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ON THE EULER CHARACTERISTIC OF MULTIPLE SELFINTERSECTION POINTS OF IMMERSED MANIFOLDS

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Abstract: Various examples of immersed codimension 1 manifolds are studied from the viewpoint of possible combinations of the Euler characteristics of the submanifolds of multiple selfintersection points. A complete answer is given for the immersed 5-manifolds in the 6-dimensional Euclidean space. Relations are discussed with other constructions in differential topology and singularity theory.

Keywords: immersion, Euler characteristic, singularity, selfintersection

1. Introduction

Let $f : M^m \to R^{m+k}$ be an immersion in general position. Denote by $f(M)_i$ the set of points $x \in R^{m+k}$ for each of which there are at least i distinct points $x_1, \dots, x_i \in M$ in the inverse image such that $f(x_j) = f(x_k)$ for all $1 \le j < k \le i$. The set $f(M)_i$ is endowed with a smooth immersed submanifold structure $g_i : \Delta_i \to R^{m+k}$ which is in most cases nongeneric, and the submanifold $g_{i+1}(\Delta_{i+1}) \subset g_i(\Delta_i)$ of points of multiplicity $i+1$ is the singular submanifold of $g_i(\Delta_i)$ in the sense that $g_i(\Delta_i)$ has a nongeneric selfintersection along $g_{i+1}(\Delta_{i+1})$. It is convenient to put $\Delta_1 = M$. Then the immersion $g_i : \Delta_i \to R^{m+k}$, $\dim(\Delta_i) = m - k(i-1)$, selfintersects along $\widetilde{\Delta}_{i+1} \subset \Delta_i$, $\operatorname{Im}(g_i|\widetilde{\Delta}_{i+1}) = g_{i+1}(\Delta_{i+1})$.

The purpose of this article is to study various examples of immersions f with different combinations of the Euler characteristics $\chi(\Delta_i)$ of the manifolds of multiple points and find some rules for constructing various interesting examples in the framework of geometric methods.

2. Calculation of the Euler Characteristics of Manifolds of Multiple Selfintersection

Let $f : S^1 \to R^2$ be an immersion in general position. Whitney [1] discovered that the index $\operatorname{Ind}(f)$ of f, i.e. the integer equal to the number of rotations of the tangent vector in the circuit along the oriented immersed curve f in the positive direction, determines the regular homotopy class of this immersed curve. In the same article, Whithey also found an important formula for restoring $\operatorname{Ind}(f)$ from the structure of Δ_2, the zero-dimensional submanifold of the plane defined by the set of double selfintersection points of the immersed curve.

Arnol′d [2] noted that Whitney's formula should admit a generalization to immersions of higher dimensions. One of the possible generalizations, for codimension 1 immersions of oriented manifolds, was found soon by Mikhalkin and Polyak [3]. Their generalization relates to integral formulas in which the measure of integration is defined as the Euler characteristic of strata of different dimensions into which the immersed submanifold $f(M)$ decomposes the ambient space R^{m+1} and into which the submanifold Δ_{i+1} decomposes the immersed submanifold Δ_i.

Independently of the problem of generalizing Whitney's formula to higher dimensions, the problem of integration with respect to Euler characteristics was studied in Viro's article [4]. This motivates the statement of the following problem.

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The Euler Characteristic Problem 2.1. *Let $f : M^m \to R^{m+k}$ be an immersion in general position. Find all possible combinations of the numbers $\chi(\Delta_i)$ for the Euler characteristics of the immersed submanifolds of selfintersection points of multiplicity i.*

Remark. The manifold Δ_i is determined by the regular homotopy class of the original immersion nonuniquely but to within an immersed cobordism with a given normal bundle structure. Therefore, a more general problem consists in calculating the cobordism class of this immersion (the cobordism class with a given normal bundle structure inclusively). In this case, the manifold Δ_i happens to be nonorientable as a rule. The parity of the Euler characteristic of the manifold serves as the simplest invariant of the cobordism class.

An exception arises when the manifold M itself is oriented and $k = 0 (\mathrm{mod}\, 2)$. In this event, the element defined by Δ_i in the cobordism group $\Omega_{m-(i-1)k}$ has an infinite order in general. The cobordism class of Δ_i was found in this case by Szücs in [5]. It turns out that for $\dim(\Delta_i) = 1 (\mathrm{mod}\, 2)$ we have $\chi(\Delta_i) = 0 (\mathrm{mod}\, 2)$.

The converse problem was studied by Ekholm in [6] and reads as follows: Can a given immersion f be restored in the regular homotopy class from the cobordism class $[\Delta_2(f)]$ in the group of cobordisms with a given normal bundle structure?

The results of [6] can be considered as a generalization of Whitney's [7] who answered the above problem in the affirmative in the case of immersions of manifolds into Euclidean space for $k = m$. Ekholm used the theory of Vasil′ev's invariants in his studies.

Surely, in the general case the answer to the converse problem is negative. For example, it turns out that the regular homotopy class of an immersion $f : S^3 \to R^5$ cannot be restored from the cobordism class of the manifold $\Delta_2(f)$. Moreover, it happens that there are infinitely many embeddings $\varphi_i : S^3 \to R^5$ no two of which are connected by a regular homotopy. More precisely, the classes of regular homotopies of immersed spheres modulo the operation of taking connected sums with embedded spheres make a cyclic group of order 24, whereas the structure of selfintersection merely allows us to recognize an element in the factor-group of order 12. A complete solution of the problem of recognizing the regular homotopy class of an immersed sphere remains an important open problem.

Problem 2.1 was studied from various viewpoints in several articles (see [5, 8–11]). For $i = 1 (\mathrm{mod}\, 2)$ the problem of calculating the parity of $\chi(\Delta_i)$ becomes essentially simpler. A canonical i-fold covering $\overline{\Delta}_i \to \Delta_i$ is well defined; moreover, the manifold $\overline{\Delta}_i$ is immersed in M. Hence, $\chi(\Delta_i) = \chi(\overline{\Delta}_i)/i$. Basing on the results of [12], define $\overline{\Delta}_i$ as representing the cobordism class of the embedded submanifold of zeros of a section in general position of the bundle $(i-1)\nu_f$ (here ν_f is the normal bundle of the immersion f) which is the direct sum of $i - 1$ isomorphic copies.

This observation readily enables us to calculate $\chi(\overline{\Delta}_i)(\mathrm{mod}\, 2)$ and so $\chi(\Delta_i)(\mathrm{mod}\, 2)$. It was noted in [12] that for $i = 1 (\mathrm{mod}\, 2)$ the value of $\chi(\Delta_i)$ depends only on the cobordism class of M and is independent of the choice of the immersion f.

In the case of an oriented M and $k = 0 (\mathrm{mod}\, 2)$, the value of $\chi(\Delta_i)$ for each $i = 0 (\mathrm{mod}\, 2)$ was calculated in [5]. A new idea was that in this case Δ_i represents an element of the group $\Omega_{m-(i-1)k}$ and the equality $\chi(\Delta_i) = 0$ can be proved by transfer also for even-fold coverings.

It turned out in particular that in the case when $m \neq 2 (\mathrm{mod}\, 4)$ we have $\chi(\Delta_i) = 0$ for all i. Note that for $i = 0 (\mathrm{mod}\, 2)$ the problem is extremely difficult. It is the case to whose consideration we now proceed.

An algebraic apparatus for solving Problem 2.1 in the most general form is exposed in the survey article [13] by Eccles. Now, we do not presume M orientable. The most lucid picture is available for calculating the parity of the 0-dimensional set of selfintersection points of maximal multiplicity $m + 1$.

In this case, the answer depends on the value of $m (\mathrm{mod}\, 4)$. For $m = 0, 2, 3, 6$ and for the more general case of $m = 2^j - 3$ if we additionally assume that in dimension $2^j - 2$ there is a manifold with Kervaire invariant 1, there exists an immersion $f : M^m \to R^{m+1}$ with an odd number $\Theta(f)$ of selfintersection points of maximal multiplicity $m + 1$ [14]. On the other hand, $\Theta(f)$ takes only even values for the other values of m.

The case of $m+1=2(\mathrm{mod}\,4)$ is most interesting. An elementary proof of triviality of the Kervaire invariants for $m \neq 2^j - 3$ was given recently in the joint article of the first author and Eccles [15]. Note that the calculation of $\chi(\Delta_i)$ becomes more complicated as m increases. Thus, a complete solution of Problem 2.1 is closely related with the main open problems of homotopy theory.

3. The Main Results

In this section we consider various examples of immersions $f : M^m \to R^{m+1}$ for $m \leq 5$ and formulate the main theorem.

Example 3.1 [16]. There exists an immersion of the projective plane RP^2 into R^3 with $\chi(\Delta_1) = \chi(\Delta_3) = 1$ (also see [10]).

Example 3.1 was generalized in [9] to the case of immersions of arbitrary even-dimensional manifolds. The idea of the proof bases on applying the Fubini theorem (a general idea is exposed in [4]). The proof itself is elementary and does not use homotopy theory.

Theorem 3.2 [9]. *If $f : M^m \to R^{m+1}$, $m = 0(\mathrm{mod}\,2)$, is an arbitrary immersion then*

$$\sum_i \chi(\Delta_{2i+1}) = 0.$$

Szücs recently obtained a complete solution of Problem 2.1 in the case when m is even by means of homotopy theory.

Theorem 3.3 [17]. *If $f : M^m \to R^{m+1}$, $m \geq 8$ or $m = 4$, is an arbitrary immersion then $\chi(\Delta_{2i+1}) = 0$ for every i. If $m = 6$ then $\chi(\Delta_7) = \chi(\Delta_5) = \chi(\Delta_3) = \chi(\Delta_1) = 0$ or 1. If $m = 2$ then $\chi(\Delta_1) = \chi(\Delta_3) = 0$ or 1.*

Remark. If $\chi(\Delta_{2i+1}) = 1$ then the submanifold Δ_{2i+1} is (up to cobordism) the projective space RP^{m-2i}.

We turn now to considering the case $m = 1(\mathrm{mod}\,2)$.

Example 3.4. There exists an immersion of a curve into the plane with a single selfintersection point, i.e., $\chi(\Delta_2) = 1$ and $\dim(\Delta_2) = 0$.

This simplest example shows that the value of $\chi(\Delta_2)$ is in general independent of the cobordism class of the original manifold M, and the analog of Theorem 3.3 for an odd m is false.

Example 3.5 [14]. There exists an immersion $f : M^5 \to R^6$ with $\chi(\Delta_6)=1$.

Construction. Choose an immersion $g : S^3 \to R^6$ with a single selfintersection point by following Whitney's construction [7]. The immersion g admits framing; i.e., there is a trivialization $\nu(g) = 3\varepsilon$ of the normal bundle. Hence, there exists an immersion $f : S^3 \times RP^2 \to R^6$ which is defined in the local coordinates of the framing by the formula $f = g \times g'$, where g' is the Boy immersion of Example 3.1.

The main result of this article is the following theorem answering the question of [9].

Theorem 3.6. *If $f : M^5 \to R^6$ is an arbitrary immersion then $\chi(\Delta_2) = \chi(\Delta_6) = 0$ or 1, but $\chi(\Delta_4) = 0$.*

Remarks. (1) In the case of $\chi(\Delta_2) = 1$ the manifold Δ_2 is cobordant to $RP^2 \times RP^2$.

(2) If M^5 is orientable then $\chi(\Delta_6) = 0$ (see [13] or [9, Remark 3]). Hence, in this case $\chi(\Delta_2) = \chi(\Delta_4) = \chi(\Delta_6) = 0$.

Corollary 3.7. *For the immersion $RP^2 \times S^3$ of Example 3.5, we have $\chi(\Delta_2) = \chi(\Delta_6) = 1$ and $\chi(\Delta_4) = 0$.*

Proving Theorem 3.6, we develop a new technique for resolution of singularities of the manifolds of multiple selfintersection points. This technique is a direct generalization of Freedman's construction in [11]. Eccles communicated to us that our result can be proved by calculations in the Dyer–Lashof algebra as it was done for similar problems in [13]. Our approach bases on elementary geometric considerations.

Theorem 3.6 generalizes the following Koschorke theorem [18] which was rediscovered and re-proved by another method in [8]. It generalizes the main theorem of [11] to the nonorientable case.

Theorem 3.8 [18]. *Let $f : M^3 \to R^4$ be an arbitrary immersion; moreover, the manifold M is not presumed orientable. Then $\chi(\Delta_2) = \chi(\Delta_4) = 0$ or 1.*

The following conjecture shows that Problem 2.1 could be difficult and not reducible to the calculation of the Euler characteristic of the zero-dimensional manifold of selfintersection points of maximal multiplicity.

Conjecture 3.9. *There exists an immersion $M^9 \to R^{10}$ with $\chi(\Delta_2) = \chi(\Delta_6) = 1$ and $\chi(\Delta_4) = \chi(\Delta_8) = \chi(\Delta_{10}) = 0$.*

4. Proof of the Main Theorem

Proof of Theorem 3.6. The equality $\chi(\Delta_2) = \chi(\Delta_6)$ ensues easily from the results of [14] wherein all assertions are formulated in algebraic terms (a geometric statement is given in [15, Theorem 1.6]). More precisely, we have $\chi(\Delta_6) = \langle e(\nu^2)^2, [\Delta_2]\rangle$, where ν^2 is the normal bundle of the immersed manifold $g_2' : \Delta_2 \to R^6$. Example 3.5 shows that there exists an immersion with $\chi(\Delta_2) = \chi(\Delta_6) = 1$.

Prove that $\chi(\Delta_4(f)) = 0$. Choose $g_2' : \Delta_2 \to R^6$ and consider $\Delta_2(g_2')$. We can naturally define $\Delta_2(g_2') = M \cup N$, where M is a 3-fold covering of the surface $\Delta_4(f)$ and N results from resolving the singularities of the immersion g_2 along the submanifold $\widetilde{\Delta}_3$ by means of a general position deformation $g_2 \to g_2'$.

More formally, let D_1 and D_2 be two sheets of Δ_2 intersecting along a sheet of the surface $l \in g_2'(D_1) \cap g_2'(D_2) \subset \Delta_2(g_2')$. By definition, $l \in M$ if $g_2(D_1) \cap g_2(D_2) \subset \Delta_4(f)$. Otherwise l lies on the surface N. Clearly, if $l \in N$ then $g_2(D_1) \cap g_2(D_2) \subset \Delta_3(f)$.

We now describe the surface N. Let $p : \overline{\Delta}_3 \to \Delta_3(f)$ be a canonical 3-fold covering ($I_3 : \overline{\Delta}_3 \subset M$ is a canonical immersion). The manifold $\widetilde{\Delta}_3 \subset \Delta_2$ is defined by resolving the singularities of triple selfintersection points ($\widetilde{\Delta}_3 \subset \Delta_2$). We identify $\widetilde{\Delta}_3$ and $\overline{\Delta}_3$ by means of the tautological diffeomorphism H. Consider an immersion $\tilde{g}_3' : \widetilde{\Delta}_3 \to R^6$ in general position which approximates $\tilde{g}$.

The canonical 2-fold covering $q : \widehat{\Delta}_3 \to \widetilde{\Delta}_3$ is induced by the canonical covering $\overline{\Delta}_2 \to \Delta_2$. Note that the manifold $\widehat{\Delta}_3$ admits a canonical immersion $h : \widehat{\Delta}_3 \to \Delta_1$ that is induced by the canonical immersion $I_2 : \overline{\Delta}_2 \to \Delta_1$. Moreover, the immersion h can be defined as the composition $h = I_3 \circ q$. In particular, the cohomology class $w_1(\nu') \in H^1(\bar{\Delta}_3)$ is well defined, where $\nu' \to \bar{\Delta}_3$ is the 1-dimensional bundle induced from the normal 1-bundle $\nu \to \Delta_1$ by the immersion $\bar{\Delta}_3 \subset \Delta_1$ (taking liberties with notation, we henceforth denote ν' again by ν).

The diffeomorphism $\bar{\Delta}_3 = H(\tilde{\Delta}_3)$ allows us to consider ν as a bundle over $\tilde{\Delta}_3$.

Define $j : N \subset \tilde{\Delta}_3$ as a submanifold representing the homological Euler class of a cross-section of the 1-bundle $\gamma \to \tilde{\Delta}_3$, where $w_1(\gamma) = w_1(q) + w_1(\nu)$. Equivalently, it can be described in terms of the covering q as follows. Consider $\widehat{\Delta}_3$ as a manifold immersed in the total space of the bundle $\nu \to \tilde{\Delta}_3$ as a result of a small general position deformation of the 2-fold covering q of the base of the bundle $\tilde{\Delta}_3$.

Thus, put $q(\widehat{\Delta}_3)$ in general position, $q \to q'$, and define $j(N) = \Delta_2(q')$. Projecting onto the base of the bundle, we can consider $j(N) \subset \widetilde{\Delta}_3$. Observe that $\dim(N) = 2$ as required in the construction. Furthermore, the immersion $j' = g \circ j : N \to \widetilde{\Delta}_3 \to R^6$ is well defined. For an arbitrary small deformation of g_2' with a suitable choice of j, the surface N is diffeormorphic to the surface $\Delta_2(g_2') \setminus M$.

Lemma 4.1. $\chi(N) = 0$.

Proof. Recall that the surface $N \subset \widetilde{\Delta}_3 = H^{-1}(\overline{\Delta}_3)$ is defined by the Euler class of the 1-bundle $\kappa \to \overline{\Delta}_3$, $w_1(\kappa) = w_1(q) + w_1(\nu)$. Moreover, the manifold $\overline{\Delta}_3$, considered as an immersed submanifold of Δ_1, can itself be defined as the submanifold of selfintersection points of the immersed submanifold $I_2(\overline{\Delta}_2)$ in Δ_1. The results of [12] readily imply that the immersed submanifold $I_2(\overline{\Delta}_2) \subset \Delta_1$ is cobordant as an immersed submanifold to the embedded submanifold represented by the zeros of a cross-section of the bundle $\nu \to \Delta_1$.

Hence, the submanifold $N \subset \widetilde{\Delta}_3$ bounds, since under a regular cobordism of $\overline{\Delta}_2$ both characteristic classes $w_1(q)$ and $w_1(\nu)$ remain defined on the three-dimensional selfintersection submanifold. This completes the proof of Lemma 4.1. Theorem 3.6 ensues from the equality $\chi(\Delta_2(g_2')) = 0$ which is a consequence of the main theorem of [19].

References

1. *Whitney H.*, "On regular closed curves in the plane," Comp. Math., **4**, 276–284 (1937).
2. *Arnol′d V. I.*, "Topological invariants of plane curves and caustics," in: Dean J. B. Lewis Memorial Lectures Rutgers Univ. Amer. Math. Soc., Providence, R. I., 1994. (Univ. Lect. Ser.; **5**).
3. *Mikhalkin G. and Polyak M.*, "Whitney formula in higher dimensions," J. Differential Geom., **44**, 583–594 (1996).
4. *Viro O. Y.*, "Some integral calculus based on Euler characteristic," in: Topology and Geometry Rohlin Seminar, Springer, Berlin; Heidelberg; New York; London; Paris; Tokyo, 1988, pp. 127–138. (Lecture Notes in Math.; **1346**).
5. *Szücs A.*, "On the multiple points of immersions in Euclidean spaces," Proc. Amer. Math. Soc., **126**, 1837–1882 (1998).
6. *Ekholm T.*, Immersions and Their Selfintersections, Uppsala Dissert. Math., **12**, (1998).
7. *Whitney H.*, "The self-intersections of a smooth n-manifold in $2n$-space," Ann. of Math. (2), **45**, 220–246 (1944).
8. *Akhmet′ev P. M.*, "An elementary proof of Freedman's theorem on immersions," St. Petersburg Math. J., **7**, 749–754 (1996).
9. *Akhmet′ev P. M., Rimanyi R., and Szücs A.*, "A generalization of Banchoff's triple point theorem," Proc. Amer. Math. Soc., **126**, 913–915 (1998).
10. *Banchoff T.*, "Triple points and singularities of projections of smoothly immersed surfaces," Proc. Amer. Math. Soc., **46**, 402–406 (1974).
11. *Freedman M. H.*, "Quadruple points of 3-manifolds in S^4," Comment. Math. Helv., **53**, 385–394 (1978).
12. *Herbert R. J.*, "Multiple point of immersed manifolds," Mem. Amer. Math. Soc., **34** (1981).
13. *Eccles P. J.*, "Characteristic numbers of immersions and self-intersection manifolds," in: Proceedings of the 7th Colloquium: Topology with Applications, Szekszárd, Hungary, 1993, János Bolyai Math. Soc., Budapest, Bolyai Soc. Math. Stud., 1995, **4**, pp. 197–216.
14. *Eccles P. J.*, "Codimension one immersions and the Kervaire invariant one problem," Math. Proc. Cambridge Philos. Soc., **90**, 483–493 (1981).
15. *Akhmet′ev P. M. and Eccles P. J.*, "A geometrical proof of Browder's result on the vanishing of the Kervaire invariant," Proc. Conf. Solitons and Geometry (60th Birthday of S. P. Novikov), Moscow, 1998, Proc. Steklov Inst., **225**, 46–51 (1999).
16. *Boy W.*, "Über die Kurvature integral und die Topologie geschlossener Flächen," Math. Ann., **57**, 151–184 (1903).
17. *Szücs A.*, On Multiple Points of Codimension One Immersions of Even Dimensional Manifolds [Preprint], Lorand Eotvos Univ., Budapest (2001).
18. *Koschorke U.*, "Multiple points of immersions and the Kahn–Priddy theorem," Math. Z., **169**, 223–236 (1979).
19. *Szücs A.*, "Double point surfaces of smooth immersions $M^m \to R^{2n-2}$," Math. Proc. Cambridge Philos. Soc., **113**, 601–613 (1993).