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Obstructions to approximating maps of n-manifolds into $\mathbb{R}^{2n}$ by embeddings

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Abstract

We prove a criterion for approximability by embeddings in $\mathbb{R}^{2n}$ of a general position map $f: K \to \mathbb{R}^{2n-1}$ from a closed n-manifold (for $n \geqslant 3$). This approximability turns out to be equivalent to the property that f is a *projected embedding*, i.e., there is an embedding $\bar{f}: K \to \mathbb{R}^{2n}$ such that $f = \pi \circ \bar{f}$, where $\pi: \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ is the canonical projection. We prove that for $n = 2$, the obstruction modulo 2 to the existence of such a map $\bar{f}$ is a product of Arf-invariants of certain quadratic forms. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper we shall work in the smooth category. A map $f: K \to \mathbb{R}^m$ is said to be *approximable by embeddings* if for each $\varepsilon > 0$ there is an embedding $\phi: K \to \mathbb{R}^m$, which is ε-close to f. This notion appeared in studies of embeddability of compacta in Euclidean spaces—for a recent survey see [15, §9] (see also [2], [8, §4], [14], [16,

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Introduction]). Let $\pi:\mathbb{R}^{m+1}\to\mathbb{R}^m$ be the canonical projection and $i:\mathbb{R}^m\to\mathbb{R}^{m+1}$ the canonical inclusion. For a smooth n-manifold K, the properties of an embedding $\bar f:K\to\mathbb{R}^{m+k}$ (e.g., the cobordism class of $\bar f$, the Euler class of the normal bundle of $\bar f(K)$, etc.) can be investigated by means of singularities of the projection $f=\pi\circ\bar f:K\to\mathbb{R}^m$ [6,7, 23]. A map $f:K\to\mathbb{R}^m$ is called a *projected embedding from* $\mathbb{R}^{m+k}$ if there is a (smooth) embedding $\bar f:K\to\mathbb{R}^{m+k}$ such that $f=\pi\circ\bar f$. Evidently, if

(P) f is a projected embedding from $\mathbb{R}^{m+k}$,

then

(A) the map $i\circ f$ is approximable by smooth embeddings.

The converse is false, as the example of a constant map shows. We conjecture that the converse is true for *general position* maps f (at least for $m+k\geqslant 3(n+1)/2$). We prove this conjecture for $k=1$ and $m=2n-1\geqslant 5$ (for $k=m=n=1$ it is obvious, cf. [18]).

Theorem 1.1. *For every integer $n\geqslant 3$, every closed n-manifold K and any general position map $f:K\to\mathbb{R}^{2n-1}$, each of the properties* (A) *and* (P) *is equivalent to the following*:

(1.1.1) *f does not contain any submap r (that is, K does not contain X^n such that $f|_{X^n}=j\circ r$, where $j:Y^n\to\mathbb{R}^{2n-1}$ is an embedding).*

Here $X^n=D^{n-1}\times I/\{(x,0)\sim(-x,1)\}$, $Y^n=X^n/\{(0,t)\sim(0,[t+\frac12])\}$ and $r:X^n\to Y^n$ is the projection. Proof of Theorem 1.1 is based on the fact that the property (A) implies the approximability of f by projected embeddings, and on the equivalence (P) $\Leftrightarrow$ (1.1.1) for $n\geqslant 3$ (this is a folklore result, see also [24]).

Corollary 1.2. *If K is an n-manifold such that $\bar w_{n-1,1}(K)\neq 0$ (this is possible only if n is a power of 2, e.g., $K=\mathbb{R}P^{2^k}$), then for any general position immersion $f:K\to\mathbb{R}^{2n-1}$ neither* (A) *nor* (P) *hold.*

Corollary 1.2 generalizes the well-known fact that the Boy immersion $\mathbb{R}P^2\to\mathbb{R}^3$ is neither projected embedding from $\mathbb{R}^4$ nor approximable by embeddings.

The implication (A) $\Rightarrow$ (1.1.1) is true even for $n=2$. The converse is false for $n=2$ (Example 1.5). The main result of this paper (Theorem 1.4) is a relation between a (complete) algebraic obstruction to (P) for $k=1$, $m=2n-1=3$ and Arf-invariants of certain quadratic forms. This is motivated by the unproved case $n=2$ of a conjecture due to Daverman [9]: is every S^n-like compactum embeddable into $\mathbb{R}^{2n}$ for $n>1$? To prove this conjecture it suffices to prove that every map $S^n\to S^n\subset\mathbb{R}^{2n}$ is approximable by embeddings for $n>1$. This is so for $n\neq 1,2,3,7$ (and thus the Daverman Conjecture is true) [2,3], this is not so for $n=1,3,7$ (and thus the Daverman Conjecture is probably untrue) [18,2], and this is unknown for $n=2$. The proof of [2] suggests the following approach to the case $n=2$:

(1) find which maps $S^2\to\mathbb{R}^3$ can be obtained by shifting a map $S^2\to S^2\subset\mathbb{R}^3$ to general position;

(2) find for which general position maps $S^2 \to \mathbb{R}^3$, their composition with the inclusion $\mathbb{R}^3 \to \mathbb{R}^4$ is approximable by embeddings.

We are now going to state the criterion for (P) in the case $m = 2n - 1 = 3$, $k = 1$. Consider the action of $\mathbb{Z}_2$ on $K \times K$, defined by exchanging the factors. For any map $f : K \to \mathbb{R}^{2n-1}$, let

$$\widetilde{\Delta}(f) = \mathrm{Cl}\{(x, y) \in K \times K \mid x \neq y,\ fx = fy\} \quad \text{and} \quad \Delta(f) = \widetilde{\Delta}(f)/\mathbb{Z}_2.$$

When f is fixed, we denote $\widetilde{\Delta}(f)$ and $\Delta(f)$ briefly as $\widetilde{\Delta}$ and Δ, respectively. If K is an n-manifold and $f : K \to \mathbb{R}^{2n-1}$ is any smooth general position map, then $\widetilde{\Delta}$ is a disjoint union of circles and Δ is a disjoint union of circles and arcs. There are maps $(x, y) \mapsto x$ from $\widetilde{\Delta}$ to K and $[(x, y)] \mapsto fx$ from Δ to $\mathbb{R}^{2n-1}$. By general position, these maps are immersions (for $n \geqslant 3$ embeddings). We shall identify $\widetilde{\Delta}$ and Δ with their images (no confusion will arise). For $n = 2$, by general position, the set of triple points of f is finite and f has no quadruple points. Note that triple points of f in $\mathbb{R}^3$ are triple self-intersection points of Δ in $\mathbb{R}^3$.

Let K be a closed orientable surface, $f : K \to \mathbb{R}^3$ a general position map and $\mathcal{T}$ an orientation on Δ. Choose an orientation on K. Every triple point d of f is the intersection of three sheets $D_1, D_2, D_3 \subset K$. Let $\{a_1, a_2, a_3\}$ be the basis in $\mathbb{R}^3$ at the point d, formed by the vectors parallel to $fD_2 \cap fD_3$, $fD_3 \cap fD_1$, $fD_1 \cap fD_2$, whose direction is defined by the orientation $\mathcal{T}$ of Δ. Let $\{b_1, b_2, b_3\}$ be the basis in $\mathbb{R}^3$ at the point d, formed by the positive normal vectors of fD_1, fD_2, fD_3. The vectors a_i and b_i are parallel, but may have opposite directions. If the number of a_i and b_i with the same directions is either 0 or 3, then call the triple point d *resolvable* (or of type A) with respect to the orientation $\mathcal{T}$. In the opposite case call d *nonresolvable* (or of type B) with respect to $\mathcal{T}$ (cf. [4, Definition 2]). It is easy to see that this definition does not depend on the choice of the orientation of K for connected K.

Note that [4, Proposition 6] can be reformulated in the spirit of the Newton binomial formula: $|3, 0| - 3|2, 1| + 3|1, 2| - |0, 3| = 0$, where $|k, 3-k|$ is the number of triple points of f for which the directions of k vectors a_i and b_i are the same and those of $(3 - k)$ are the opposite.

Theorem 1.3 [4]. *For $k = 1$, any closed orientable surface K and any general position map $f : K \to \mathbb{R}^3$, the property* (P) *is equivalent to the following*:

(1.3.1) *there is an orientation $\mathcal{T}$ on Δ such that all triple points of f are resolvable with respect to $\mathcal{T}$.*

We conjecture that $(1.3.1) \Leftrightarrow (\mathrm{A})$. Remark that although (1.3.1) obviously generalizes to maps $K^{2n} \to \mathbb{R}^{3n} \subset \mathbb{R}^{4n}$ of an orientable manifold K, it is no longer necessary to (A) or (P) by [2, Remark 4b on p. 9].

Now we are going to relate the condition of Theorem 1.3 and Arf-invariants of certain quadratic forms (Theorem 1.4). Let $\beta(f, \mathcal{T}) \in \mathbb{Z}_2$ be the number mod 2 of nonresolvable triple points with respect to $\mathcal{T}$. Then $\beta(f) = \prod_{\tau} \beta(f, \mathcal{T}) \in \mathbb{Z}_2$ is an (incomplete) obstruction to approximability of $i \circ f$ by embeddings. Take any $x \in H_1(K, \mathbb{Z}_2)$ and a simple (embedded) closed curve $L \subset K$, representing x and avoiding singular points. By general position, we may assume that $f|_L$ is an embedding. Let ξ be a unit vector field, normal to $f(K)$ (it exists since L avoid the singular points). Define

$$q(f, \mathcal{T})(x) = \operatorname*{lk}_{\text{mod } 2}(L, \xi) + |L \cap s_{\mathcal{T}} \Delta| \text{ mod } 2.$$

Here the map $s_{\mathcal{T}} : \Delta \to K$ is defined as follows. Take any point $(d_1, d_2) \in \widetilde{\Delta}$ and take the basis $\{b_1, b_2, a\}$ of $\mathbb{R}^3$ at the point $f d_1 = f d_2$, formed by the positive normal vectors b_1, b_2 of the two sheets of $f K$, corresponding to d_1 and d_2, and the vector a parallel to the intersection of those sheets and directed along the orientation $\mathcal{T}$ of Δ. If this basis is positive, then set $s_{\mathcal{T}}[(d_1, d_2)] = d_1$. In the opposite case set $s_{\mathcal{T}}[(d_1, d_2)] = d_2$.

Theorem 1.4. *Let K be a closed orientable surface and $f : K \to \mathbb{R}^3$ a general position smooth map. Then $q(f, \mathcal{T})$ is a well-defined quadratic form* (*i.e., it does not depend on the choice of L*)*, associated to the intersection form $\cap$ on $H^1(K, \mathbb{Z}_2)$* (*i.e., $x \cap y = q(x) + q(y) + q(x + y)$ for each $x, y \in H_1(K, \mathbb{Z}_2)$*)*, $\beta(f, \mathcal{T}) \equiv \operatorname{Arf} q(f, \mathcal{T})$* (mod 2) *and $\beta(f) = \prod_{\mathcal{T}} \operatorname{Arf} q(f, \mathcal{T})$* (mod 2).

For the case of $K = S^2$ and a connected $\Delta(f)$, Theorem 1.4 is due to Akhmetiev [4, Theorem 3]. Our proof is an extension of [4, proof of Theorem 1.3]. It is based on the fact that q coincides with a certain form, defined for a characteristic surface of some 4-manifold. In Corollary 2.1 we relate the quadratic form $q(f, \tau)$ to the standard quadratic forms of an immersed surface in $\mathbb{R}^3$ [11].

Example 1.5. There is a closed orientable surface K and an immersion $f : K \to \mathbb{R}^3$ such that $\beta(f) = 1$ (and hence neither (A) nor (P) hold).

Example 1.5 will be constructed by a modification of [1, Proposition 4], via surgery of immersed surfaces along 1-handles (Section 2).

Note that our results are valid even if we replace $\mathbb{R}^3$ by any 3-submanifold of $\mathbb{R}^4$. In Section 3 we conjecture polyhedral versions of our results and present some related problems.

2. Proofs

Proof of Theorem 1.1. It suffices to prove (P) $\Leftrightarrow$ (1.1.1) $\Leftarrow$ (A). The implication (1.1.1) $\Rightarrow$ (P) was actually proved in [2]. To prove that (P) $\Rightarrow$ (1.1.1), observe that (P) implies that for each double point $x \in \Delta$ we can define the ordering on the two sheets of $f K$, intersecting at x, so that this ordering depends on x continuously, and such an ordering does not exists near any connected component of Δ, corresponding to the abbreviation r of f. The implication (A) $\Rightarrow$ (1.1.1) follows from the non-approximability by embeddings of the map $i \circ j \circ r$. In fact, every map, close to $j \circ r$, contains a submap r and hence is not a projected embedding. $\square$

Proof of Corollary 1.2. By Theorem 1.1, it sufficies to prove that (P) does not hold. If $f : K \to \mathbb{R}^{2n-1}$ is an immersion and a projected embedding (for $k = 1$), then $K \times I$ embeds into $\mathbb{R}^{2n}$ (this is proved either analogously to [17, proof of Theorem 1.5] or using the equivalence of normal bundles of an immersion $i \circ f$ and of an embedding $K \to \mathbb{R}^{2n}$, projected to f). If K is an n-manifold such that $\bar{w}_{n-1,1}(K) \neq 0$, then $K \times I$ does not embed into $\mathbb{R}^{2n}$.

Also, (P) does not hold by (1.1.4) (see Section 3) and the formula $\varepsilon w_1(p) = \bar{w}_{n-1,1}(K)$, where $\varepsilon : H^1(\Delta; \mathbb{Z}_2) \to \mathbb{Z}_2$ is the augmentation homomorphism [24, Theorem 1]. □

Proof of Theorem 1.4. Fix an orientation $\mathcal{T}$ on Δ. Analogously to [4, proof of Criterion 1], we can construct a general position map $\bar{f} : K \to \mathbb{R}^4$ such that $\pi \circ \bar{f} = f$ and each self-intersection point $\bar{d}_i \in \mathbb{R}^4$ of $\bar{f}$ is projected onto a nonresolvable triple point $d_i \in \mathbb{R}^3$ of f. For each point $\bar{d}_i$ take a small ball $B_i^4 \subset \mathbb{R}^4$, centered at $\bar{d}_i$. For each $x \in H_1(K; \mathbb{Z}_2)$, take a simple closed curve $\partial D^2 \subset K - \bar{f}^{-1}(\bigcup_i B_i)$ representing x. Since $\mathbb{R}^4 - \bigcup_i B_i^4$ is simply connected, there is a generic immersion $g : D^2 \to \mathbb{R}^4 - \bigcup_i B_i$ such that $g|_{\partial D^2} = f|_{\partial D^2}$. Let ξ be a normal vector field of ∂D^2 in K. Then ξ defines a section over $g(\partial D^2)$ of the normal bundle of $g(D^2)$ in $\mathbb{R}^4 - \bigcup_i B_i$. Let $e \in \mathbb{Z} = H^2(D^2, \partial D^2)$ be the obstruction to the extension of this section to $g(D^2)$. Then $e = \mathrm{lk}(\partial D^2, \xi)$ and $|g(D^2) \cap f(K)| = |\partial D^2 \cap s_{\mathcal{T}} \Delta|$. Therefore $q(f, \mathcal{T})(x) = e + |g(D) \cap f(M)| \pmod 2$. This is a well-defined quadratic form, associated with the intersection [11]. Therefore $q(f, \mathcal{T})$ is a well-defined quadratic form, associated with the intersection. On the other hand, $\mathrm{Arf}(q)$ modulo 2 is the number of nonresolvable triple points d_i of f with respect to $\mathcal{T}$ (the proof is a straightforward extension of [4, proof of Theorem 1.3]). □

Construction of Example 1.5. Recall from [1, Proposition 4] the construction of an immersion $g : K \to \mathbb{R}^3$ (the Konstantinov torus) of the torus $K = T^2$, of an orientation $\mathcal{T}$ on $\Delta(g)$ and proof of $\beta(g, \mathcal{T}) = 1$ (Fig. 1). The critical points curve of the projection of the Konstantinov torus into the plane is shown on Fig. 1 [1, Figure 2B]. The immersion itself is constructed by gluing of the upper surface (the torus with one hole) S_1, $\partial S_1 = \Sigma_1$, the middle cylinder S_2, $\partial S_2 = \Sigma_0 \cup \Sigma_1$, and the lower disk S_3, $\partial S_3 = \Sigma_0$, along the folded curves Σ_0 and Σ_1. The curve $\widetilde{\Delta}$ intersects each cycle on S_1 in an even number of points. The immersion g is invariant under the rotation on the angle $\frac{2\pi}{3}$ with respect to the axis, perpendicular to the plane of the projection in the central point of the Fig. 1. Take the orientation $\mathcal{T}$ of Δ invariant under this rotation. For an arbitrary cycle $L \subset S_1$ we have $|L \cap s_{\mathcal{T}} \Delta| = 0 \pmod 2$, because L intersects only lower component of $s_{\mathcal{T}} \Delta$. Therefore by [11] or by Corollary 2.1 below, $q(g, \mathcal{T}) = \bar{q}(g)$. By an easy calculation (cf. [1, Corollary 22]), we have $\mathrm{Arf}\,\bar{q} = 1$, hence by Theorem 1.4, $\beta(g, \mathcal{T}) = 1$. For a direct proof see [1, Theorem II, B].

The set $\Delta(g)$ is a union of circles (possibly, intersecting and self-intersecting). If there is only one such circle, then there are exactly two (opposite) orientations $\mathcal{T}, \overline{\mathcal{T}}$ of $\Delta(g)$ and therefore $\beta(g, \mathcal{T}) = \beta(g, \bar{\mathcal{T}}) = \beta(g) = 1$. However, for the Konstantinov torus there are 3 circles in $\Delta(g)$. We shall made a modification of g which will have the effect of

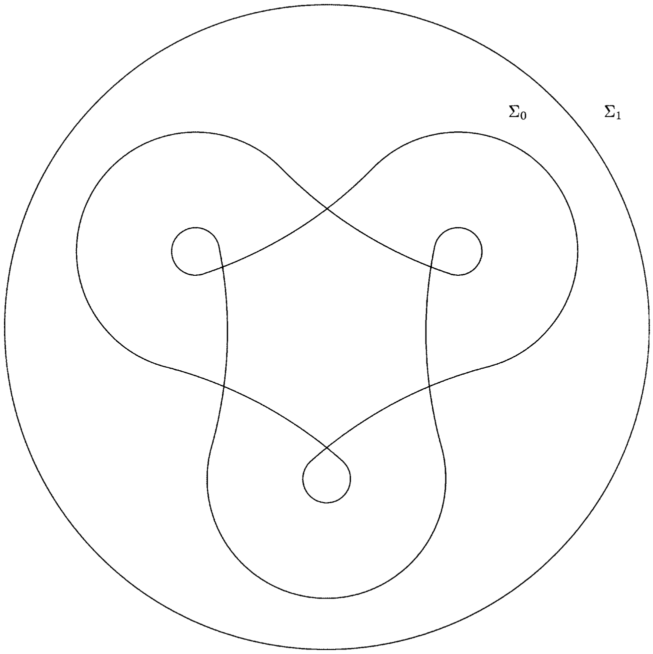

Fig. 1.

decreasing the number of circles in Δ, containing triple points. In fact, the modification will *add* several circles to Δ, but they will not contain triple points.

Let $a, b \in \Delta \subset \mathbb{R}^3$ be any two points on distinct circles of $\Delta(g)$. Take an arc $l \subset \mathbb{R}^3$ joining a to b. By general position, $l \cap gK = \{a = c_1, c_2, \dots, c_{m-1} = b\}$ and $l \cap \Delta(g) = \{a, b\}$. Let U be a small neighborhood of l. Then $U \cap \Delta(g) = A \cup B$, where A and B are small arcs containing a and b, respectively. Now, $U \cap g(T^2) = D_0 \cup D_1 \cup \cdots \cup D_m$, where $D_0, \dots, D_m$ are embedded 2-disks, $D_0 \cap D_1 = A$, $D_{m-1} \cap D_m = B$ and $D_u \cap D_v = \emptyset$ for $(u, v) \neq (0, 1), (m-1, m)$. Take a pair of 1-handles $H_1, H_2 \subset U$, such that

$$H_1 \cong H_2 \cong S^1 \times I, \qquad H_1 \cap H_2 = l_1 \sqcup l_2,$$

$$\partial H_1 = \partial D_0 \sqcup \partial D_s, \qquad \partial H_2 = \partial D_1 \sqcup \partial D_t,$$

where l_1 and l_2 are two arcs parallel to l and $\{s, t\} = \{m-1, m\}$ (the choice of these two possibilities will be specified below). Having made a surgery $g(K) \to g_1(K_1)$ by the handles H_1 and H_2, we get

$$g_1(K_1) = g\big[(K \cup H_1 \cup H_2) - \big(\mathring{D}_0 \cup \mathring{D}_1 \cup \mathring{D}_{m-1} \cup \mathring{D}_m\big)\big].$$

The curve $\Delta(g)$ is modified by a surgery by the handle $l_1 \sqcup l_2$: we have $\Delta(g_1) = \Delta' \cup \Delta_2 \cup \cdots \cup \Delta_{m-2}$, where $\Delta' = (\Delta(g) \cup l_1 \cup l_2) - (\mathring{A} \cup \mathring{B})$ and Δ_u is a pair of small curves, immersed in the disk D_u. We choose $(\partial H_1, \partial H_2)$ (see above) so that the orientation $\mathcal{T}$ on $\Delta(g)$ induce an orientation on Δ'. Note that in the neighborhood of every disk D_u there are two triple points of g_1.

Let $g \to g_1 \to \cdots \to g_k = f$ be a sequence of such surgeries, where $k+1$ equals to the number of connected components in $\Delta(g)$. We have $\Delta(f) = \Delta' \cup \bigcup_u \Delta_u$, where Δ' is a circle, Δ_u is a union of two circles and $\Delta_u \cap \Delta_v = \emptyset$ for $u \neq v$. The numbers of nonresolvable triple points of f on the two circles of Δ_u (with respect to any of the four orientations on Δ_u) are the same. (In fact, we may assume that there is a plane α such that under the mirror symmetry with respect to α, Δ_u is invariant and the two triple points x, y of Δ_u exchange their positions. Recall the definition of resolvable and nonresolvable triple points. Fix any of the four orientations on the two circles of Δ_u. Under the above mirror symmetry vectors a_{1x}, a_{2x}, a_{3x} at the point x goes to vectors a_{1y}, a_{2y}, a_{3y} at the point y, and vectors b_{1x}, b_{2x}, b_{3x} at the point x goes to vectors $-b_{1y}, -b_{2y}, -b_{3y}$ at the point y. Therefore points x and y are resolvable or nonresolvable simultaneously.) The number mod 2 of nonresolvable triple points of f on Δ' outside $\bigcup_u \Delta_u$ (with respect to any of the two opposite orientations of Δ') equals to $\beta(g, \mathcal{T}) = \beta(g, \bar{\mathcal{T}}) = 1$. So for an arbitrary orientation $\mathcal{T}$ on $\Delta(f)$, we have $\beta(f, \mathcal{T}) = 1$, and therefore $\beta(f) = 1$. □

Corollary 2.1. *Let $f : K \to \mathbb{R}^3$ be an immersion of a closed orientable surface and $\mathcal{T}$ an orientation on $\Delta(f)$ such that $[s_{\mathcal{T}}\Delta(f)] = 0 \in H^1(K; \mathbb{Z}_2)$. Then* $\mathrm{Arf}(q(f, \mathcal{T})) = \beta(f, \mathcal{T}) = [f]$, *where $[f] \in \pi_2^S \cong \mathbb{Z}_2$ corresponds to f.*

Proof. By [26], the cobordism group of such immersions $f : K \to \mathbb{R}^3$ can be identified with π_2^s. Given such an immersion $f : K \to \mathbb{R}^3$, one can define the quadratic function $q_f : H_1(K; \mathbb{Z}_2) \to \mathbb{R}$ as follows: $q_f(x) = \mathrm{lk}(L, \xi)$, where L is an embedded curve in K representing x, and ξ is the normal field of the immersion. Pontrjagin proved that the Arf invariant of this q_f is a cobordism invariant and gives an isomorphism between π_2^s and $\mathbb{Z}_2$. We shall identify the cobordism class $[f] \in \pi_2^s$ of f with $\mathrm{Arf}\, q_f$. Earlier we have defined $q(f, \mathcal{T})$ as $\mathrm{lk}(L, \xi) + [L \cap s_{\mathcal{T}}]$. By the assumption of the corollary, $[L \cap s_{\mathcal{T}}] = 0$, hence $q_f = q(f, \mathcal{T})$. By Theorem 1.4, $\beta(f, \mathcal{T}) = \mathrm{Arf}(q(f, \mathcal{T}))$. So finally we have $\beta(f, \tau) = [f]$ for an orientation $\mathcal{T}$ on Δ such that $[s_{\mathcal{T}}(\Delta)] = 0$. □

3. Epilogue: open problems on approximability by embeddings

3.1. The polyhedral analogue of Theorem 1.1

Theorem 1.1 is probably false for polyhedra K (for $n = 1$ see [16, Example 1.6]). But we conjecture that Theorem 1.1 is true for polyhedra K and $n \neq 2$, if we replace (1.1.1) by either of the following conditions (equivalent to (1.1.1) for the case of manifolds):

(1.1.2) There is a continuous equivariant map $\widetilde{\Delta} \to S^0 = \{+1, -1\}$.

(1.1.3) Any two distinct points $x, y \in K$ such that $fx = fy$ cannot exchange their positions moving continously and preserving the conditions '$x \neq y$' and '$fx = fy$'.

(1.1.4) $w_1(p) = 0 \in H^1(\Delta; \mathbb{Z}_2)$, where $p : \widetilde{\Delta} \to \Delta$ is the projection.

It is easy to see that (1.1.2) ⇔ (1.1.3) ⇔ (1.1.4) and (P) ⇒ (1.1.3). Also for general position maps f, (A) ⇒ (1.1.3).

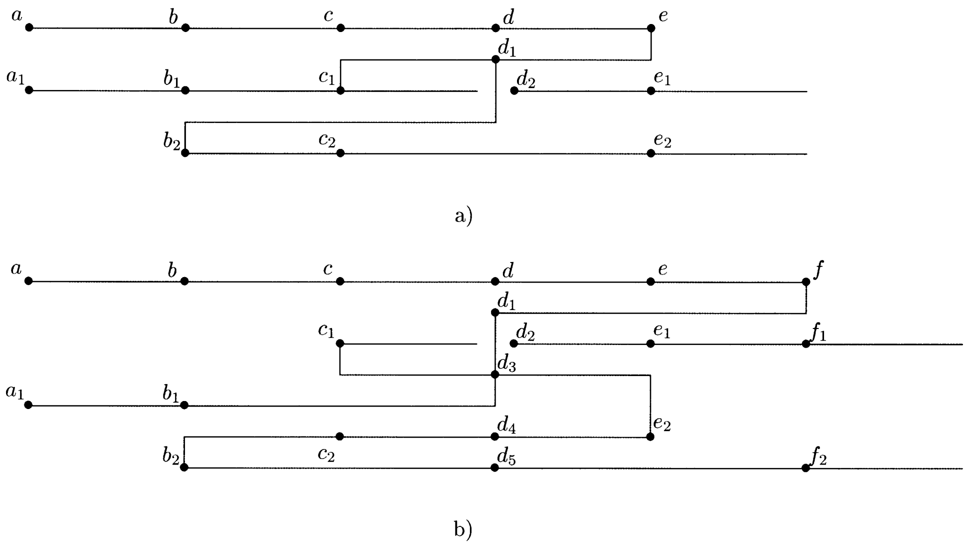

Fig. 2.

Let us show that (1.1.3) is not true for some maps $f : K \to \mathbb{R}$ of a graph K. Let H be the 'letter H' and $f : H \to \mathbb{R}$ the map defined in [18] (this map can be understood from Fig. 2(a), where a general position map $g : H \to \mathbb{R}^2$, close to $i \circ f$, is shown). Denote some points of H as on Fig. 2(a). Then the following sequence shows that (1.1.3) is not true:

$$aa_1,\ ee_1,\ d_1d_2,\ b_2b_1,\ c_2c_1,\ e_2e,\ b_2b,\ d_1d,\ c_1c,\ a_1a.$$

Analogously, let X be the 'letter X' and $f : X \to \mathbb{R}$ the map defined in [18] (Fig. 2(b)). Denote some points of X as on Fig. 2(b). Then the following sequence shows that (1.1.3) is not true:

$$aa_1,\ dd_3,\ cc_1,\ ff_1,\ d_1d_2,\ e_2e_1,\ c_2c_1,\ d_4d_3,\ b_2b_1,$$
$$d_5d_3,\ f_2f,\ b_2b,\ e_2e,\ d_3d,\ a_1a.$$

3.2. Approximability by embeddings of maps $K \to \mathbb{R}^{2n}$

We can add to (1.1.2)–(1.1.4) the formally weaker conditions on approximability by embeddings of any map $g : K \to \mathbb{R}^{2n}$ (not necessarily of the form $g = i \circ f$). Let us give necessary definitions. In (3.2) we assume that in (A), $i \circ f$ is replaced to g. For a polyhedron K with a fixed triangulation T and a map $g : K \to \mathbb{R}^{2n}$ (such that $g|_\sigma$ is an embedding for each $\sigma \in T$) let

$$\widetilde{K} = \bigcup \{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \emptyset\} \quad \text{and}$$

$$\widetilde{K}^g = \bigcup \{\sigma \times \tau \in T \times T \mid g\sigma \cap g\tau = \emptyset\}.$$

Clearly, $\widetilde{K}^{i\circ f} = \widetilde{K}^f$. Note that $\widetilde{K}^g$ is an equivariant retract of $\widetilde{K} - \widetilde{\Delta}$, that $\widetilde{K} - \widetilde{K}^g$ is a regular neighborhood of $\widetilde{\Delta}$ in $\widetilde{K}$, and hence $\mathrm{Cl}(\widetilde{K} - \widetilde{K}^g) \cong \operatorname{Map}\psi$ for some equivariant map $\psi : A \to \widetilde{\Delta}$ ($A \subset \widetilde{K}^g$). Define $\bar{g} : \widetilde{K}^g \to S^{2n-1}$ by $\bar{g}(x, y) = (gx - gy)/|gx - gy|$. We omit $\mathbb{Z}$-coefficients (with the involution $k \to (-1)^n k$) from (symmetric) cohomology groups. Let $K^* = \widetilde{K}/\mathbb{Z}_2$ and $K^{*g} = \widetilde{K}^g/\mathbb{Z}_2$.

Let us construct a generalization of the van Kampen obstruction $v(g) \in H^{2n}_S(\widetilde{K}, \widetilde{K}^g)$ for approximability by embeddings of a map $g : K \to \mathbb{R}^{2n}$ [16]. Take a general position PL map $h : K \to \mathbb{R}^{2n}$, sufficiently close to g. Fix an orientation of $\mathbb{R}^{2n}$ and on n-simplices of K. For any two disjoint n-simplices $\sigma, \tau \in T$, count an intersection $h\sigma \cap h\tau$, where the orientation of $h\sigma$ followed by that of $h\tau$ agrees with that of $\mathbb{R}^{2n}$ as $+1$, and -1 otherwise. Then $v(g)$ is the class of the cocycle $v_h(g)(\sigma, \tau)$ which counts the intersections of $h\sigma$ and $h\tau$ algebraically in this fashion. Clearly, this definition is correct. Remark that $H^{2n}_S(\widetilde{K}, \widetilde{K}^g) \cong H^{2n}(K^*, K^{*g})$ for even n. Sometimes it is useful to consider $\rho_2 v(f) \in H^{2n}_S(\widetilde{K}, \widetilde{K}^g; \mathbb{Z}_2) \cong H^{2n}(K^*, K^{*g}; \mathbb{Z}_2)$.

The difference element $\omega(g) \in H^{2n-1}(K^{*g})$ for arbitrary PL map $g : K \to \mathbb{R}^{2n}$ (not necessarily an embedding) is defined as $\omega(g) = (g^* : K^{*g} \to \mathbb{R}P^{2n-1})^*(1)$, where $1 \in H^{2n-1}(\mathbb{R}P^{2n-1})$ is the generator. The geometric interpretation of this definition is as follows [8]. Take a point $x \in S^{2n-1}$ that is regular for $\bar{g}$. Fix an orientation of S^{2n-1} and on n- and $(n-1)$-simplices of T. For any two disjoint n- and $(n-1)$-simplices $\sigma, \tau \in T$ (where $g\tau \cap g\sigma = \emptyset$) let $\omega_x(g)(\sigma, \tau)$ be the degree of $\widetilde{h} : \sigma \times \tau \to S^{2n-1}$ at x. Then $\omega(g)$ is the class of the cocycle $\omega_x(g)(\sigma, \tau)$. Clearly, this definition is correct. Remark that the choice of x can be replaced here by the choice of a general position map h, close to g and such that $1 \in S^{2n-1}$ is a regular point of $\widetilde{h}$.

(1.1.5) $v(g) = 0$.

(1.1.6) There is an equivariant homotopic extension $\widetilde{K} \to S^{2n-1}$ of the map $\bar{g} : \widetilde{K}^g \to S^{2n-1}$.

(1.1.6′) There exists an embedding $\varphi : K \to \mathbb{R}^{2n}$ such that $\widetilde{\varphi} \simeq_{eq} \bar{g}$ on $\widetilde{K}^g$.

(1.1.7) There exists an element $\omega \in H^{2n-1}(K^*)$ such that $\omega|_{K^{*g}} = \omega(g)$.

(1.1.7′) There exists an embedding $\varphi : K \to \mathbb{R}^{2n}$ such that $\omega(\varphi)|_{K^{*g}} = \omega(g)$.

Clearly, (1.1.5) is the first (and the only) obstruction to equivariant extension of $\bar{g} : \widetilde{K}^g \to S^{2n-1}$ to $\widetilde{K}$, so (1.1.5) $\Leftrightarrow$ (1.1.6). For even n, it is easy to see that $v(g) \in H^{2n}(K^*, K^{*g})$ is the boundary of $\omega(g)$, hence from the exact sequence of the pair (K^*, K^{*g}) it follows that (1.1.5) $\Leftrightarrow$ (1.1.7). Evidently, (1.1.6′) $\Rightarrow$ (1.1.6) and (1.1.7′) $\Rightarrow$ (1.1.7). For $n \geqslant 3$ the converse is true by [25]. For $n = 1$ the converse (and (1.1.6) $\Rightarrow$ (A), (1.1.7) $\Rightarrow$ (A)) is not true by [16, Example 1.6].

The implications (A) $\Rightarrow$ (1.1.5), (A) $\Rightarrow$ (1.1.6) and the converse for $n \geqslant 3$ were proved in [16]. Remark that in the proof [16, §4] it was used the property that φ is a join on the preimage of $\delta_{\sigma\tau} \cong S_{\sigma\tau} * D^r_{\sigma\tau}$. This is not true for *arbitrary* general position PL map φ. But the assumption does not affect the proof. In fact, the assumption holds before application of Proposition 3.1 (when f is linear on simplices of T), it is preserved under the modifications from Section 3 (since the map f on each simplex is modified by an ambient isotopy), and so the assumption holds before application of Proposition 4.1. In the proof of Proposition 4.1 the required property is preserved under modifications of φ for the same ambient isotopy reason. Similar modification should be done in [19, Proof of Theorem 1.2], for detailed account see [20].

Note that there is a mistake in [16, Example 1.7], [15, Example 9.5.b] and [8, Example 4.4.b]. It was asserted there that for examples from (3.1), the condition (1.1.6) is true. But (1.1.6) is not true for these examples by (3.1) and (1.1.6) $\Rightarrow$ (1.1.4) below.

Sketch of the proof of (1.1.2) $\Rightarrow$ (1.1.6). Recall that $\mathrm{Cl}(\widetilde{K} - \widetilde{K}^f) \cong \mathrm{Map}(\psi : A \to \widetilde{\Delta})$. Represent $+1$, -1 and S^{2n-2} as the north and the south poles and the equator of S^{2n-1}. Then we can extend the maps $K^f \xrightarrow{\bar{f}} S^{2n-2} \subset S^{2n-1}$ and $\widetilde{\Delta} \to S^0 \subset S^{2n-1}$ 'linearly' to an equivariant map $\widetilde{K} \to S^{2n-1}$, and (1.1.6) follows.

Sketch of the proof of (1.1.6) $\Rightarrow$ (1.1.4) *for* $n = 1$. We have $\widetilde{K}^f = f^{-1}(1) \sqcup f^{-1}(-1)$, so $A = A^+ \sqcup A^-$. If (1.1.4) does not hold, then there is an equivariant circle $C \subset \widetilde{\Delta}$. It is easy to see that then there is an equivariant circle $C' \subset A^+$ such that $\psi C' \subset C$. If the map $C' \xrightarrow{\bar{f}} 1 \in S^1$ can be extended to an equivariant map $\Phi : \mathrm{Map}(\psi|_{C'} : C' \to C) \to S^1$, then $\Phi|_C$ is null-homotopic and equivariant, which is impossible. So (1.1.6) does not hold.

It would be interesting to know if either of (1.1.5)–(1.1.7′) implies (A) or (P). Interesting partial cases of this problem are $n = 1$, $g = f \circ i$ and/or g monotone and/or K and $g(K)$ trees. The partial case, important for dynamical systems, is when $n = 1$, K and $g(K)$ are wedges of p and q circles, respectively, g is represented by p words of q letters and $\mathbb{R}^2$ is replaced by an arbitrary 2-manifold [27]. E.g.,

- (Smale) The map $S^1 \vee S^1 \to S^1 \vee S^1$, defined by $a \mapsto aba$ and $b \mapsto ab$ is embeddable into torus but not into plane.
- (Wada–Plykin) The map $S^1 \vee S^1 \vee S^1 \to S^1 \vee S^1 \vee S^1$, defined by $a \mapsto aca^{-1}$, $b \mapsto bab^{-1}$ and $c \mapsto b$ is embeddable into plane.
- (Zhirov) The map $S^1 \vee S^1 \vee S^1 \vee S^1 \to S^1 \vee S^1 \vee S^1 \vee S^1$, defined by $a \mapsto ac$, $b \mapsto ad$, $c \mapsto bac$ and $d \mapsto c$ is embeddable into pretzel but not into torus.

An interesting an perhaps easier analogue of these problems is their *link map* analogue: given a map $g : K \sqcup L \to \mathbb{R}^m$, under what conditions it can be approximated by maps g' with $g'(K) \cap g'(L) = \emptyset$ (cf. [13] and references there). If $K \sqcup L$ is an n-polyhedron and g is PL, then analogous to (1.1.6) necessary condition can be introduced (for $m = 2n$—also those analogous to (1.1.5) and (1.1.7), for $m = 2n$ and $g(K \sqcup L) \subset \mathbb{R}^{2n-1}$—to (1.1.2)–(1.1.4)). This condition is sufficient for higher-dimensional case [22, Theorem 1.3], but the case $(m, n) = (2, 1)$ is unknown. For the case $K = L$ and g a composition of the identification of the two copies and an embedding $K \to \mathbb{R}^2$ see [17] and references there.

The manifold analogue of Theorem 1.1 (cf. the remark at the end of Section 1) is false for $n = 1$ [16, Example 1.6] (and Theorem 1.3 just does not make sense for $n = 1$). For every map $f : I \to S^1$ or $f : I \to \mathbb{R}$ and for every general position map $f : S^1 \to \mathbb{R}$, both (A) and (P) hold [18]. For every general position map $f : S^1 \to S^1$, (P) is equivalent to (A) and to the condition that the degree of f is 0, $+1$ or -1 [18]. The condition that f is in general position is unnecessary for (A) in this assertion, but it is necessary for (P) (as the example of the constant map shows). To understand the non-general position case, it would be interesting to characterize maps $S^1 \to \mathbb{R}$ and $S^1 \to S^1$, for which (P) holds.

3.3. *The polyhedral analogue of Theorem 1.3*

Observe that for an orientable 2-surface K and a general position map $f : K \to \mathbb{R}^3$, the equivariant maps $\widetilde{\Delta} \to \{+1, -1\}$ are in 1–1 correspondence with the orientations on Δ. In fact, for an orientation $\mathcal{T}$ on Δ define a map $\tau : \widetilde{\Delta} \to \{+1, -1\}$ as follows. Take any point $(d_1, d_2) \in \widetilde{\Delta}$ and take the basis $\{b_1, b_2, a\}$ of $\mathbb{R}^3$ at the point $fd_1 = fd_2$, formed by the positive normal vectors b_1, b_2 of the two sheets of fK, corresponding to d_1 and d_2, and the vector a parallel to the intesection of those sheets and directed along the orientation of Δ. If this basis is positive, then set $\tau(d_1, d_2) = -1$. If it is negative, then set $\tau(d_1, d_2) = +1$. It is easy to see that the correspondence $\mathcal{T} \mapsto \tau$ is 1–1. For an equivariant map $\tau : \widetilde{\Delta} \to \{+1, -1\}$ and a triple point d of f with preimages d_1, d_2, d_3 define the relation '$<$' on $\{d_1, d_2, d_3\}$ by $d_i > d_j$ if $\tau(d_1, d_2) = +1$ and $d_i < d_j$ if $\tau(d_1, d_2) = -1$. Evidently, the point d is resolvable if and only if the relation '$<$' is transitive.

The condition (1.3.1) can be reformulated so that it will be a strengthening of (1.1.2):

(1.3.2) There is a continuous equivariant map $\tau : \widetilde{\Delta} \to \{+1, -1\}$ such that all triple points of f are resolvable with respect to τ.

We conjecture that in this form Theorem 1.3 is true even for polyhedra. We conjecture that (1.3.1) and (1.3.2) can be reformulated in terms of the *deleted cube*.

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