Mathematical Notes, vol. 71, no. 4, 2002, pp. 455–463.
Translated from Matematicheskie Zametki, vol. 71, no. 4, 2002, pp. 496–507.

On Milnor's Invariants of 4-Component Links

P. M. Akhmet′ev, J. Malešič, and D. Repovš

Received November 29, 2000

Abstract—We study the behavior of Milnor's μ-invariants of three- and four-component links with respect to the discriminant determined by Δ-moves of links. We introduce a new type of Δ-move, balanced Δ-moves, or, briefly, $B\Delta$-moves. Since each four-component link is equivalent to a standard link under a sequence of balanced Δ-moves, Δ-moves that involve at most two components, and Reidemeister moves, we manage to define axiomatically μ-invariants of length 3 for arbitrary semibounding links.

Key words: *Milnor's μ-invariants, semibounding links, Δ-moves, balanced Δ-moves, Reidemeister moves.*

1. INTRODUCTION

In his famous paper [1], Milnor introduced the notion of integer-valued μ-invariants of oriented multicomponent links with ordered set of components satisfying the additional condition that the link is semibounding. The μ-invariants are partially defined; this means that a μ-invariant of order k is well defined only if all μ-invariants of order not greater than $k-1$ vanish on all sublinks (determined by a subset of components) of the given link. Milnor's μ-invariants are ambient isotopy invariants of a link. Moreover, they are homotopy invariants with respect to homotopy allowing the components to self-intersect, but not to intersect other components.

Invariants of length 3 are invariants of finite order (of order not greater than 3, to be precise) in the sense of Vassiliev. There is exactly one μ-invariant which is Vassiliev of length 2; it is well defined for arbitrary three-component link with pairwise nonlinked components. There are precisely two distinct invariants of length 3 (see [1]), and each of these invariants is well defined for four-component semibounding links (see Definition 2.1).

There is an equivalent elementary definition of μ-invariants of length 2 and 3 in geometric terms of Seifert surfaces. By the Porter–Turaev Theorem [2, 3], Milnor's μ-invariants coincide with iterated Massey invariants. The Massey invariants are defined in terms of products of cocycles in the complement to the link in the sphere S^3 (or even in an arbitrary homology sphere, as in [3]). The dual construction exploits only geometric properties of Seifert surfaces; it was discovered by Cohran in [4].

There is no way to extend any Milnor invariant of length 2 or 3 to any finite order, in the sense of Vassiliev, integer-valued homotopy invariant of arbitrary links (see [5]). If we omit the first requirement, that is, if we do not require homotopy invariance, but require the finiteness of the order, then, generally speaking, such an extension is possible. The simplest example of a Milnor invariant, which is an invariant of order 3 well defined on isotopy classes of arbitrary oriented two-component links with unordered components, is given by the generalized Sato-Levine invariant (see [6, 7]). The Sato-Levine invariant for two-component link with zero linking number is studied, as a Milnor $\overline{\mu}$-invariant, in [8].

In the general case, the problem of extending the integer-valued Milnor μ-invariant to an ambient isotopy link invariant preserving the Vassiliev order of the invariant is very important for physical applications; in the simplest setting (without using Vassiliev's construction) it was studied in [9]. No such extension for the invariant of order 2 is known, although we do not have the corresponding nonexistence theorem. On the contrary, both invariants of order 3 admit such an extension; this statement will be proved in the second part of the paper, which is now in preparation.

We solve this problem by giving an axiomatic definition of the invariants of order 3 by means of a new kind of Δ-moves of link diagrams, the so-called *balanced* Δ*-moves*. The notion of a Δ-move understood as a transformation of a link was introduced in [10]. It was intensively studied in the last years from various points of view (see [11–13]). The study of μ-invariants naturally leads to generalizations of this notion. For example, invariants of order 3 are studied in [13, p. 58] by means of the generalized Δ-move such that all four components of the link are close to the singular point of the move. Under this move the value of the invariant of order 3 is changed by ± 1.

Our approach is based on a different generalization. First of all, a balanced Δ-move (for the sake of brevity we use below the term $B\Delta$*-move*) preserves the class of semibounding links. On the other hand, if we allow to pass from one link to another by means of $B\Delta$-moves, then the space of semibounding links becomes connected. This means that if we know how an invariant of order 3 changes under balanced Δ-moves, then it can be calculated for arbitrary semibounding link. We set the value of the invariants on links whose components can be bounded by pairwise nonintersecting balls equal to 0. In part II of the present paper, which is in preparation now, we intend to generalize the definition of μ-invariants of order 3 and define them for arbitrary, not necessarily semibounding, four-component links.

Let us describe the contents of the present paper in more detail. In Sec. 2, we give a new definition of a semibounding three- or four-component link; in contrast to the definition above, it does not assume that each component of the link is connected. We define the notion of a bordism of a multicomponent link (which simulates, as it was remarked in [9], the process of reclosing of trajectories of magnetic fields in a medium with high magnetic conductivity). This new notion is an analog of the well-known notion of homotopy of links; the latter is much better studied.

In Sec. 3, we switch to the study of three-component links (with connected components), recall the definition of the μ-invariant of order 2, and prove its main properties. We conclude this section by illustrating our approach to invariants of order 3 on a simpler example and introduce the notion of a balanced $\times$-move, which preserves the class of semibounding three-component links. The space of three-component semibounding links becomes connected with respect to these moves. Note that we did not manage to extend invariants of order 2 to total invariants by means of the balanced $\times$-moves.

In Sec. 4, we formulate the theorem stating that the space of four-component semibounding links (with connected components) is the unique equivalence class of links modulo $B\Delta$-moves. We recall the definition of invariants of order 3 and prove their main properties. The jump rule of a μ-invariant under a $B\Delta$-move is formulated. The concluding Sec. 5 is devoted to the proofs.

2. MULTICOMPONENT SEMIBOUNDING LINKS (IN THE GENERALIZED SENSE) AND THEIR CLASSIFICATION PROBLEM

Let $I = \{1, \dots, s\}$ be a segment of integers; we shall usually have $s = 2$ or 3. Let $L \subset \mathbb{R}^3$ be a smooth closed oriented one-dimensional submanifold, and suppose an element of I is assigned to each component of L (the components can well be disconnected or even empty). The submanifold L is called a *multicomponent* (in the generalized sense) *link*. We denote the components (generally speaking, disconnected) of a generalized multicomponent link by L_i, $i \in I$. For the sake of brevity we omit below the adjective "generalized." Now let us recall the standard notion of a cobordism of submanifolds. In our case a cobordism is a sequence of regular homotopy transformations of submanifolds and Morse bifurcations or the components' surgery (see [9]). The reconstructions preserve both the orientation of the components and the value of I.

Let L_i be the submanifold corresponding to a given $i \in I$. We mark the connected components of this submanifold by the second subscript j, $j \in \{1, \dots, s_i\}$, e.g., $L_{2,4} \subset L_2 \subset L$.

Definition 2.1. A two- or three-component link is said to be *semibounding* if for any two pairs of indices i, j, $i \in \{1, 2\}$ or $i \in \{1, 2, 3\}$, $j \in \{1, \dots, s_i\}$, we have $\operatorname{lk}(L_{i_1,j_1}; L_{i_2,j_2}) = 0$.

A four-component link is said to be *semibounding* if the following two conditions are satisfied. First, for any two pairs of indices i, j, $i \in \{1, 2, 3, 4\}$, $j \in \{1, \dots, s_i\}$, we have $\operatorname{lk}(L_{i_1,j_1}; L_{i_2,j_2}) = 0$. Second, for any three pairs of indices, i, j we have $\mu(L_{i_1,j_1}; L_{i_2,j_2}; L_{i_3,j_3}) = 0$, where μ denotes Milnor's integer-valued invariant of length 2; we recall the definition of this invariant in Sec. 3.

Definition 2.2. Two s-component semibounding links L and L', $s = 2, 3, 4$, are called *bordant* if they are bordant in the class of semibounding links. In other words, if there is a bordism preserving the semiboundedness of the link under each surgery.

The elementary argument from [9] easily shows that two general multicomponent links (not necessarily semibounding) are bordant if the symmetric $s \times s$-matrices formed by the sums (over the pairs of the second subscripts) of linking numbers of each pair of components with given pair of first subscripts coincide. Semibounding bordisms are more complicated. The s-component links with zero pairwise linking numbers of the components form a subspace, and the discriminant in this subspace consists of components resulting from a number of surgeries (here a *surgery* is a Morse critical point of the time function on the cobordism manifold; see the details in [9]). It seems that the classification problem for Vassiliev invariants in this theory was not studied yet. On the other hand, nontrivial invariants exist. The generalized Sato-Levine invariant provides the simplest example.

These invariants arise in the classification of divergence-free vector fields. Suppose a vector field is the union of s vector subfields with pairwise nonintersecting supports (each summand in this sum models a set of magnetic tubes). We wish to study the behavior of invariants under evolution of the system of magnetic tubes with a weak reclosing of the vector field trajectories. It is important, for example, to predict the statistics of the space spectrum of the field's spirality (i.e., the magnitude of the vector field in domains with dominating positive or negative linking numbers of the magnetic tubes) if we know something about higher invariants of the linking of the trajectories.

In the context of the present paper, below we use only standard multicomponent links, meaning that each component of a link is connected and nonempty. Conjecturally, our constructions can be generalized to the case of multicomponent (in the generalized sense) semibounding links and provide some knowledge of finite order invariants in the theory above. For example, the corresponding μ-invariants can be extended to multicomponent links by additivity. The behavior of such invariants under bordisms requires additional study.

3. THREE-COMPONENT SEMIBOUNDING LINKS

Let $L = L_1 \cup L_2 \cup L_3$ be a three-component semibounding link with connected and nonempty components, and let S_1, S_2 and S_3 be Seifert surfaces of the components L_1, L_2 and L_3, respectively, such that $L_i \cap S_j = \varnothing$ for $i \neq j$. The intersection $S_1 \cap S_2 \cap S_3$ is finite. Let us fix the order of the components (i.e., a permutation ijk of the indices 1, 2 and 3). Recall that the components of the link are oriented, and therefore the Seifert surfaces also are oriented by the standard consistency rule for orientations of a manifold and the boundary. This means that we can assign to each intersection point of the surfaces a *sign* according to the standard choice of orientation on the transversal intersection submanifold of an ordered set of submanifolds in an oriented ambient space. Define the function $\mu_{ijk}(L)$ as the sum of signs over all intersection points.

Proposition 3.1. *The value $\mu_{ijk}(L)$ is well defined, i.e., it is independent of the choice of Seifert surfaces. It changes sign under each change of orientation, as well as under the change of parity of the permutation ijk.*

Remark. Proposition 3.1 and the equivalence of the definition of μ_{ijk} to the algebraic definitions from [14] were established in [8] (see also [4]).

Consider, for example, the two Borromean links from Fig. 1.

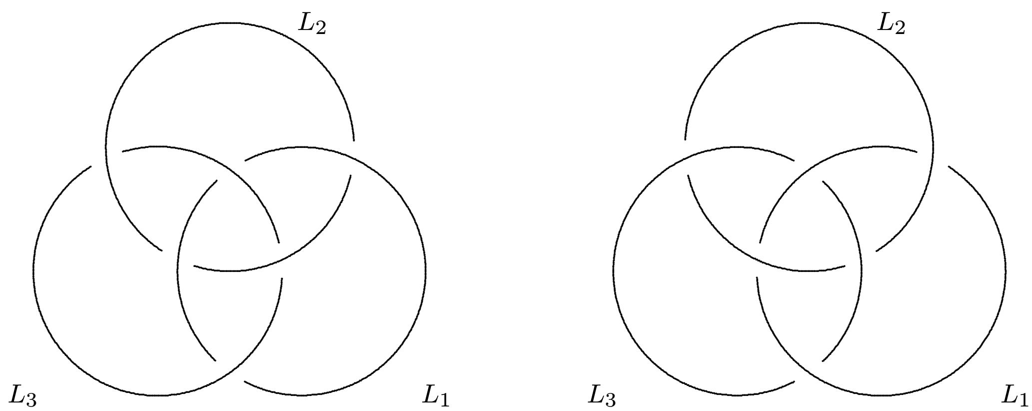

Fig. 1. Borromean links

Let us compute the invariant $\mu_{123}(L)$ for the link on the left-hand side with positive orientation of the projection of each component and the chosen order L_1, L_2, L_3. Following Gauss, the preimages of the intersection points of a link projection are studied in their order along the projection axis (the left-hand side of Fig. 2).

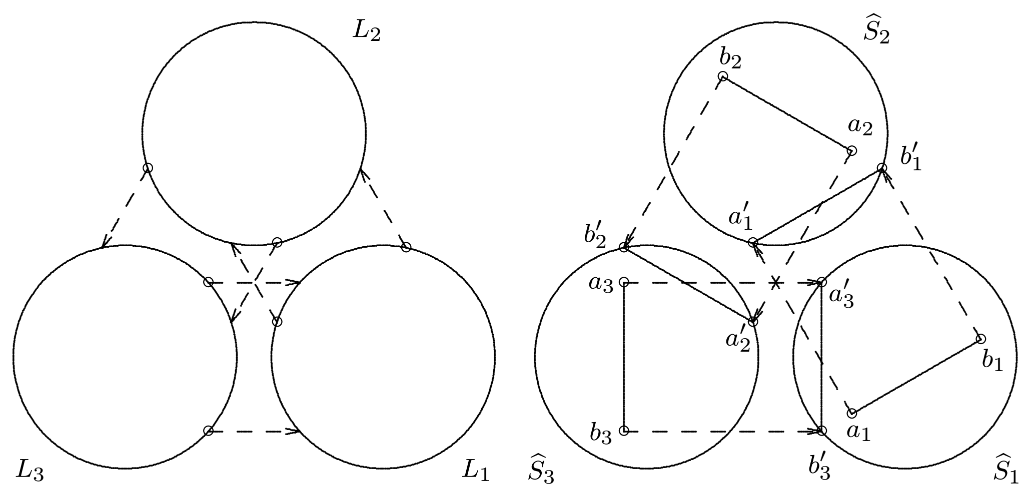

Fig. 2. Gauss diagrams

Each arrow connects the two preimages of the same double point; the end of the arrow marks the lower preimage. Similarly, consider the disks $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_3$, bounded by L_1, L_2 and L_3. These disks intersect pairwise along the segments $[a_ib_i]$, $i = 1, 2, 3$. The arrows connect pairs of segments projected to a single segment.

In order to obtain Seifert surfaces S_1, S_2, and S_3, satisfying the condition $S_i \cap L_j = \varnothing$ for $i \neq j$, let us make a reconstruction of the disks $\widehat{S}_1$, $\widehat{S}_2$, and $\widehat{S}_3$. We cut two small disks centered at a_1 and b_1 from $\widehat{S}_1$ and attach to it a small tube concentric to the arc $a_1'b_1'$ of the circle L_2. Similar surgery is applied to $\widehat{S}_2$ and $\widehat{S}_3$. (The methods of constructing Seifert surfaces in the general case are described in [4].)

Now we can compute the algebraic sum of intersection points $S_1 \cap S_2 \cap S_3$ as the algebraic sum of intersection points of the surface S_3 with the curve $S_1 \cap S_2$, or, what is the same, as the linking number of the oriented curves L_3 and $S_1 \cap S_2$. The closed curve $S_1 \cap S_2$ consists of the segment $[b_1 a_1]$, lying on the disc, and the arc $[a_1 b_1]$, lying on the tube. The linking number is 1, whence $\mu_{123}(L) = 1$. In the same way we obtain $\mu_{123}(L) = -1$ for the Borromean link on the right-hand side of Fig. 1.

In [10] Murakami and Nakanishi introduced a special type of link transformations, called a Δ-*move*. In terms of the link projection, this transformation looks like a fake Reidemeister move; it does not correspond to the projection of a link isotopy. This move is shown in the central and the left parts of Fig. 3.

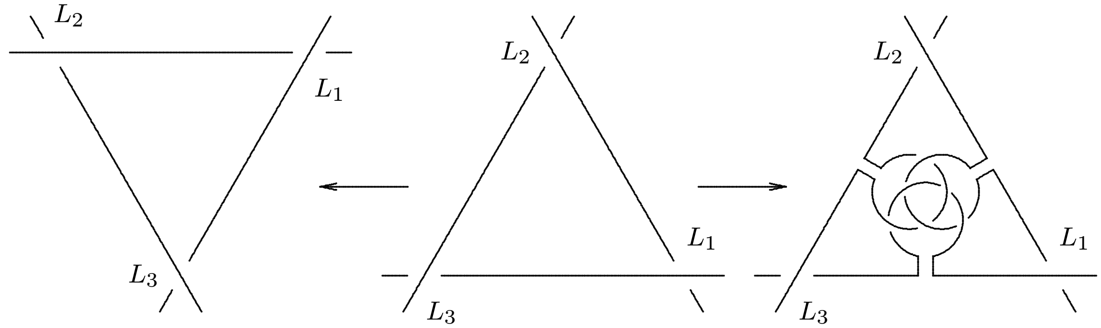

Fig. 3. A Δ-move

The central picture is called the *vanishing triangle*. Define the sign of the vanishing triangle whose sides belong to pairwise distinct components of the link according to the following rule. First, let ρ be the number of the sides of the triangle with orientation opposite to the positive orientation of the triangle on the plane (i.e., the counter clockwise orientation), and set $\varepsilon_1 = (-1)^\rho$. Let ε_2 be the parity of the permutation of the indices 1, 2, and 3, determined by the positive orientation of the triangle (i.e., $+1$ for an even permutation, and -1 for an odd one). Let x be a double point of the link projection, and let ξ and η be the tangent vectors at x, ξ being tangent to the upper strand. The sign $\mathcal{O}(x)$ of the point x is defined as the sign of the frame (ξ, η). The product of the signs of the vertices of a vanishing triangle does not depend on the choice of the components' orientation, and it determines the *type* of the triangle. We denote this product by ε_3. Finally, we call the product $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$ the *sign* of the vanishing triangle.

Since the signs of double points are invariant under Δ-moves, the linking numbers of the components also remain the same. Hence any semibounding link remains semibounding after a Δ-move.

The statement below was proved recently in [11] in the algebraic framework. We present an essentially shorter proof based on the geometry of Seifert surfaces.

Proposition 3.2. *Under a Δ-move, the sign of the vanishing triangle is added to the invariant μ_{ijk}.*

Proof. The right-hand side of Fig. 3 shows that a Δ-move can be realized as the direct sum of a Borromean link inside the vanishing triangle and three consequent Reidemeister moves, each leading to the birth and the death of a pair of close double points with distinct values of $\mathcal{O}$. The vanishing triangle in the center of the Borromean link must coincide with the external triangle. Since the new Seifert surface is obtained simply by attaching the Seifert surface of the Borromean link, the sign of the vanishing triangle in the center of the Borromean link is added to the invariant μ. □

In the final part of this section we study some constructions that are not used in the proof of the main result concerning μ-invariants of order 3 stated in the next section. Nevertheless, these constructions give a nice illustration to the general idea of our approach.

We introduce a new transformation, which we call a *balanced* $\times$*-move*. Under a balanced $\times$-move, the projection of a link remains unchanged, while the value of the function $\mathcal{O}$ at a pair of self-intersection points of a component projection changes. We assume that the values of the function $\mathcal{O}$ at the pair of points under consideration are distinct. Obviously, no balanced $\times$-move changes the linking numbers of the components. In particular, it takes a semibounding link to a semibounding one.

By definition, a $B\times$-move involves only two components that form a pair of *diangles*. The signs of the vertices of each diangle are always opposite.

Let ijk be a permutation of the indices 1, 2, and 3. Suppose that two components, L_i and L_j, participate in a given balanced $\times$-move. Denote by L_i^+ and L_j^+ the arcs of the curves L_i and L_j such that before the move their starting points are at the positive vertices, while their ends are at the negative ones taking the orientation of the components into account. Similarly, the two other arcs are denoted by L_i^- and L_j^-.

Proposition 3.3. *Under a balanced $\times$-move of a semibounding three-component link $L = L_1 \cup L_2 \cup L_3$ the linking number $\operatorname{lk}(L_i^+ \cup L_j^-, L_k)$ is added to the value of the invariant $\mu_{ijk}(L)$.*

Proof. Let S_i and S_j be Seifert surfaces of the curves L_i and L_j, respectively, that satisfy the condition $S_i \cap L_j = S_j \cap L_i = \varnothing$ before a balanced $\times$-move. In order to construct Seifert surfaces $\widehat{S}_i$ and $\widehat{S}_j$ that satisfy a similar requirement after the move, it suffices to cut a pair of disks from each surface S_i and S_j and attach tubes instead of them. It is easy to verify that the intersection curves $\widehat{S}_i \cap \widehat{S}_j$ and $S_i \cap S_j$ differ in a single closed component K such that $\operatorname{lk}(K, L_k) = \operatorname{lk}(L_i^- \cup L_j^+, L_k)$. Therefore, the value $-\operatorname{lk}(L_i^- \cup L_j^+, L_k)$ is added to $\mu_{ijk}(L)$ and, because of the equations $\operatorname{lk}(L_i, L_k) = \operatorname{lk}(L_j, L_k) = 0$, this value coincides with $\operatorname{lk}(L_i^+ \cup L_j^-, L_k)$. $\square$

Using the proposition above, we can suggest the following axiomatic construction of the invariant μ_{ijk} for arbitrary three-component semibounding links. It is easy to see that adding singular elements arising in balanced $\times$-moves to the space of semibounding three-component links, we obtain a connected space. Let us set μ_{ijk} equal to 0 on arbitrary link whose components can be bounded by pairwise nonintersecting balls. The jump of the invariant under a balanced $\times$-move is described by Proposition 3.3. Now the proof of the proposition becomes the proof of the fact that the invariant with required properties is well defined, and the uniqueness of the invariant follows from the construction.

4. FOUR-COMPONENT SEMIBOUNDING LINKS

Let $L = L_1 \cup L_2 \cup L_3 \cup L_4 \subset \mathbb{R}^3$ be an oriented four-component semibounding link with connected nonempty components. For each permutation $ijkl$ of the indices $\{1, 2, 3, 4\}$ the Milnor invariant $\mu_{ijkl}(L) \in \mathbb{Z}$ is uniquely determined (see [1]). Only two of the invariants corresponding to all permutations are independent, say μ_{1234} and μ_{1342}; all other invariants can be expressed in terms of these two.

Now let us define the invariants μ_{ijkl} by means of families of Seifert surfaces. In Proposition 4.2 we state and prove some properties of the constructed invariants and then switch to the study of these invariants using balanced Δ-moves.

Let S_1, S_2, S_3, S_4 be a family of Seifert surfaces for each of the components, and suppose the following geometric equalities hold for arbitrary i, j, l, k:

$$S_i \cap L_j = S_j \cap L_k = S_j \cap S_l = \varnothing.$$

The existence of a family of surfaces with such a property is a consequence of the definition of the semiboundedness of a link. If the Seifert surfaces are in general position, then the curves of their pairwise transversal intersections $\Gamma_{i,j} = S_i \cap S_j$, $\Gamma_{k,l} = S_k \cap S_l$ are embedded and do not intersect.

Definition 4.1. We define the value of the invariant μ_{ijkl} as the difference of linking numbers by the formula

$$\mathrm{lk}(\Gamma_{ik}\,;\Gamma_{jl}) - \mathrm{lk}(\Gamma_{il}\,;\Gamma_{jk}) = \mu_{ijkl}(L). \tag{1}$$

Proposition 4.2 (cf. [4]).

1. *The invariants above are well defined, and their values do not depend on the choice of Seifert surfaces.*
2. *The following identities hold for arbitrary four-component semibounding link L:*

$$\mu_{ijkl}(L) + \mu_{iljk}(L) + \mu_{iklj}(L) = 0, \tag{2}$$

$$\mu_{ijkl}(L) = -\mu_{jikl}(L) = \mu_{ijlk}(L) = -\mu_{ijkl}(L), \tag{3}$$

$$\mu_{ijkl}(L) = \mu_{klij}(L). \tag{4}$$

As was explained in the previous section, there are two kinds of moves taking a semibounding three-component link to semibounding ones: Δ-moves and balanced $\times$-moves. The higher analog of a Δ-move for four-component links is the move described in [13]. Under this move, all four components of the link are present in a neighborhood of the point of surgery, and whence the projection of the link to the plane is not generic in contrast to the Δ-move. Depending on the type of the move, the Milnor invariants are changed by ± 1.

The analog of a balanced $\times$-move is a *balanced Δ-move* or, briefly, a *$B\Delta$-move*, which we define in the following way. A $B\Delta$-move is a pair of simultaneous Δ-moves such that precisely three arcs of the same components participate in each move, and the signs of the vanishing triangles of the two moves are opposite. Therefore, each balanced Δ-move takes a semibounding four-component link to a semibounding link.

Now we are going to define the jump of the invariant μ_{ijkl} under a balanced Δ-move. In order to do this, introduce the following notation. Let L_i^+, L_j^+, and L_k^+ be the arcs of the curves L_i, L_j, and L_k starting at the positive vanishing triangle and ending at the negative one with respect to the given orientations of the components L_i, L_j, and L_k, respectively. Let L_i^-, L_j^-, and L_k^- be the complemental arcs.

Theorem 4.3.

1. *Under a balanced Δ-move of a semibounding four-component link L, in which arcs of the curves L_i, L_j, L_k participate, the difference of the linking numbers*

$$\mathrm{lk}(L_i^+ \cup L_j^-, L_l) - \mathrm{lk}(L_i^+ \cup L_k, L_l) \tag{5}$$

is added to the invariant $\mu_{ijkl}(L)$. Under a balanced Δ-move, in which arcs of the components $L_{i'}$, $L_{j'}$, $L_{k'}$ participate, the jump of the invariant is determined by the same rule, but is additionally multiplied by the sign of the permutation $\binom{i\ j\ k}{i'\ j'\ k'}$.

2. *Under an (unbalanced) Δ-move, in which only arcs of two or one component participate, the value μ_{ijkl} remains unchanged.*

Definition 4.4. We say that two four-component links L and L' are *Δ-homotopy equivalent* if there is a chain of Reidemeister moves and Δ-moves connecting there projections. Further, two four-component links L and L' are *balanced Δ-homotopy equivalent* if there is such a chain of Reidemeister moves, Δ-moves, in which at most two components of the link participate, and balanced Δ-moves.

Theorem 4.5. *Any two semibounding four-component links are balanced Δ-homotopy equivalent.*

Theorem 4.3 allows us to give an axiomatic definition of the invariant μ_{ijkl} by means of Eq. (5), which defines the jump of the invariant under the change of the isotopy class of the link. The value of the invariant on a link whose components can be bounded by pairwise nonintersecting balls is set to zero. The uniqueness of the invariant possessing properties (2)–(4) follows from Theorem 4.5, and its existence follows from Proposition 4.2.

5. PROOFS COMPLETED

In this section we complete the proofs of Propositions 3.1 and 4.2, and of Theorems 4.3 and 4.5.

The proof of Proposition 3.1 is obvious. Details can be found in [8]. □

Proof of Proposition 4.2. A Seifert surface S_i, $i = 1, 2, 3, 4$, for a given component of the link L is determined uniquely up to an embedded cobordism and a disjoint union of embedded nonintersecting tori that enclose the other three components of the link; this means that each of these tori is the boundary of a small tubular neighborhood and it is endowed with some coorientation. When adding one such torus, which encloses, say, the jth component, we add to the self-intersection curve Γ_{ij} the component parallel (taking the orientation into account) to the component L_j. Then the linking number $\operatorname{lk}(\Gamma_{ij}\,;\Gamma_{kl})$ remains unchanged since the link is semibounding, and whence $\mu_{ijk}(L) = \mu_{ijl}(L) = 0$. Therefore, neither of the μ-invariants changes.

Now let us verify that under an embedded cobordism of one of the Seifert surfaces, say, of S_i, expression (1) and all other expressions obtained from it by permuting the indices remain unchanged. If the cobordism contains no points of intersection of all four Seifert surfaces, then the statement is obvious. At a point of quadruple intersection, both linking numbers entering the difference (1) change, and the jump of both of them is either $+1$, or -1 simultaneously. Therefore, the value of μ is independent of the choice of the family of Seifert surfaces. Properties (2)–(4) follow from the definition. The proposition is proved. □

Proof of Theorem 4.3. If less than three components participate in a Δ-move, then all linking numbers are preserved. Consider the case of a balanced Δ-move. To be definite, we assume that the components participating in the move are L_2, L_3, L_4, and we study the behavior of the invariant μ_{1234}.

It was explained in Sec. 3 that in a single Δ-move for a three-component semibounding link with components i, j, k it is convenient to choose the system of Seifert surfaces in such a way that a point of triple intersection arises after the move, or, in other words, three cycles $\gamma_{ij} \subset \Gamma_{ij}$ each linked with the component L_k with coefficient ± 1 depending on the sign of the corresponding vanishing triangle under the Δ-move arise. Similarly, a balanced Δ-move causes the birth of a pair of triple points, and each of the two points belongs to a neighborhood of the corresponding intersection surgery point after the balanced Δ-move.

Note that a neighborhood of the pair of triples of closed cycles on the curves Γ_{23}, Γ_{34}, Γ_{41}, each of which is linked with the corresponding component of the link, contains, generally speaking, components of the curves Γ_{13} and Γ_{14}. Obviously, starting with arbitrary Seifert surfaces (even if neighboring surfaces intersect each other) we can obtain, after a number of tube surgeries as explained in Sec. 3, Seifert surfaces such that the chosen components are situated only in a neighborhood of a single singular point in parallel to the component L_2. The algebraic number of arcs of corresponding components is precisely the linking numbers $\operatorname{lk}(L_2^+ \cup L_3^-, L_1)$ and $\operatorname{lk}(L_2^+ \cup L_4, L_1)$, which, according to (5) determine the jump of the invariant μ_{1234}. The theorem is proved. □

Proof of Theorem 4.5. Below we only give an outline of the proof and postpone a more general construction until further publications. It is shown in Fig. 3 that a Δ-move can be realized by attaching the Borromean link inside the vanishing triangle and using three Reidemeister moves of the type two (we follow the terminology from [15]). A direct verification shows that two operations of attaching two Borromean links mutually commute up to a simultaneous attachment of some pairs of Borromean links having opposite signs to the same triples of components, and a number of Borromean links to one or two components of the link (i.e., each operation of attaching a pair of Borromean links can be transformed into another one under an isotopy and some number of moves described above). This means that replacing Δ-homotopy with balanced Δ-homotopy we can change the sequence of Δ-moves in the initial homotopy in arbitrary way.

It is known that any two four-component links with coinciding matrices of linking numbers of pairs of corresponding components are Δ-homotopy equivalent (see [10]), i.e., there is a sequence of Reidemeister moves and Δ-moves connecting the projections of the links. Since the links are semibounding, by Proposition 3.2, the sum of signs of all Δ-moves in which given three components participate, is zero. Now let us change the order of the Δ-moves in the sequence in such a way that it splits into pairs of consecutive Δ-moves with opposite signs and Δ-moves with less than three participating components. The resulting homotopy is the required one. $\square$

ACKNOWLEDGMENTS

The authors wish to express their gratitude to S. A. Melikhov, V. P. Leksin, and A. I. Rez for useful discussions.

This research was supported by the Russian Foundation for Basic Research under grants no. 99-01-0009, no. 96-01-00211, and no. 00-05-64015, by INTAS-CNES under grant no. 1048, and by the Ministry of Science and Technology of Republic Slovenia.

REFERENCES

1. J. W. Milnor, "Link groups," *Ann. of Math.*, **59** (1954), no. 2, 177–195.
2. R. Porter, "Milnor μ-invariants and Massey products," *Trans. Amer. Math. Soc.*, **257** (1980), 39–71.
3. V. G. Turaev, "Milnor's invariants and Massey's products," *Ucheb. Zapiski LOMI*, **66** (1976), 189–203.
4. T. D. Cochran, "Derivatives of links: Milnor's concordance invariants and Massey's products," *Mem. Amer. Math. Soc.*, **84** (1990), no. 427, 1–73.
5. B. Mellor, *Finite Type Homotopy Invariants, I*, E-print `math.GT/9807162u4`, 1999.
6. P. Kirk and C. Livingston, "Vassiliev invariants of two components links and the Casson–Walker invariant," *Topology*, **36** (1997), 1333–1353.
7. P. M. Akhmet′ev and D. Repovš, "Generalization of the Sato-Levine invariant," *Trudy Mat. Inst. Steklov* [*Proc. Steklov Inst. Math.*], **221** (1998), 69–80.
8. T. D. Cochran, "Concordance invariance of the coefficients of Conway's link polynomial," *Invent. Math.*, **82** (1985), 527–541.
9. P. M. Akhmet'ev and A. Ruzmaikin, "Borromeanism and Bordism," in: *Topological Aspects of the Dynamics of Fluids and Plasma* (H. K. Moffatt et al., editor), Kluwer, Dordrecht, 1992, pp. 249–260.
10. H. Murakami and Y. Nakanishi, "On a certain move generating link-homology," *Math. Ann.*, **284** (1989), 75–89.
11. B. Mellor, *Finite Type Homotopy Invariants, II: Milnor's $\overline{m}$u-Invariants*, E-print `math.GT/9812119`, 1998.
12. T. Stanford, *Braid Commutators and Δ Finite-type Invariants*, E-print `math.GT/9907071`, 1999.
13. K. Habiro, "Claspers and finite type invariants of links," *Geom. Topol.*, **4** (2000), 1–83.
14. J. P. Levine, "An approach to homotopy classification of links," *Trans. Amer. Math. Soc.*, **306** (1988).
15. P. M. Akhmet′ev, J. Malešič, and D. Repovš, "A formula for the generalized Sato-Levine invariant," *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*] (to appear).

(P. M. Akhmet′ev) IZMIRAN, Troitsk
E-mail: akhmetev@izmiran.rssi.ru
(J. Malešič, D. Repovš) Ljubljana University, Slovenia
E-mail: (J. Malešič) joze.malesic@uni-lj.si, (D. Repovš) dusan.repovs@fmf.uni-lj.si