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A formula for the generalized Sato–Levine invariant

P. M. Akhmet'ev, I. Maleshich, and D. Repovš

Abstract. Let W be the generalized Sato–Levine invariant, that is, the unique Vassiliev invariant of order 3 for two-component links that is equal to zero on double torus links of type $(1,k)$. It is proved that

$$W=\beta-\frac{k^3-k}{6},$$

where β is the invariant of order 3 proposed by Viro and Polyak in the form of representations of Gauss diagrams and k is the linking number.

Bibliography: 9 titles.

§ 1. Introduction

In [1] Kirk and Livingston proposed the construction of an integer invariant (which we denote by W) defined on two-component oriented links $f(L_1')\cup f(L_2')\subset\mathbb{R}^3$. Here L_1' and L_2' are disjoint oriented circles, and $f\colon L_1'\cup L_2'\to\mathbb{R}^3$ is an embedding. Independently of the results in [1], the first author [2] defined the invariant W starting out from considerations in hydrodynamics. An elementary approach to the definition of W was proposed in the joint paper [3] by the first and third authors (different notation is used in the paper cited). We prove that W generalizes the well-known Sato–Levine invariant β (see [4]). The invariant β was originally defined only under the assumption that $\operatorname{lk}(f(L_1');f(L_2'))=0$. In the first half of the 1990s Viro and Polyak [5] proposed a generalization of β in the framework of the Vassiliev theory by using the concept of a Gauss diagram. The generalization amounted to defining β not as a partial invariant but as a complete invariant on the isotopy class of an oriented two-component link.

The invariants W and β are Vassiliev invariants of order 3. Moreover, W serves as a higher analogue of the linking number. Just as the linking coefficient of two closed curves is connected with the self-linking coefficient and acts as a measure of the loss of reflective symmetry in a hydrodynamical system, the generalized Sato–Levine invariant serves as a measure of the loss of symmetry under the additional condition of 'knottedness' of the trajectories of the system [2] (see also the formula (1) and

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the computations after Fig. 8). This property of W enables us to look at the invariant in the asymptotic limit for systems with open lines of force of a field, and it also enables us to assume that the analytic expression connected with W (more precisely, with the generalized Milnor invariant of order 3 in the Vassiliev sense with which W is associated) has the form of a multiple integral over the space coordinates. The original construction for the asymptotic linking number (the Hopf invariant) was proposed by Arnol'd in Chapter 28 of [6]. An integral expression of the desired form for the classical Sato–Levine invariant was constructed in the joint paper [7] by the first author and Ruzmaikin.

An analogue of the invariant W with the linking coefficient can be expressed for a framed knot (K, ξ) by the formula

$$W(K, K') = 2v_2(K)\,\mathrm{lk}(K; K'), \tag{1}$$

where K' is obtained from K by a shift along the vectors of the framing ξ, and v_2 is the Vassiliev invariant of order 2 for knots.

Let us recall the method in [2] for the axiomatic construction of the invariant W. Let $\varphi\colon L_1' \cup L_2' \to \mathbb{R}^2$ be the immersion obtained by the projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ of a link $f\colon L_1' \cup L_2' \to \mathbb{R}^3$ in general position onto the plane, that is, $\varphi = \pi \circ f$. Let $L_i = \varphi(L_i')$, $i = 1, 2$, and denote by $\triangle$ the set of crossing points of the components of $L_1 \cap L_2$, including the self-crossing points of L_1 and L_2. We define a function $O\colon \triangle \to \{+1; -1\} \subset \mathbb{Z}$ according to the following rule. At each point $x \in \triangle$ we consider an ordered frame (ξ_1, ξ_2) composed of vectors tangent to the branches of the oriented curve $L_1 \cup L_2$. The order of the vectors in the frame is chosen so that in a neighbourhood of x the branch of $L_1 \cup L_2$ containing the vector ξ_1 is above the branch containing the vector ξ_2 with respect to the direction of the axis of the projection π. We define $O(x) = +1$ if the frame (ξ_1, ξ_2) is positively oriented in the plane, and $O(x) = -1$ otherwise. We shall call the pair (φ, O) the *diagram of the link* f, and the number $O(x)$ the *sign of the point.*

Three types of transformations of diagrams of knots and links are defined in the theory of knots and links, called Reidemeister moves. For transformations of types 1 or 2, one or two crossing points arise in a small neighbourhood of a critical point of the diagram in $\mathbb{R}^2$,

$$\varphi_{t_0}\colon \bigl(\varphi(t_0 - \varepsilon), O(t_0 - \varepsilon)\bigr) \to \bigl(\varphi(t_0 + \varepsilon), O(t_0 + \varepsilon)\bigr),$$

according to the rule pictured in Fig. 1.

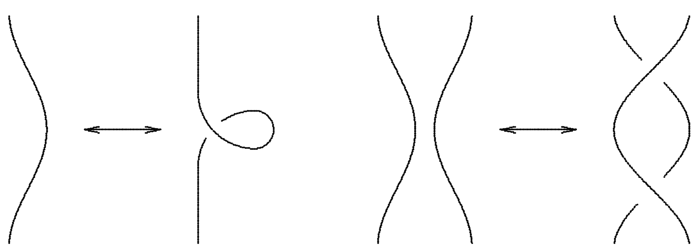


Figure 1. Reidemeister moves of type 1 and 2

For a transformation of type 3, pictured in Fig. 2, the only change is the triple $\{x, y, z\} \subset \triangle$ of points such that there are no other points of $\triangle$ inside or on the boundary of the triangle determined by the vertices x, y, and z. We call such a triangle a *disappearing triangle*. In addition, a disappearing triangle must have a side lying above the other two sides (then one of the other two sides automatically lies below the others). For a transformation of type 3 a new triangle with vertices $x' = x_{t_0+\varepsilon}$, $y' = y_{t_0+\varepsilon}$, and $z' = z_{t_0+\varepsilon}$ appears in a small neighbourhood of a critical point in place of the disappearing triangle, and:

- the signs of the vertices are preserved, that is, $O(x_{t_0-\varepsilon}) = O(x_{t_0+\varepsilon})$, $O(y_{t_0-\varepsilon}) = O(y_{t_0+\varepsilon})$, and $O(z_{t_0-\varepsilon}) = O(z_{t_0+\varepsilon})$;
- each pair of vertices of the triangle which appeared lies on the same arc of the diagram as in the disappearing triangle;
- the order of each pair of vertices is reversed, that is, for example, if the direction of the vector $\overrightarrow{xy}_{t_0-\varepsilon}$ corresponded to the orientation on the diagram, then it will be the opposite for $\overrightarrow{xy}_{t_0+\varepsilon}$.

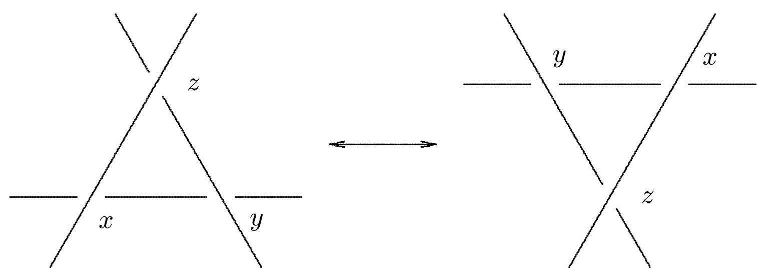


Figure 2. Reidemeister move of type 3

In addition to the Reidemeister moves described above we shall need the so-called false transformation of type 3; see Fig. 3. Instead of the condition that the disappearing triangle have a side lying above the other two sides, we consider the negation of this condition: there is no side lying above the other two sides. Let the rule for transforming a disappearing triangle stay the same as for a Reidemeister move of type 3.

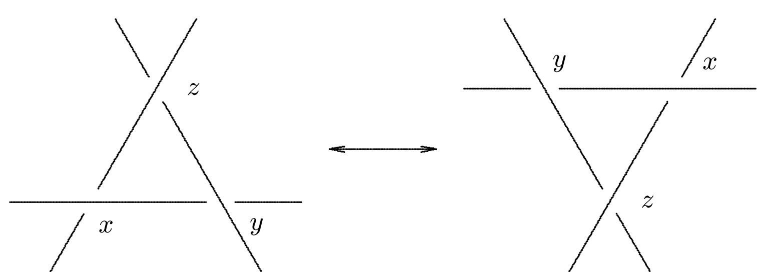


Figure 3. A false Reidemeister move, a δ-move

The above variant of a false Reidemeister move was first defined in [8], where this transformation was called a δ-move. (The authors thank L. Plakhta for this important remark.) Matveev studied δ-moves of a special form independently in [9], where these moves are called Borromeo transformations.

We consider the strata of spaces of projections of links

$$R \subset R_A \subset R_B.$$

Here

- R is the space of regular projections of two-component links;
- R_A is the completion of R by singular projections obtained in the course of Reidemeister moves of types 1, 2, and 3;
- R_B is the completion of the space R_A by singular projections obtained in the course of a δ-move.

The main problem in the theory of knots and links is to compute the groups $\pi_0(R_A)$.

Let x, y, and z be the vertices of a disappearing triangle. Suppose that one of the vertices is a self-crossing point of one component, and the other two are crossings of different components. The self-crossing point will be denoted by x. The link component for which x is a self-crossing point will be denoted by L_x, and the other by L'_x. For a disappearing triangle corresponding to a δ-move there is no side lying above the other sides, and thus there is exactly one vertex of the trangle at which the component L_x passes below the component L'_x. Denote this vertex by y, and the remaining one by z. The point x divides the curve L_x into two loops; denote by L_x^+ the loop containing y, and by L_x^- the other loop. Then $L_x^+ \cup L'_x$ and $L_x^- \cup L'_x$ are diagrams of links, and hence the numbers $\mathrm{lk}(L_x^+; L'_x)$ and $\mathrm{lk}(L_x^-; L'_x)$ exist.

Theorem 1 [2]. *On the completion of the space T_A of diagrams there is a unique function $W\colon T_A \to \mathbb{Z}$ satisfying the following two conditions*:

1. *in a small neighbourhood of a critical point of a δ-move the jump of $W(f)$ is given by the formula*

$$W(f)_{t_0+\varepsilon} - W(f)_{t_0-\varepsilon} = O(x)O(y)\big(\mathrm{lk}(L_x^+; L'_x) - \mathrm{lk}(L_x^-; L'_x) - O(x)\big)\big|_{t_0-\varepsilon}; \qquad (2)$$

2. *if $f(L'_1) \cup f(L'_2)$ is a double torus knot of type $(1,k)$, then $W(f) = 0$.*

Corollary 1. *W is an invariant of order 1 in the space R with distinguished discriminant R_B.*

Remark. We particularly emphasize the fact that the invariants W and β have order 3 in the Vassiliev sense if we consider the usual discriminant connected with double non-degenerate crossing points in the space of singular links.

Remark. In (2) the jump of W is expressed in terms of the signs of the points and the linking numbers at the time $t_0 - \varepsilon$, that is, before the δ-move. The jump of W can be expressed using the values of these numbers considered at the time $t_0 + \varepsilon$, that is, after the δ-move. To this end we note first that the signs of the vertices do not change in the transition $t_0 - \varepsilon \to t_0 + \varepsilon$. By considering the diagram of a link in a neighbourhood of a disappearing triangle it is easy to see that

$$\mathrm{lk}(L_x^+; L'_x)|_{t_0+\varepsilon} = \mathrm{lk}(L_x^-; L'_x)|_{t_0-\varepsilon} + O(y),$$
$$\mathrm{lk}(L_x^-; L'_x)|_{t_0+\varepsilon} = \mathrm{lk}(L_x^+; L'_x)|_{t_0-\varepsilon} + O(y),$$

and hence

$$W(f)_{t_0+\varepsilon} - W(f)_{t_0-\varepsilon} = O(x)O(y)\big(\mathrm{lk}(L_x^-; L'_x) - \mathrm{lk}(L_x^+; L'_x) - O(x)\big)\big|_{t_0+\varepsilon}.$$

This means that (2) is invariant under a change in the value of the deformation parameter in a neighbourhood of a critical value of the δ-move to the opposite.

Viro and Polyak [5] gave an example of an explicit formula for an invariant of order 3 for two-component links, starting out from the representation space of Gauss diagrams. The Viro–Polyak formula is recalled below and the invariant it determines is denoted by β. The following is our main theorem.

Theorem 2. *β is an invariant of order 1 in the space R with respect to the discriminant R_B. This invariant is connected with the invariant W by the formula*

$$\beta - W = \frac{k^3 - k}{6}, \tag{3}$$

where $k = \mathrm{lk}(f(L'_1); f(L'_2))$.

Corollary 2. *The invariants β and W do not change when the enumeration of the components is changed. Under a change of orientation on one of the components the invariants do not change if and only if $k = \pm 1, 0$.*

§ 2. Definition of the Sato–Levine invariant β

In this section we recall the main constructions associated with representations of Gauss diagrams obtained by Polyak and Viro in [5].

Let $\varphi\colon L'_1 \cup L'_2 \to \mathbb{R}^2$ be the immersion obtained as a result of the projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ of a link $f\colon L'_1 \cup L'_2 \to \mathbb{R}^3$ in general position onto the plane, that is, $\varphi = \pi \circ f$. Instead of the projection $\varphi(L'_1 \cup L'_2)$ of a link it is more convenient for combinatorial investigations to use the Gauss diagram, which is constructed as follows: it is formed by disjoint circles $L'_1 \cup L'_2$ together with a finite set of their chords and a finite set of segments having end-points on both circles in such a way that the chords and segments connect precisely those points of $L'_1 \cup L'_2$ that are mapped by the immersion φ into double points of the link projection. The chords and segments are oriented according to the following rule: the initial point of a chord or segment lies above the terminal point. The circles $L'_1 \cup L'_2$ are oriented in the standard way. As an example we give the projection of the Whitehead link and its Gauss diagram (see Fig. 4).

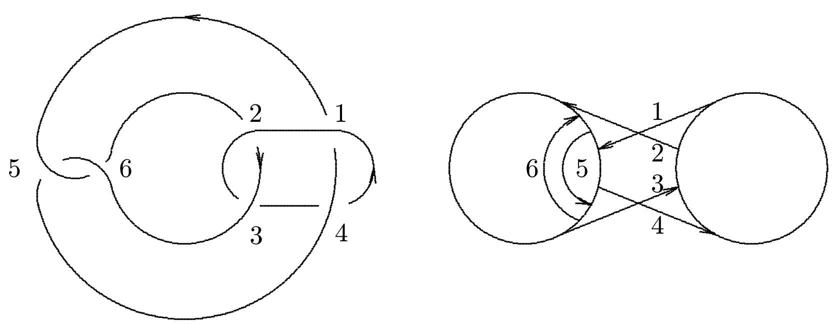


Figure 4. The Whitehead link

Any diagram consisting of two disjoint circles oriented in the standard way together with a finite set of oriented chords of them and a finite set of oriented segments having end-points on both circles is called an *arrow diagram* (hence, Gauss diagrams are particular cases of arrow diagrams). An embedding $f\colon A \hookrightarrow G$ of arrow diagrams A in G is called a *representation of the diagram A in the diagram G* if it preserves all orientations (of circles, arrows, and chords).

We denote by C_A the set of all segments and chords of a given arrow diagram A. If $f\colon A \hookrightarrow G$ is a representation of A in the Gauss diagram G, then $f(x) \in C_G$ for each chord or segment $x \in C_A$, and hence the sign $O(f(x)) \in \{-1, +1\}$ is defined. The *sign* $O(f)$ of the *representation* f is given by the formula

$$O(f) = \prod_{x \in C_A} O(f(x)).$$

We define the *weight* $\langle A, G\rangle$ of the arrow diagram A in the Gauss diagram G to be the sum of the signs of all the representations of A in G. Let $\langle A, G\rangle = 0$ in the case when representations do not exist. As an example we compute the weight of the arrow diagrams , , , and in the diagram of the Whitehead link.

According to Fig. 4 there exist two representations of the arrow diagram in the Gauss diagram of the Whitehead link (see Fig. 5).

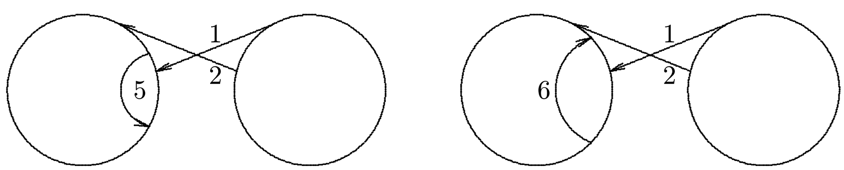


Figure 5. Representations of the diagram

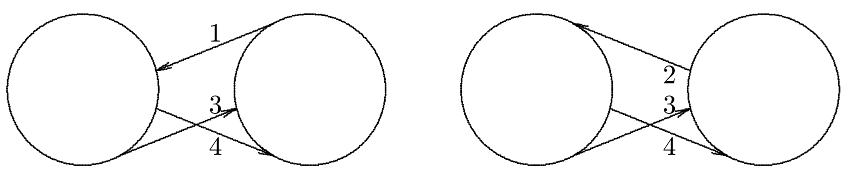
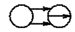


Figure 6. Representations of the diagram

Since $O(1)O(2)O(5) = +1$ and $O(1)O(2)O(6) = +1$, we get that $\langle\ , G\rangle = +2$. Obviously, the representations of the diagrams and in the diagram of the Whitehead link are represented by the empty set, and thus $\langle\ , G\rangle = 0$ and $\langle\ , G\rangle = 0$. It is easy to verify (see Fig. 6) that there are exactly two representations of the arrow diagram . Since $O(1)O(3)O(4) = +1$ and $O(2)O(3)O(4) = -1$, we have $\langle\ , G\rangle = 0$.

Viro and Polyak showed that it is possible to express a number of invariants of knots and links in terms of representations of arrow diagrams in the Gauss diagram of a knot or link. They announced that the following expression is an invariant (of the isotopy class) of a two-component link:

$$\beta(G) = \frac{\langle\ \text{[diagram 1]} + \text{[diagram 2]} + \text{[diagram 3]} - \text{[diagram 4]}, G\rangle}{2}\,. \tag{4}$$

For example, it follows from the preceding computations that $\beta = 2/2 = 1$ for the Whitehead link.

§ 3. Proof of Theorem 2

We make use of the fact that the expression for $\beta(G)$ defined above really is an invariant of the isotopy class of a link; in other words, it does not change under Reidemeister moves of types 1–3. The formula (2) determining the jump of the value of W is invariant with respect to renaming the components and changing the orientation on one of them. To prove Theorem 2 and Corollary 2 with Theorem 1 taken into account it suffices to verify that:

- the jump of the invariant $\beta(G)$ under an arbitrary δ-move is equal to the jump of the invariant W, that is,

$$\beta_{t_0+\varepsilon} - \beta_{t_0-\varepsilon} = O(x)O(y)\big(\mathrm{lk}(L_x^+; L_x') - \mathrm{lk}(L_x^-; L_x') - O(x)\big)\big|_{t_0-\varepsilon}; \tag{5}$$

- if $f(L_1') \cup f(L_2')$ is a double torus knot of type $(1, k)$ and G is its Gauss diagram, then

$$\beta(G) = \frac{k^3 - k}{6}\,; \tag{6}$$

- for the Gauss diagram G_{12} of a double torus knot $f(L_1') \cup f(L_2')$, the diagram G_{21} obtained from G_{12} by renaming the components, and the diagram G_{12}^* obtained from G_{12} by changing the orientation on the first component we have the equalities

$$\beta(G_{12}) = \beta(G_{21}), \tag{7}$$
$$\beta(G_{12}^*) = 0. \tag{8}$$

Let us verify the formula (5), putting off the verification of the identity (6) until the next section. We represent the δ-move taking place on the diagram of the link at the critical time t_0 as the composition $\delta_t = A_2 \circ \alpha(t) \circ A_1$, where A_i $(i = 1, 2)$ is a transformation changing the order of the branches at the point x (at the distinguished vertex of the disappearing triangle) and $\alpha(t)$ is an isotopy, as a result of which the projection of the link transforms in the same way as under a δ-move. We note that the transformations A_1 and A_2 are mutually inverse, but the successive composition of them with the isotopy $\alpha(t)$ no longer simplifies. We introduce some notation:

$$\beta(A_1(L(t_0 - \varepsilon))) - \beta(L(t_0 - \varepsilon)) = \Delta A_1,$$
$$\beta(A_2(L(t_0 + \varepsilon))) - \beta(L(t_0 + \varepsilon)) = \Delta A_2.$$

We claim that

$$\Delta A_1 - \Delta A_2 = O(x)O(y)\big(\mathrm{lk}(L_x^+; L'_x) - \mathrm{lk}(L_x^-; L'_x) - O(x)\big)\big|_{t_0-\varepsilon}. \tag{9}$$

It is easy to see that the values of the third and fourth terms in the formula (4) for $\beta(G)$ do not change under the transformations A_1 and A_2; only the values of the first two terms change. Moreover, if the point x is a self-crossing point of the component L_1 (L_2), then only the first (second) term changes. It is also obvious that the only changes are in the values of the terms composed from representations containing the vertices x and y simultaneously.

The algebraic number of terms occurring in the expression ΔA_1 (ΔA_2) and containing the points x and y is equal to $O(x)O(y)\,\mathrm{lk}(L_x^+; L'_x)$ $(O(x)O(y)\,\mathrm{lk}(L_x^-; L'_x))$. The computations carried out prove the formulae (9) and (5). We conclude the proof of Theorem 2 by verifying the formulae (6)–(8).

§ 4. Examples of computations of the Viro–Polyak invariant β

Double torus knot. Taking a torus knot of type $(1, k)$ with a framing along the radius directed toward the middle line of the torus, we double it. As a result we get the link pictured in Fig. 7.

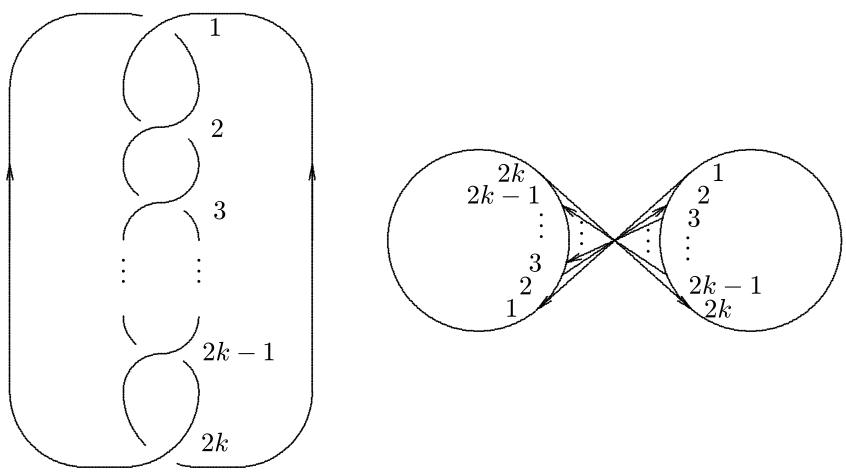
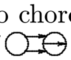
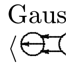
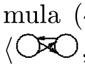
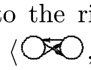
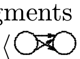


Figure 7. Double torus knot

Since there are no chords in the Gauss diagram, the first two terms in the formula (4), that is, $\langle$$, G\rangle$ and $\langle$$, G\rangle$, are equal to zero. For the term $\langle$$, G\rangle$ all triples of segments work that have the following direction: one segment with direction to the left and two with direction to the right. There are k segments directed to the left, and $\binom{k}{2}$ directed to the right. Since the signs of all the segments are equal to $+1$, it follows that $\langle$$, G\rangle = (k^3 - k^2)/2$. For the term $\langle$$, G\rangle$ any triple of segments with direction to the right works.

Hence, $\langle\ , G\rangle = \binom{k}{3}$. In the end we get that $\beta = (k^3 - k)/6$. This proves the formula (6). The formulae (7) and (8) can be proved by analogous but simpler computations.

Let us go through the computations explaining the formula (1).

The double trefoil. We double the right-hand trefoil (the torus knot $(2,3)$) along a framing parallel to the plane of projection, and we obtain the link pictured in Fig. 8.

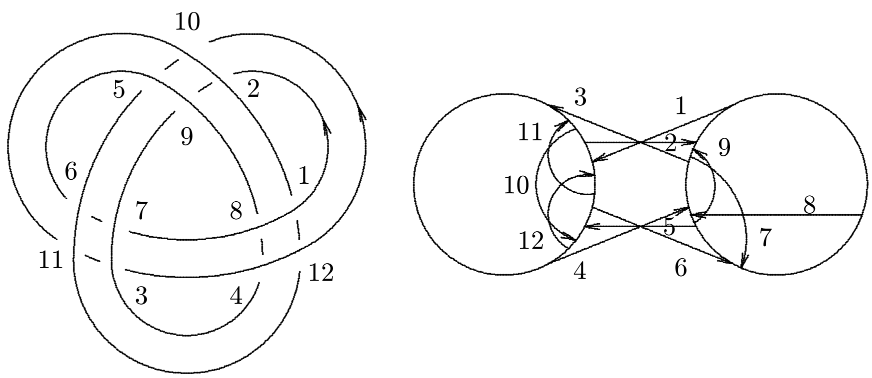


Figure 8. The double trefoil

It is easy to verify that $\langle\ , G\rangle = 6$, $\langle\ , G\rangle = 6$, $\langle\ , G\rangle = 9$, and $\langle\ , G\rangle = 1$. As a result, $\beta = 10$. The same result follows from the formulae (1) and (3) if we take into account that $v_2 = 1$ for the trefoil and that the linking number for the double trefoil is equal to 3.

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Bibliography

[1] P. Kirk and C. Livingston, “Vassiliev invariants of two-component links and the Casson–Walker invariant”, *Topology* **36** (1997), 1333–1353.

[2] P. M. Akhmetiev [Akhmet'ev], “On a higher analog of the linking number of two curves”, *Amer. Math. Soc. Transl.* (2) **185** (1998), 113–127.

[3] P. M. Akhmet'ev and D. Repovš, “Generalization of the Sato–Levine invariant”, *Trudy Mat. Inst. Steklov.* **221** (1998), 69–80; English transl., *Proc. Steklov Inst. Math.* **221** (1998), 60–70.

[4] N. Sato, “Cobordism of semiboundary links”, *Topology Appl.* **18** (1984), 225–234.

[5] M. Polyak and O. Viro, “Gauss diagram formulas for Vassiliev invariants”, *Internat. Math. Res. Notices* **1994**, no. 11, 445–453.

[6] V. I. Arnol'd, *Selecta*–60, Fazis, Moscow 1997. (Russian)

[7] P. M. Akhmetiev [Akhmet'ev] and A. Ruzmaikin, “A fourth-order topological invariant of magnetic or vortex lines”, *J. Geom. Phys.* **15**:2 (1995), 95–101.

[8] H. Murakami and Y. Nakanishi, “On a certain move generating link-homology”, *Math. Ann.* **284** (1989), 75–89.

[9] S. V. Matveev, "Generalized surgeries of three-dimensional manifolds and representations of homology spheres", *Mat. Zametki* **42** (1987), 268–278; English transl., *Math. Notes* **42** (1987), 651–656.

Institute of Earth's Magnetism, Ionosphere, and Radio Wave Propagation,
Russian Academy of Sciences

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