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Embedding products of low-dimensional manifolds into $\mathbb{R}^m$ ☆

P.M. Akhmetiev [a], D. Repovš [b,*], A.B. Skopenkov [c]

[a] IZMIRAN, Moscow region, Troitsk, 142092 Russia
[b] Institute for Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 2964, 1001 Ljubljana, Slovenia
[c] Department of Mathematics, Kolmogorov College, Kremenchugskaya 11, Moscow, 121357 Russia

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Abstract

Let X be a Cartesian product of s circles, p orientable 2-manifolds, q non-orientable 2-manifolds, r orientable 3-manifolds and t non-orientable 3-manifolds (all of them are closed). We prove that if either some of these r orientable 3-manifolds embed into $\mathbb{R}^4$ or $p+q+s+t>0$, then the lowest dimension of Euclidean space in which X is smoothly embeddable is $s+2p+3(q+r)+4t+1$. If none of the closed orientable 3-manifolds $R_1,\ldots,R_r$ embed into $\mathbb{R}^4$, then their product is embeddable into $\mathbb{R}^{3r+2}$ and, at least for some cases, non-embeddable into $\mathbb{R}^{3r+1}$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper we shall work in the smooth category. A classical problem in topology is to find the lowest possible dimension m such that a given manifold N embeds into $\mathbb{R}^m$. The class of manifolds N for which such an m is known is not very large, although there exist many criteria for embeddability of N into $\mathbb{R}^m$ for a given m (for surveys see [5, 13]). The following is our main result.

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* Corresponding author.
E-mail addresses: akhmetiev@izmiran.rssi.ru (P.M. Akhmetiev), dusan.repovs@fmf.uni-lj.si (D. Repovš), skopenko@aesc.msu.ru, skopenko@nw.math.msu.su (A.B. Skopenkov).

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Theorem 1.1. *Let $P_1, \ldots, P_p$ be orientable 2-manifolds, $Q_1, \ldots, Q_q$ non-orientable 2-manifolds, $R_1, \ldots, R_r$ orientable 3-manifolds, $T_1, \ldots, T_t$ non-orientable 3-manifolds (all closed). If either some R_i is embeddable into $\mathbb{R}^4$ or $p + q + s + t > 0$, then the lowest dimension of the Euclidean space into which the product*

$$(S^1)^s \times P_1 \times \cdots \times P_p \times Q_1 \times \cdots \times Q_q \times R_1 \times \cdots \times R_r \times T_1 \times \cdots \times T_t$$

is embeddable is $s + 2p + 3(q + r) + 4t + 1$.

If no R_i is embeddable into $\mathbb{R}^4$, then the product $R_1 \times \cdots \times R_r$ is embeddable into $\mathbb{R}^{3r+2}$.

The embeddability is based on classical results on embeddabilty and immersability of low-dimensional manifolds in $\mathbb{R}^m$ and on the Brown lemma on embeddings of products (Lemma 2.1). The non-embeddability follows from the calculation of the normal Stiefel–Whitney classes. Theorem 1.1 should be compared with [1, Corollary 2.2]. Example 1.2 below shows that the dimension $3r + 2$ in the second part of Theorem 1.1 is the best possible for *some* $R_1, \ldots, R_r$ (the proof, based on analysis of the cohomology ring of the complement, is due to Rees, who kindly permitted us to include it in this paper). We conjecture that nevertheless this dimension is not the best possible for *all* $R_1, \ldots, R_r$, i.e., that for each $r > 1$ there exist closed orientable 3-manifolds $R_1, \ldots, R_r$ which are non-embeddable in $\mathbb{R}^4$ whereas their product $R_1 \times \cdots \times R_r$ embeds into $\mathbb{R}^{3r+1}$.

Example 1.2 (*for* $r = 1$ [7, Theorem 3], *for* $r > 1$ [12]). $(\mathbb{R}P^3)^r$ does not embed into $\mathbb{R}^{3r+1}$ for any r.

The following graph analogue of Theorem 1.1 was announced without proof in [3]. (We tried to check whether a proof could be found in Galecki's thesis [4]. However, after an extensive search Daverman kindly informed us that there is no longer any copy of it available at the University of Tennessee.)

Conjecture 1.3 [3]. Let $G_1, \ldots, G_u$ be connected graphs, distinct from I and S^1. If either some G_i is planar (i.e., contains neither of the Kuratowski graphs K_5 or K_{33}) or $k > 0$ or $k = s = u = 0$, then the lowest dimension of the Euclidean space into which the product $I^k \times (S^1)^s \times G_1 \times \cdots \times G_u$ is embeddable, is $k + s + 2u$. If no G_i is planar and $s + u > 0$, then the lowest dimension of the Euclidean space into which the product $(S^1)^s \times G_1 \times \cdots \times G_u$ is embeddable, is $s + 2u + 1$.

2. Proofs and related results

Lemma 2.1.

(a) [1, Lemma 2.1] *Let M and N be any manifolds (possibly, nonclosed). If M embeds into $\mathbb{R}^e$, N immerses in $\mathbb{R}^i$ (or $i = \dim N$ and $N \times I$ immerses into $\mathbb{R}^{i+1}$) and $e + i > 2\dim N$, then $M \times N$ embeds into $\mathbb{R}^{e+i}$.*

(b) *Let $M, N_1, \ldots, N_d$ be any manifolds (possibly, nonclosed). If M embeds into $\mathbb{R}^e$, N_l immerses in $\mathbb{R}^{i_l}$ (or $i_l = \dim N_l$ and $N_l \times I$ immerses into $\mathbb{R}^{i_l+1}$) and $e + i_1 + \cdots + i_l > 2\dim N_l$, for each $l = 1, \ldots, d$, then $M \times N_1 \times \cdots \times N_d$ embeds into Euclidean space of dimension $e + i_1 + \cdots + i_d$.*

Note that it was not assumed in [1, Lemma 2.1] that $i = \dim N$ and $N \times I$ immerses into $\mathbb{R}^{i+1}$ is possible, however the proof is the same under this assumption. Since Lemma 2.1(a) plays a key role in our proof, we sketch the idea of its proof here. Lemma 2.1(b) follows by applying Lemma 2.1(a) consecutively for

$$(M, N) = (M, N_1),\ (M \times N_1, N_2), \ldots, (M \times N_1 \times \cdots \times N_{d-1}, N_d).$$

Idea of the proof of Lemma 2.1(a). To illustrate the idea, we show how to embed $\mathbb{R}P^3 \times \mathbb{R}P^2$ into $\mathbb{R}^7$. Take a composition of an immersion $\mathbb{R}P^3 \times I \to \mathbb{R}^4$ and the inclusion $\mathbb{R}^4 \to \mathbb{R}^7$. We obtain an immersion $\mathbb{R}P^3 \to \mathbb{R}^7$ with normal bundle $1 \oplus 3$ (this bundle is the Whitney sum of the two trivial bundles $\mathbb{R}P^3 \times \mathbb{R}$ and $\mathbb{R}P^3 \times \mathbb{R}^3$ over $\mathbb{R}P^3$). Shift this immersion to general position to get an embedding $\mathbb{R}P^3 \to \mathbb{R}^7$ with the same normal bundle. We obtain an embedding $\mathbb{R}P^3 \times \mathbb{R}^4 \to \mathbb{R}^7$. Since $\mathbb{R}P^2$ embeds into $\mathbb{R}^4$, it follows that $\mathbb{R}P^3 \times \mathbb{R}P^2$ embeds into $\mathbb{R}^7$. □

Proof of embeddability in Theorem 1.1. Recall that $S^1 \times I$ embeds into $\mathbb{R}^2$, $P_l \times I$ embeds into $\mathbb{R}^3$, Q_l immerses into $\mathbb{R}^3$, R_l and T_l embed into $\mathbb{R}^5$ [17,14], $R_l \times I$ immerses into $\mathbb{R}^4$ [8], and T_l immerses into $\mathbb{R}^4$ [2]. The normal bundle of any orientable 3-manifold, embedded into $\mathbb{R}^5$, is trivial [9,16]. Hence for every orientable 3-manifold R, $R \times I^2$ embeds into $\mathbb{R}^5$. So in the case when R_1 embeds into $\mathbb{R}^4$, embeddability in Theorem 1.1 follows from Lemma 2.1(b) for

$$(M, N_1, \ldots, N_d) = (R_1, \ldots, R_r, P_1, \ldots, P_p, S^1, \ldots, S^1, Q_1, \ldots, Q_q, T_1, \ldots, T_t),$$

where there are s copies of S^1. Note that the order of the manifolds in the above formula is important. Embeddability of $R_1 \times \cdots \times R_r \times I^2$ into $\mathbb{R}^{3r+2}$ follows by embeddability of $R_i \times I^2$ into $\mathbb{R}^5$. For the case when $p + q + s + t > 0$, embeddability in Theorem 1.1 follows by embeddability of $R_1 \times \cdots \times R_r \times I^2$ into $\mathbb{R}^{3r+2}$ and of

$$(S^1)^s \times P_1 \times \cdots \times P_p \times Q_1 \times \cdots \times Q_q \times T_1 \times \cdots \times T_t$$

into $\mathbb{R}^{s+2p+3q+4t+1}$. □

Proof of Example 1.2 [12]. Let $N = (\mathbb{R}P^3)^r$. Suppose to the contrary that $N \subset S^{3r+1}$ is an embedding. Let A_1 and A_2 be the closures of the connected components of $S^{3r+1} - N$ and let $i_1 : N \to A_1$, $i_2 : N \to A_2$ be the inclusions. Using the Mayer–Vietoris sequence for $S^{3r+1} = A_1 \cup A_2$, one sees that $i_1^* + i_2^* : H^r(A_1) \oplus H^r(A_2) \to H^r(N)$ is an isomorphism. We have

$$H^*(N, \mathbb{Z}_2) = \langle x_1, \ldots, x_r \mid x_i^4 = 0 \rangle.$$

Therefore by relabeling, if necessary, we can assume that there is an element $a \in H^r(A_1)$ such that $i_1^* a = x_1 \cdot \cdots \cdot x_r + \cdots$, where dots denote summands containing x_l^2 for some l.

So, $i_1^* a^2 = (x_1 \cdot \dots \cdot x_r)^2$ and $i_1^* a^3 = (x_1 \cdot \dots \cdot x_r)^3 \neq 0$. But from the above Mayer–Vietoris sequence it follows that $H^{3r}(A_1) = 0$, which is a contradiction. □

Note that $Q \times I$ does not embed into $\mathbb{R}^4$ for any closed surface Q with an odd Euler characteristic (this shows that Lemma 2.1 is indeed necessary in the proof of embeddability in Theorem 1.1). In fact, although Q is non-orientable, the normal Euler class $\bar{e}(Q) \in \mathbb{Z}$ of an *embedding* $Q \subset \mathbb{R}^4$ is well-defined and $\bar{e}(Q) = 2\chi(Q) \bmod 4$ [18], see also [11,15,6, p. 98]. Hence the normal bundle of an embedding $Q \subset \mathbb{R}^4$ has no cross-sections. Note that $Q \times I$ embeds into $\mathbb{R}^4$ for any closed non-orientable surface Q with an *even* Euler characteristic. For the Klein bottle K^2, this is evident by the usual immersion $K^2 \to \mathbb{R}^3$, and the general case can easily be proved by attaching handles. Also note that if Q is a closed n-manifold such that $\overline{w}_{1,n-1}(Q) = 1$ (in this case n is a power of 2, e.g., $N = \mathbb{R}P^{2^k}$), then $Q \times I$ does not embed into $\mathbb{R}^{2n}$ [10].

In the rest of the paper we show that one cannot construct examples of closed orientable 3-manifolds $R_1, \dots, R_r$ such that $R_1 \times \dots \times R_r$ does not embed into $\mathbb{R}^{3r+1}$ (cf. Example 1.2) by means of the following necessary condition for embeddability in codimension 1 [7, Theorem 3]: If a closed orientable n-manifold N embeds into $\mathbb{R}^{n+1}$, then the lth Betti number of N is even for $n = 2l$ and all the lth torsion coefficients are even for $n = 2l + 1$. Observe that for n even this result is true under a weaker assumption that N is the boundary of a compact orientable manifold, but the example $N = \mathbb{R}P^3$ shows that for odd n this result is false under the weaker assumption. Now, if $N_1, \dots, N_r$ are closed orientable manifolds (not necessarily 3-dimensional), some of which are boundaries of compact orientable manifolds, and $\dim(N_1 \times \dots \times N_r) = 2l$, then the product $N_1 \times \dots \times N_r$ is a boundary of a compact orientable manifold, hence the lth Betti number of this product is even, therefore [7, Theorem 3] does not apply to even-dimensional examples. It follows from Theorem 2.2 below that it also does not apply to odd-(> 1)-dimensional examples. Note that Theorem 2.2 is false for $r = 1$, as shown by the example $N = \mathbb{R}P^3$.

Theorem 2.2. *Let $r > 1$ be any integer and $N_1, \dots, N_r$ any closed orientable manifolds of even Euler characteristic. If* $\dim(N_1 \times \dots \times N_r) = 2l + 1$, *then* $\operatorname{Tors} H_l(N, \mathbb{Z}) \cong G \oplus G$, *for some Abelian group G.*

Proof. For any n-dimensional polyhedron N such that

$$H_l(N, \mathbb{Z}) = \mathbb{Z}^{b_l} \oplus \bigoplus_{i,j} \mathbb{Z}_{p_i^j}^{t_l^{ij}}$$

($p_1, p_2, \dots$ are distinct prime numbers) define the *complete Poincaré polynomial* of N as follows:

$$P_N\bigl(x, \{y_{ij}\}\bigr) = F_N(x) + \sum_{i,j} T_N^{ij}(y_{ij}),$$

where

$$F_N(x) = b_0 + b_1 x + \dots + b_n x^n \quad \text{and} \quad T_N^{ij}(y_{ij}) = t_1^{ij} y_{ij} + \dots + t_n^{ij} y_{ij}^n.$$

The proof of Theorem 2.2 is based on the following representation of the Künneth formula: $P_{M\times N} = P_M * P_N$, where $*$ is the (unique) commutative distributive (*Künneth*) product defined on generators by $x^a * x^b = x^{a+b}$, $x^a * y_{ij}^b = y_{ij}^{a+b}$, $y_{ij}^a * y_{ik}^b = (1+y_{ij})y_{ij}^{a+b}$ for $j \leqslant k$ and $y_{ij}^a * y_{i'k}^b = 0$ for $i \neq i'$. Equivalently,

$$\begin{aligned} P_{M\times N}\bigl(x,\{y_{ij}\}\bigr) &= P_M\bigl(x,\{y_{ij}\}\bigr) * P_N\bigl(x,\{y_{ij}\}\bigr) \\ &= F_M(x)F_N(x) + \sum_{ij}\Bigl[\bigl(F_M T_N^{ij} + T_M^{ij} F_N\bigr) \\ &\qquad + (1+y_{ij})\Bigl(T_M^{ij}\sum_{k\geqslant j} T_N^{ik} + T_N^{ij}\sum_{k>j} T_M^{ik}\Bigr)\Bigr](y_{ij}). \end{aligned}$$

Consider the complete Poincaré polynomials modulo 2. Since $F_{N_i}(1) = \chi(N_i) = 0 \bmod 2$, it follows from the Künneth Formula that $T^{ij}_{N_1\times\cdots\times N_r}(1) = 0$ for $r > 1$. Theorem 2.2 now follows, since by duality and the Universal Coefficients Formula one has $t^{ij}_{l+r} = t^{ij}_{l-r}$ for all $r \geqslant 0$. $\square$

Note that Theorem 2.2 can also be proved by localization, i.e., from the Künneth formulae with $\mathbb{Z}_{p_k}$-coefficients. By the Universal Coefficients Formula, the complete Poincaré polynomial of N with $\mathbb{Z}_{p_i^j}$-coefficients is

$$P_N^{ij}(y_1,\ldots,y_j) = F_N(y_j) + \sum_{k=1}^{j}(1+y_k)T_N^{ik}(y_k),$$

where y_k is the shorthand for y_{ik} from above. Then we have

$$T^{i1}_{N_1\times\cdots\times N_r}(1) = \frac{(F_{N_1} + (1+y_1)T^{i1}_{N_1})\cdots(F_{N_r} + (1+y_1)T^{i1}_{N_r}) - F_{N_1}\cdots F_{N_r}}{1+y_1}\bigg|_{y_1=1},$$

where all the polynomials are of y_1 (the polynomial in the denominator of the above fraction is clearly divisible by $1+y_1$). This is zero when $F_{N_s}(1) = 0$. For $j > 1$ the proof of $T^{i1}_{N_1\times\cdots\times N_r}(1) = 0$ is analogous, but it is not easier than the direct proof above (since we have to apply the Künneth Formula with coefficients $\mathbb{Z}_{p_i^j}$, which is not a field).

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References

[1] R.L.W. Brown, Immersions and embeddings up to cobordism, Canad. J. Math. 23 (1971) 1102–1115.

[2] R.L. Cohen, The immersion conjecture for differentiable manifolds, Ann. of Math. (2) 122 (1985) 237–328.
[3] M. Galecki, On embeddability of CW-complexes in Euclidean space, Univ. of Tennessee, Knoxville, TN, Preprint, 1992.
[4] M. Galecki, Enhanced cohomology and obstruction theory, Doctoral Dissertation, Univ. of Tennessee, Knoxville, TN, 1993.
[5] S. Gitler, Immersion and embedding of manifolds, in: Algebraic Topology, Proc. Symp. Pure Math., Vol. 22, Amer. Math. Soc., Providence, RI, 1971, pp. 87–96.
[6] L. Guillou, A. Marin (Eds.), A la Récherche de la Topologie Perdue, Progress in Math., Vol. 62, Birkhäuser, Basel, 1986.
[7] W. Hantzsche, Einlagerung von Mannigfaltigkeiten in euklidische Räume, Math. Z. 43 (1937) 38–58.
[8] M.W. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93 (1959) 242–276.
[9] M.A. Kervaire, On higher-dimensional knots, in: S.S. Cairns (Ed.), Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, NJ, 1965, pp. 105–119.
[10] M. Mahowald, On the normal bundle of a manifold, Pacific J. Math. 14 (1964) 1335–1341.
[11] W.S. Massey, Pontryagin squares in the Thom space of a bundle, Pacific J. Math. 31 (1969) 133–142.
[12] E.G. Rees, Private communication, 1998.
[13] D. Repovš, A.B. Skopenkov, Embeddability and isotopy of polyhedra in Euclidean spaces, Proc. Steklov Math. Inst. 212 (1996) 163–178.
[14] V.A. Rokhlin, The embedding of non-orientable three-manifolds into five-dimensional Euclidean space, Dokl. Akad. Nauk SSSR 160 (1965) 549–551 (in Russian).
[15] V.A. Rokhlin, The normal Euler numbers of the projective plane and Klein bottle in four-dimensional Euclidean space, Dokl. Akad. Nauk SSSR 191 (1970) 27–29 (in Russian).
[16] D. Rolfsen, Knots and Links, Math. Lecture Ser., Vol. 7, Publish or Perish, Berkeley, CA, 1976.
[17] C.T.C. Wall, All 3-manifolds imbed in 5-space, Bull. Amer. Math. Soc. 71 (1965) 564–567.
[18] H. Whitney, On the topology of differentiable manifolds, in: Lectures on Topology, Univ. of Michigan Press, Ann Arbor, MI, 1941, pp. 101–141.