{V_0} + h(\varepsilon)\},$$

where $h(\varepsilon) = \sup_{y \in M} |\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}|$. By Lemma 6.1, we know that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. By definition of $h(\varepsilon)$, we can deduce that for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$. Moreover, we have the following lemma.

Lemma 6.3. *For any $\delta > 0$, the following holds*

$$\lim_{\varepsilon \to 0} \sup_{u \in \widetilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, there exists $\{u_n\} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u \in \widetilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it suffices to prove that there exists $\{y_n\} \subset M_\delta$ such that

$$\lim_{n \to \infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{6.15}$$

Thus, recalling that $\{u_n\} \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can deduce that

$$d_{V_0} \leq c_{\varepsilon_n} \leq \mathcal{I}_{\varepsilon_n}(u_n) \leq d_{V_0} + h(\varepsilon_n)$$

which implies that $\mathcal{I}_{\varepsilon_n}(u_n) \to d_{V_0}$. By Proposition 6.1, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n sufficiently large. Thus,

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int\limits_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n](|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,\mathrm{d}z}{\int\limits_{\mathbb{R}^N} (|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,\mathrm{d}z}.$$

Since $u_n(\cdot+\tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n z + y_n \to y \in M$, we can deduce that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$, that is (6.15) holds. □

Now we show that (P_ε) admits at least $cat_{M_\delta}(M)$ solutions. In order to achieve our aim, we recall the following result for critical points involving Ljusternik–Schnirelmann category. For more details, one can see [10].

Theorem 6.1. *Let U be a $C^{1,1}$ complete Riemannian manifold (modeled on a Hilbert space). Assume that $h \in C^1(U,\mathbb{R})$ is bounded from below and satisfies $-\infty < \inf_U h < d < k < \infty$. Moreover, suppose that h satisfies the Palais-Smale condition on the sublevel $\{u \in U : h(u) \leq k\}$ and that d is not a critical level for h. Then*

$$card\{u \in h^d : \nabla h(u) = 0\} \geq cat_{h^d}(h^d),$$

where $h^d = \{u \in U \,:\, h(u) \leq d\}$.

With a view to apply Theorem 6.1, the following abstract lemma provides a very useful tool since relates the topology of some sublevel of a functional to the topology of some subset of the space $\mathbb{R}^N$; see [10].

Lemma 6.4. *Let Ω, Ω_1 and Ω_2 be closed sets with $\Omega_1 \subset \Omega_2$ and let $\pi : \Omega \to \Omega_2$, $\psi : \Omega_1 \to \Omega$ be continuous maps such that $\pi \circ \psi$ is homotopically equivalent to the embedding $j : \Omega_1 \to \Omega_2$. Then $cat_\Omega(\Omega) \geq cat_{\Omega_2}(\Omega_1)$.*

Since $\mathcal{N}_\varepsilon$ is not a C^1 submanifold of $\mathbb{X}_\varepsilon$, we cannot directly apply Theorem 6.1. Fortunately, by Lemma 3.5, we know that the mapping m_ε is a homeomorphism between $\mathcal{N}_\varepsilon$ and $\mathbb{S}_\varepsilon$, and $\mathbb{S}_\varepsilon$ is a C^1 submanifold of $\mathbb{X}_\varepsilon$. So we can apply Theorem 6.1 to $\Psi_\varepsilon(u) = \mathcal{I}_\varepsilon(\hat{m}_\varepsilon(u))|_{\mathbb{S}_\varepsilon} = \mathcal{I}_\varepsilon(m_\varepsilon(u))$, where Ψ_ε is given in Lemma 3.6. In the light of the above observations, we are ready to give the proof of the main result of this work.

Proof of Theorem 1.1. For any $\varepsilon > 0$, we define $\alpha_\varepsilon : M \to \mathbb{S}_\varepsilon$ by setting $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. By using Lemma 6.1 and the definition of Ψ_ε, we can see that

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \quad \text{ uniformly in } y \in M.$$

Set $\tilde{\mathbb{S}}_\varepsilon = \{w \in \mathbb{S}_\varepsilon : \Psi_\varepsilon(w) \leq d_{V_0} + h(\varepsilon)\}$, where $h(\varepsilon) = \sup_{y\in M} |\Psi_\varepsilon(\alpha_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$. Thus, $\alpha_\varepsilon(y) \in \tilde{\mathbb{S}}_\varepsilon$ for all $y \in M$, and this yields $\tilde{\mathbb{S}}_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.

Taking into account Lemma 6.1, Lemma 3.5, Lemma 3.6 and Lemma 6.3, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \widetilde{\mathcal{N}}_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \tilde{\mathbb{S}}_\varepsilon \xrightarrow{m_\varepsilon} \widetilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined for any $\varepsilon \in (0,\bar{\varepsilon})$. By using Lemma 6.2, there exists a function $\theta(\varepsilon, y)$ with $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$, for all $\varepsilon \in (0,\bar{\varepsilon})$, such that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$. We can see that $H(t,y) = y + (1-t)\theta(\varepsilon,y)$, with $(t,y) \in [0,1] \times M$, is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \alpha_\varepsilon$ and the inclusion map $id : M \to M_\delta$. This fact and Lemma 6.4 imply that $cat_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon) \geq cat_{M_\delta}(M)$. On the other hand, let us choose a function $h(\varepsilon) > 0$ such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $d_{V_0} + h(\varepsilon)$ is not a critical level for $\mathcal{I}_\varepsilon$. For $\varepsilon > 0$ small enough, we deduce from Proposition 5.1 that $\mathcal{I}_\varepsilon$ satisfies the Palais-Smale condition in $\widetilde{\mathcal{N}}_\varepsilon$. So, by (ii) of Lemma 3.6, we infer that Ψ_ε satisfies the Palais-Smale condition in $\tilde{\mathbb{S}}_\varepsilon$. Hence, by using Theorem 6.1, we obtain that Ψ_ε has at least $cat_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon)$ critical points on $\tilde{\mathbb{S}}_\varepsilon$. Then, in view of (iii) of Lemma 3.6, we can infer that $\mathcal{I}_\varepsilon$ admits at least $cat_{M_\delta}(M)$ critical points. □

7. Concentration of solutions to (P_ε)

Let us start with the following result which plays a fundamental role in the study of the behavior of maximum points of solutions to (P_ε).

Lemma 7.1. *Let v_n be a weak solution of the problem*

$$\begin{cases} -\Delta_p v_n - \Delta_q v_n + V_n(x)(|v_n|^{p-2}v_n + |v_n|^{q-2}v_n) = f(v_n) & \text{in } \mathbb{R}^N \\ v_n \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ v_n > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P_{V_n}}$$

where $V_n(x) \geq V_0$ and $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ for some $v \not\equiv 0$. Then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \leq C$ for all $n \in \mathbb{N}$. Moreover,

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof. We follow some ideas in [3,13] by developing a suitable Moser iteration argument [19]. For any $R > 0$, $0 < r \leq \frac{R}{2}$, let $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $\mathbb{R}^N \setminus B_R(0)$, $\eta = 0$ in $\overline{B_{R-r}(0)}$ and $|\nabla\eta| \leq 2/r$. For each $n \in \mathbb{N}$ and for $L > 0$, let

$$z_{L,n} = \eta^q v_n v_{L,n}^{q(\beta-1)} \quad \text{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\beta-1},$$

where $v_{L,n} = \min\{v_n, L\}$ and $\beta > 1$ to be determined later. Choosing $z_{L,n}$ as a test function in (P_{V_n}), we have

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla z_{L,n} + |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla z_{L,n} + V_n(v_n^{p-1} + v_n^{q-1})z_{L,n}\,\mathrm{d}x = \int_{\mathbb{R}^N} f(v_n)z_{L,n}\,\mathrm{d}x.$$

By assumptions (f_1) and (f_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|f(t)| \leq \xi|t|^{p-1} + C_\xi|t|^{q^*-1} \text{ for all } t \in \mathbb{R}.$$

Hence, using (V_1) and choosing $\xi \in (0, V_0)$, we have

$$\int_{\mathbb{R}^N} \eta^q v_{L,n}^{q(\beta-1)}|\nabla v_n|^q\,\mathrm{d}x \leq C_\xi \int_{\mathbb{R}^N} v_n^{q^*}\eta^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x - q\int_{\mathbb{R}^N} \eta^{q-1}v_{L,n}^{q(\beta-1)}v_n|\nabla v_n|^{q-2}\nabla v_n \cdot \nabla\eta\,\mathrm{d}x.$$

For each $\tau > 0$, we can use Young's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^q v_{L,n}^{q(\beta-1)}|\nabla v_n|^q\,\mathrm{d}x &\leq C_\xi \int_{\mathbb{R}^N} v_n^{q^*}\eta^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x + q\tau\int_{\mathbb{R}^N} |\nabla v_n|^q v_{L,n}^{q(\beta-1)}\eta^q\,\mathrm{d}x \\ &\quad + qC_\tau \int_{\mathbb{R}^N} v_n^q|\nabla\eta|^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x\end{aligned}$$

and taking $\tau > 0$ sufficiently small, we get

$$\int_{\mathbb{R}^N} \eta^q v_{L,n}^{q(\beta-1)}|\nabla v_n|^q\,\mathrm{d}x \leq C\int_{\mathbb{R}^3} v_n^{q^*}\eta^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x + C\int_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x. \tag{7.1}$$

On the other hand, using the Sobolev inequality and the Hölder inequality, we can infer

$$\begin{aligned}|w_{L,n}|_{q^*}^q &\leq C\int_{\mathbb{R}^N} |\nabla w_{L,n}|^q\,\mathrm{d}x = C\int_{\mathbb{R}^N} |\nabla(\eta v_{L,n}^{\beta-1}v_n)|^q\,\mathrm{d}x \\ &\leq C\beta^q\left(\int_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x + \int_{\mathbb{R}^N} \eta^q v_{L,n}^{q(\beta-1)}|\nabla v_n|^q\,\mathrm{d}x\right).\end{aligned} \tag{7.2}$$

Combining (7.1) and (7.2), we find

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q(\beta-1)} \, \mathrm{d}x + \int\limits_{\mathbb{R}^N} v_n^{q^*} \eta^q v_{L,n}^{q(\beta-1)} \, \mathrm{d}x \right). \tag{7.3}$$

We claim that $v_n \in L^{\frac{(q^*)^2}{q}}(|x| \geq R)$ for R large enough and uniformly in n. Let $\beta = \frac{q^*}{q}$. From (7.3), we have

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q^*-q} \, \mathrm{d}x + \int\limits_{\mathbb{R}^N} v_n^{q^*} \eta^q v_{L,n}^{q^*-q} \, \mathrm{d}x \right)$$

or equivalently

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q^*-q} \, \mathrm{d}x + \int\limits_{\mathbb{R}^N} v_n^q \eta^q v_{L,n}^{q^*-q} v_n^{q^*-q} \, \mathrm{d}x \right).$$

Using the Hölder inequality with exponents $\frac{q^*}{q}$ and $\frac{q^*}{q^*-q}$, we obtain

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q^*-q} \, \mathrm{d}x \right) + C\beta^q \left(\int\limits_{\mathbb{R}^N} (v_n \eta v_{L,n}^{\frac{q^*-q}{q}})^{q^*} \mathrm{d}x \right)^{\frac{q}{q^*}} \left(\int\limits_{|x| \geq \frac{R}{2}} v_n^{q^*} \, \mathrm{d}x \right)^{\frac{q^*-q}{q^*}}.$$

From the definition of $w_{L,n}$, we have

$$\left(\int\limits_{\mathbb{R}^N} (v_n \eta v_{L,n}^{\frac{q^*-q}{q}})^{q^*} \mathrm{d}x \right)^{\frac{q}{q^*}} \leq C\beta^q \left(\int\limits_{\mathbb{R}^N} |\nabla\eta|^q v_n^q v_{L,n}^{q^*-q} \, \mathrm{d}x \right)$$
$$+ C\beta^q \left(\int\limits_{\mathbb{R}^N} (v_n \eta v_{L,n}^{\frac{q^*-q}{q}})^{q^*} \mathrm{d}x \right)^{\frac{q}{q^*}} \left(\int\limits_{|x| \geq \frac{R}{2}} v_n^{q^*} \, \mathrm{d}x \right)^{\frac{q^*-q}{q^*}}.$$

Since $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, for $R > 0$ sufficiently large, we get

$$\int\limits_{|x| \geq \frac{R}{2}} v_n^{q^*} \, \mathrm{d}x \leq \epsilon \quad \text{uniformly in } n \in \mathbb{N}.$$

Hence,

$$\left(\int\limits_{|x| \geq R} (v_n \eta v_{L,n}^{\frac{q^*-q}{q}})^{q^*} \mathrm{d}x \right)^{\frac{q}{q^*}} \leq C\beta^q \int\limits_{\mathbb{R}^N} v_n^q v_{L,n}^{q^*-q} \, \mathrm{d}x \leq C\beta^q \int\limits_{\mathbb{R}^N} v_n^q \, \mathrm{d}x \leq K < \infty.$$

Using Fatou's lemma, as $L \to \infty$, we deduce that

$$\int\limits_{|x| \geq R} v_n^{\frac{(q^*)^2}{q}} \, \mathrm{d}x < \infty$$

and therefore the assertion holds. Next, choosing $\beta = q^* \frac{t-1}{qt}$ with $t = \frac{(q^*)^2}{q(q^*-q)}$, we have $\beta > 1$, $\frac{qt}{t-1} < q^*$ and $v_n \in L^{\frac{\beta q t}{t-1}}(|x| \geq R-r)$. From (7.3), we find

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{R\geq|x|\geq R-r} v_n^q v_{L,n}^{q(\beta-1)}\,\mathrm{d}x + \int\limits_{|x|\geq R-r} v_n^{q^*} v_{L,n}^{q(\beta-1)}\,\mathrm{d}x \right)$$

or equivalently

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{R\geq|x|\geq R-r} v_n^{q\beta}\,\mathrm{d}x + \int\limits_{|x|\geq R-r} v_n^{q^*-q} v_n^{q\beta}\,\mathrm{d}x \right).$$

Using the Hölder inequality with exponents $\frac{t}{t-1}$ and t, we get

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left\{ \left[\int\limits_{R\geq|x|\geq R-r} v_n^{\frac{q\beta t}{t-1}}\,\mathrm{d}x \right]^{\frac{t-1}{t}} \left[\int\limits_{R\geq|x|\geq R-r} \mathrm{d}x \right]^{\frac{1}{t}} + \left[\int\limits_{|x|\geq R-r} v_n^{(q^*-q)t}\,\mathrm{d}x \right]^{\frac{1}{t}} \left[\int\limits_{|x|\geq R-r} v_n^{\frac{q\beta t}{t-1}}\,\mathrm{d}x \right]^{\frac{t-1}{t}} \right\}.$$

Since $(q^*-q)t = (q^*)^2$, we deduce that

$$|w_{L,n}|_{q^*}^q \leq C\beta^q \left(\int\limits_{R\geq|x|\geq R-r} v_n^{\frac{q\beta t}{t-1}}\,\mathrm{d}x \right)^{\frac{t-1}{t}}.$$

Note that

$$\begin{aligned}
|v_{L,n}|_{L^{q^*\beta}(|x|\geq R)}^{q\beta} &\leq \left(\int\limits_{|x|\geq R-r} v_{L,n}^{q^*\beta}\,\mathrm{d}x \right)^{\frac{q}{q^*}} \\
&\leq \left(\int\limits_{\mathbb{R}^N} \eta^q v_n^{q^*} v_{L,n}^{q^*(\beta-1)}\,\mathrm{d}x \right)^{\frac{q}{q^*}} = |w_{L,n}|_{q^*}^q \\
&\leq C\beta^q \left(\int\limits_{R\geq|x|\geq R-r} v_n^{\frac{q\beta t}{t-1}}\,\mathrm{d}x \right)^{\frac{t-1}{t}} \\
&= C\beta^q |v_n|_{L^{\frac{q\beta t}{t-1}}(|x|\geq R-r)}^{\beta q}
\end{aligned}$$

which combined with Fatou's lemma with respect to L gives

$$|v_n|_{L^{q^*\beta}(|x|\geq R)}^{q\beta} \leq C\beta^q |v_n|_{L^{\frac{q\beta t}{t-1}}(|x|\geq R-r)}^{\beta q}.$$

Taking $\chi = \frac{q^*(t-1)}{qt}$ and $s = \frac{qt}{t-1}$, it follows from the above inequality that

$$|v_n|_{L^{\chi^{m+1}s}(|x|\geq R)}^{q\beta} \leq C^{\sum_{i=1}^m \chi^{-i}} \chi^{\sum_{i=1}^m i\chi^{-i}} |v_n|_{L^{q^*}(|x|\geq R-r)}$$

which implies that $|v_n|_{L^\infty(|x|\geq R)} \leq C|v_n|_{L^{q^*}(|x|\geq R-r)}$. Since $v_n \to v$ in $W^{1,q}(\mathbb{R}^N)$, for all $\epsilon > 0$ there exists $R > 0$ such that

$$|v_n|_{L^\infty(|x|\geq R)} < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

This completes the proof of the lemma. □

Lemma 7.2. *There exists $\delta > 0$ such that $|v_n|_\infty \geq \delta$ for all $n \in \mathbb{N}$.*

Proof. Assume to the contrary that $|v_n|_\infty \to 0$ as $n \to \infty$. By (f_2), there exists $n_0 \in \mathbb{N}$ such that $\frac{f(|v_n|_\infty)}{|v_n|_\infty^{p-1}} < \frac{V_0}{2}$ for all $n \geq n_0$. Therefore, in view of (f_5), we can see that

$$|\nabla v_n|_p^p + |\nabla v_n|_q^q + V_0(|v_n|_p^p + |v_n|_q^q) \leq \int_{\mathbb{R}^N} \frac{f(|v_n|_\infty)}{|v_n|_\infty^{p-1}} |v_n|^p \mathrm{d}x \leq \frac{V_0}{2} |v_n|_p^p,$$

which leads to a contradiction. □

End of the proof of Theorem 1.1. Let u_{ε_n} be a solution to (P_{ε_n}). Then $v_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$ is a solution to (P_{V_n}) with $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$, where $\{\tilde{y}_n\}$ is given by Proposition 6.1. Moreover, in view of Proposition 6.1, up to subsequence, $v_n \to v \neq 0$ in $\mathbb{Y}_{V_0}$ and $y_n = \varepsilon_n \tilde{y}_n \to y \in M$. If p_n denotes a global maximum point of v_n, we can use Lemma 7.1 and Lemma 7.2 to see that $p_n \in B_R(0)$ for some $R > 0$. Consequently, $z_{\varepsilon_n} = p_n + \tilde{y}_n$ is a global maximum point of u_{ε_n}, and then $\varepsilon_n z_{\varepsilon_n} = \varepsilon_n p_n + \varepsilon_n \tilde{y}_n \to y$ because $\{p_n\}$ is bounded. This fact and the continuity of V yield $V(\varepsilon_n z_{\varepsilon_n}) \to V(y) = V_0$ as $n \to \infty$.

Finally, we prove the exponential decay of u_{ε_n}. We use some arguments from [13]. Since $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and using (f_1), we can find $R > 0$ such that

$$f(v_n(x)) \leq \frac{V_0}{2}(v_n^{p-1}(x) + v_n^{q-1}(x)) \quad \text{for all } |x| \geq R.$$

Then, by using (V_1), we obtain

$$\begin{aligned} -\Delta_p v_n - \Delta_q v_n + \frac{V_0}{2}(v_n^{p-1} + v_n^{q-1}) &= f(v_n) - \left(V_n - \frac{V_0}{2}\right)(v_n^{p-1} + v_n^{q-1}) \\ &\leq f(v_n) - \frac{V_0}{2}(v_n^{p-1} + v_n^{q-1}) \leq 0 \quad \text{for } |x| \geq R. \end{aligned} \tag{7.4}$$

Let $\phi(x) = Me^{-c|x|}$ with $c, M > 0$ such that $c^p(p-1) < \frac{V_0}{2}$, $c^q(q-1) < \frac{V_0}{2}$ and $Me^{-cR} \geq v_n(x)$ for all $|x| = R$. We can see that

$$\begin{aligned} &-\Delta_p \phi - \Delta_q \phi + \frac{V_0}{2}(\phi^{p-1} + \phi^{q-1}) \\ &= \phi^{p-1}\left(\frac{V_0}{2} - c^p(p-1) + \frac{N-1}{|x|}c^{p-1}\right) + \phi^{q-1}\left(\frac{V_0}{2} - c^q(q-1) + \frac{N-1}{|x|}c^{q-1}\right) > 0 \quad \text{for } |x| \geq R. \end{aligned} \tag{7.5}$$

Using $\eta = (v_n - \phi)^+ \in W_0^{1,q}(\mathbb{R}^N \setminus B_R)$ as a test function in (7.4) and (7.5), we find

$$\begin{aligned} 0 \geq \int_{\{|x|\geq R\}\cap\{v_n > \phi\}} & \left[(|\nabla v_n|^{p-2}\nabla v_n - |\nabla \phi|^{p-2}\nabla \phi) \cdot \nabla \eta + (|\nabla v_n|^{q-2}\nabla v_n - |\nabla \phi|^{q-2}\nabla \phi) \cdot \nabla \eta\right] \\ &+ \frac{V_0}{2}\left[(v_n^{p-1} - \phi^{p-1}) + (v_n^{q-1} - \phi^{q-1})\right]\eta \, \mathrm{d}x. \end{aligned}$$

Since for $t > 1$ the following holds (see formula (2.10) in [22])

$$(|x|^{t-2}x - |y|^{t-2}y) \cdot (x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^N,$$

and U, v_n are continuous in $\mathbb{R}^N$, we deduce that $v_n(x) \leq \phi(x)$ for all $|x| \geq R$. Recalling that $u_{\varepsilon_n}(x) = v_n(x - \tilde{y}_n)$ and $\{p_n\}$ is bounded, we conclude that $u_{\varepsilon_n}(x) \leq C_1 e^{-C_2|x - z_{\varepsilon_n}|}$ for all $x \in \mathbb{R}^N$. This completes the proof of Theorem 1.1. □

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