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Some Algebraic Properties of Cerf Diagrams of One-Parameter Function Families*

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Abstract. We obtain results concerning Arnold's problem about a generalization of the Pontryagin–Thom construction in cobordism theory to real algebraic functions. The Pontryagin–Thom construction in the Wells form is transferred to the space of real functions. The relation of the problem with algebraic K-theory and the h-principle due to Eliashberg and Mishachev is revealed.

Key words: wrinkle, Pontryagin–Thom construction, h-principle.

The Cerf diagram of a family of functions f_λ, $\lambda \in [0,1]^k$, on a manifold is the hypersurface with singularities in $[0,1]^k \times \mathbb{R}$ consisting of all possible pairs (λ, x), where x is one of the critical values of f_λ. Akhmet′ev [3] obtained some topological restrictions on the global structure of Cerf diagrams that have proper local singularities and correspond to two-parameter families. In the present paper, we prove a similar result for Cerf diagrams of one-parameter families.

1. Introduction

The starting point of our study is Arnold's paper [1] (see also [2, Problem 1988-23], where a reference to an earlier paper is given), which states the problem of generalizing the Pontryagin–Thom construction [16, 19] to a new class of problems related to function spaces.

Simultaneously with Arnold's work, there were papers in which the Pontryagin–Thom construction was transferred from the cobordism category of manifolds to the cobordism category of maps with singularities. The idea of this generalization was suggested for the first time by Szűcs [18]; later it was developed for maps with more complex singularities in the joint paper [17] of the same author with Rimányi. Recently, the Pontryagin–Thom construction has been applied to problems of approximation of maps with singularities by smooth embeddings [5].

Arnold [1] noticed that the computation of the fundamental group of the space of real-valued functions defined on the line, taking constant values in a neighborhood of infinity, and having no singularities of type A_3 or more complex types is equivalent to the computation of the cobordism group of plane curves without points of horizontal inflection. This group was proved to be isomorphic to the additive group of integers. To solve Problem 1988-23, one should generalize these computations to a function space on a higher-dimensional manifold. A study of this kind was carried out by the first author in [3], where the Pontryagin–Thom construction is used in the Wells form [20]. The considerations involved only the first two homotopy groups (i.e., were carried out for the case in which there are at most two parameters in the problem); furthermore, higher singularities on the function space, except for Morse singularities and singularities of birth-death of a pair of Morse singular points, were prohibited.

Recently the authors became aware that the exclusion theorem in [3], whose proof is rather complicated, can be obtained in a simpler way as a corollary to the Igusa–Laudenbach theorem [11–13] cited below and the h-principle. As applied to pseudoisotopy theory, the h-principle was developed by Eliashberg and Mishachev in the series of papers [7, 8]. A comprehensive survey of

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the main results concerning the h-principle can be found in the book [9]. We intend to reprove (and generalize) the theorem in [3] in the framework of this method elsewhere.

In the present paper, we consider an analog of the exclusion theorem. In the one-dimensional case, there are no restrictions on the Cerf diagram, but one encounters the important notion of a false wrinkle, or a twisted wrinkle, if we use the term “wrinkle” in the sense of [7]. We mean a component of the curve of singular points such that the image of this component in the space of values and parameters is represented by the simplest Cerf diagram (a lens) with distinct incidence coefficients of the separatrix disks at the birth-death points. It is shown below in Fig. 1β (see p. 168). An odd number of such false wrinkles cannot form a Cerf diagram.

The Cerf diagram of a one-parameter family of functions on the manifold $M \times I$ is a curve with cusps that lies with self-intersections in the space of values and parameters. At each cusp of the diagram, the birth or death of a pair of Morse critical points with neighboring indices occurs. Suppose that the stable separatrix disks of the Morse critical points are equipped with a compatible family of orientations. At the cusps of the diagram, there is an additional structure related to the intersection number of the stable and unstable separatrix disks that are born or die in a neighborhood of the birth or death point.

To define this additional structure, consider an arbitrary separatrix joining two Morse points of neighboring indices $i+1$ and i. Such a separatrix is a curve lying in the intersection of the stable separatrix disk of the point of index $i+1$ and the unstable separatrix disk of the point of index i under the assumption that these disks meet each other transversally. Consider an arbitrary orientation of the manifold $M \times I$ in a neighborhood of a Morse point x of index i. This orientation permits one to define an orientation of the unstable separatrix disk at the critical point in question in such a way that the orientation of the manifold in a neighborhood of x is the sum of orientations of two separatrix disks at this point, namely, the stable disk (which bears an orientation by assumption) and the unstable disk.

The choice of the above-mentioned orientations determines the incidence coefficient ± 1 of the separatrix disks. One can readily verify that the coefficient does not change if the orientation of the manifold at x is changed to the opposite. By carrying out this argument for a pair of critical points in a neighborhood of a birth or death point, one can assign a sign ± 1 to the corresponding cusp in the Cerf diagram. The choice of a sign at the cusp depends on the choice of the family of orientations of stable separatrix disks and hence is ambiguous. It is determined modulo a simultaneous change of the coefficients in an arbitrary pair of cusps joined by a common segment of Morse critical values in the diagram. Hence each Cerf diagram can be characterized as nontwisted or twisted depending on the parity (0 or $1 \pmod 2$, respectively) of the number of cusps with the same coefficients. Note that wrinkles correspond to nontwisted diagrams.

It turns out that a Cerf diagram is twisted if and only if it has an odd number of points of transversal self-intersection. This is proved in the theorem in Sec. 3. For example, the diagram in Fig. 1γ is twisted, since it has a single point of self-intersection.

We give two independent proofs of this theorem. One of the proofs uses K-theoretic methods. Our constructions are simple in that we do not consider higher levels of K-theory and deal solely with the K_1-functor, i.e., the determinant. The use of this functor provides a nontrivial algebraic relation satisfied by Cerf diagrams of a function family representing an element of $\pi_1(P(M \times I), E(M \times I))$. This approach is motivated by a visualization of the higher algebraic K-functor, whose properties for integral group rings of the fundamental group have not been comprehensively studied yet. (For possible applications in topology, see [4].)

The outline of the paper is as follows. In Sec. 2, we recall (in minimum generality) the main definitions, which were actually explained above. Next, the main theorem is stated in Sec. 3 and proved from the viewpoint of K-theory in Sec. 4 and by the Eliashberg–Mishachev method in Sec. 5.

2. Main Spaces

Let M be a closed manifold, and let $I = [0, 1]$. The pseudoisotopy space $E(M \times I)$ was studied by Cerf [6]. (We retain the notation adopted in that paper.) It was defined there as the space of functions $f\colon M \times I \to I$, $f(M \times \{0\}) = 0$, $f(M \times \{1\}) = 1$, such that f has no critical points and coincides in some neighborhood (whose size is not fixed) of the boundary with the standard projection onto the second factor. This space has the homotopy type of the space of self-diffeomorphisms of $M \times I$ whose restriction to the lower base is the identity map.

The space of nonzero sections of the tangent bundle $T(M \times I)$ coinciding in some neighborhood of the boundary with a section in the direction of I serves as a formal analog $hE(M \times I)$ of $E(M \times I)$. We assume M to be equipped with a Riemannian metric. Then we can consider the map $h_E\colon E(M \times I) \to hE(M \times I)$ that takes each function f to its gradient field. If M is simply connected, then $E(M \times I)$ is connected by the Cerf theorem. For an arbitrary M, the map h_E is homotopic to the map into a point. This result is referred to in [7] as the Igusa–Laudenbach theorem.

To study the homotopy type of $E(M \times I)$, one defines the space $P(M \times I)$ of functions with generalized Morse singularities, i.e., Morse singularities and singularities of the type of birth or death of a pair of Morse singular points. We define this space as well as its formal analog $hP(M \times I)$ in the next section. It was proved in [7] that the natural inclusion $E(M \times I) \subset P(M \times I)$ is contractible. However, a close result on the contractibility of this map up to some dimension was originally proved by Igusa [12]. It was proved in [8] that $P(M \times I)$ is weakly homotopically equivalent to its formal analog $hP(M \times I)$. The proof is rather complicated and uses a special technique developed in [7].

Now consider a manifold N of dimension $m + 1$ with boundary (in general, nonempty). By $F(N, \varphi)$ we denote the space of all smooth real-valued functions defined on N and coinciding in a neighborhood of the boundary with a given function $\varphi\colon N \to \mathbb{R}^1$ regular in this neighborhood.

Let $\Re \subset J^k(N)$ be an open subspace in the jet manifold such that $\Re$ is invariant with respect to left changes of coordinates and the complement of $\Re$ is a semialgebraic set. The jet manifold has the bundle structure $\kappa\colon J^k(N) \to N$. We define the space $A(N, \Re, \varphi)$ of functions with $\Re$-moderate singularities as the subspace of $F(N, \varphi)$ consisting of functions whose jets lie in $\Re$ and which coincide with φ in a sufficiently small neighborhood of the boundary ∂N. If $N = M \times I$, $p\colon M \times I \to I$ is the natural projection, and $\Re$ coincides with the entire space $J^k(M \times I)$, then the space $A(M \times I, \Re, p)$ coincides with the space $F(M \times I, p)$.

For each given value of k, the formal analog of $A(N, \Re, \varphi)$ is the space $hA(N, \Re, \varphi)$ of sections of κ that range in $\Re$ and have a k-jet coinciding with the k-jet of a given regular map in a sufficiently small neighborhood of the boundary. For $N = \mathbb{S}^m \times I$, the formal analog hA can be defined as the subspace of maps with given boundary conditions in the space $\mathrm{Map}[\mathbb{S}^m \times I; \Re]$ of maps ranging in $\Re$.

There is a map $A(N, \Re, \varphi) \to hA(N, \Re, \varphi)$ that takes each smooth function to its k-jet extension.

The original problem of computing the homotopy type of A can be solved with the use of methods of algebraic topology by computing the homotopy type of hA. This reduction is known as the h-principle. In many problems, the formal analog hA proves to be weakly homotopically equivalent to the original function space A; see Vasil′ev's papers [14, 15]. It follows from the Eliashberg–Mishachev theorem that the h-principle holds for the space of functions that have only A_1 and A_2 singularities.

Definition of the space $P(M \times I)$ of generalized Morse functions. We say that a function $f\colon M \times I \to I$, $f \in F(M \times I, p)$, is a generalized Morse function if for some $i \in \{0, \dots, m\}$ the singularities of f are given by the following formulas in some local coordinate system:

$$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_{m+1}^2, \qquad (A_1)$$

$$-x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_m^2 + x_{m+1}^3. \qquad (A_2)$$

Points of the type A_1 are called Morse singular points of index i. Points of the type A_2 are called birth-death points of a pair of Morse critical points of indices i and $i+1$.

The space $P(M\times I)$ of functions with generalized Morse singularities is defined as the subspace of $F(M\times I,p)$ formed by functions that have only A_1 and A_2 singularities. According to the preceding constructions, we have the formal analog $hA(M\times I,\Re,p)$, where $\Re$ is the subspace of the jet space formed by jets without A_3 and more complex singularities and p is the standard projection onto the second factor. For brevity, we denote $hA(M\times I,\Re,p)$ by $hP(M\times I)$.

The space $hol\,P(M\times I)$. In the space $hP(M\times I)$ of formal functions, we define the subspace $hol\,P(M\times I)$ of formal functions $hf\colon M\times I_x\to J^3(M\times I,\mathbb{R})$ satisfying the following additional conditions: there exists an open subset $U\subset M\times I_x$ such that $hf|_{M\times I\setminus U}$ has no critical points and $hf|_U$ coincides with the jet extension of some function $f\colon M\times I_x\to I_y$ in $P(M\times I)$. In other words, the formal function $hf\colon M\times I_x\to I_y$ should be holonomic in a neighborhood of its critical points. This space is included in the diagram

$$P\overset{i_{hol}}{\subset} hol\,P\overset{i_h}{\subset} hP. \tag{1}$$

The space $E(M\times I)$ and its formal analog $hE(M\times I)$ for a stably parallelizable manifold M. If M is a Riemannian manifold and $k=1$, then $hE(M\times I)$ can be naturally identified with the space of nonzero sections of the tangent bundle $T(M\times I)$ with the standard conditions on the boundary. For a stably parallelizable M, the bundle $T(M\times I)$ is trivial, and $hE(M\times I)$ can be described as the space of maps $M\times I\to\mathbb{S}^m$ that take the boundary of $M\times\{0\}\cup M\times\{1\}$ to a marked point on $\mathbb{S}^m$. By the Igusa–Laudenbach theorem, the forgetful map $h_E\colon E(M\times I)\to hE(M\times I)$ is contractible. (This map is not one-to-one on connected components if M is not simply connected, since in this case $hE(M\times I)$ is connected but $E(M\times I)$ is not.) The connected components of $hE(M\times I)$ are described in [10]. Although the spaces $E(M\times I)$ and $hE(M\times I)$ proved to be homotopically different, Eliashberg and Mishachev [7] managed to construct a function space whose formal analog is $hE(M\times I)$. This is the space of functions that have only Morse and generalized Morse critical points with some additional structure determining a coordinate system in some disk containing a pair of critical points with neighboring indices or a birth-death point. The Cerf diagrams of functions belonging to this subspace have the simplest form and are referred to as *wrinkles*. The standard wrinkle for a one-parameter family of functions is shown in Fig. 1α.

The spaces constructed above are included in the commutative diagram

$$\begin{array}{ccccc} P(M\times I) & & \supset & & E(M\times I)\\ {\scriptstyle h_P}\downarrow & & & & \downarrow{\scriptstyle h_E}\\ h(P(M\times I)) & \supset & hol\,P(M\times I) & \supset & hE(M\times I). \end{array} \tag{2}$$

3. Statement of the Main Result

Recall that a representative of an element of the group $\pi_1(P)$, or, which is the same, of the group $\pi_0(\Omega P)$, can be studied with the use of a Cerf diagram; see [6, 7]. Consider the Cerf diagrams α, β, γ, and δ depicted in Fig. 1.

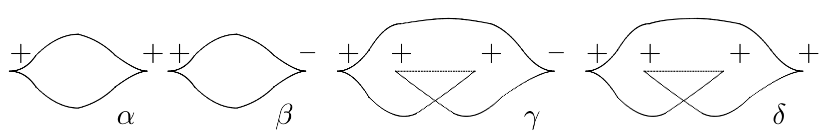

Fig. 1

One can readily construct examples of families of functions f_λ and g_λ, $\lambda\in[0,1]$, with Cerf diagrams α (wrinkle) and γ, respectively, equipped with a family of orientations $O(\lambda)$ (e.g., see [6]). Each of the families represented in this figure has only critical points with two neighboring indices,

$i+1$ and i. In what follows, the critical point of these families will be denoted by the letter x with a subscript.

Let us choose some family of orientations $O(\lambda)$ on the set of stable separatrix disks of Morse critical points for each of the families α–δ. Here we assume that the orientation continuously depends on the parameter of the family.

Consider the families $O(\lambda)$ in a neighborhood of the birth-death points. Every critical point x of this type (say, every singularity of the birth-death type in Fig. 1α) is equipped with a sign $o(x) = \pm 1$ in accordance with the sign of the incidence coefficient for the pair of Morse critical points that are born or die at this singular point. The family α obviously satisfies the relation

$$o(x_1)o(x_2) = +1, \tag{3α}$$

whereas for the family γ one has

$$o(x_1)o(x_2)o(x_3)o(x_4) = -1. \tag{3γ}$$

The diagrams β (a false wrinkle) and δ can also be viewed from the formal viewpoint as the Cerf diagrams for families of functions lying in $hol\, P(M \times I)$. For the diagram β, we have

$$o(x_1)o(x_2) = -1, \tag{3β}$$

while

$$o(x_1)o(x_2)o(x_3)o(x_4) = +1 \tag{3δ}$$

for the diagram δ.

Cerf diagrams can also be defined for the formal families $hol\,\alpha$ and $hol\,\beta$, which model the corresponding families in the group $\pi_1(hol\,P)$. If we somehow choose families of orientations $hO(\lambda)$ on the stable separatrix disks of the critical points of the formal families, then for each birth-death critical point hx the sign $o(hx) = \pm 1$ is also defined. Since the families of functions in $hol\,P$ are holonomic in a neighborhood of their critical points, it is meaningful to speak of Cerf diagrams for such families.

Consider an element ϕ of the homotopy group $\pi_1(P, E)$ and a Cerf diagram representing this element. Let us choose an additional structure $O(\lambda)$ of orientations of separatrix manifolds. Neglecting the additional structure O, the Cerf diagram is represented by the curve S of critical values in the two-dimensional space of values and parameters. The curve S has singularities at the points of the set Σ of birth-death critical values. By smoothing S along Σ, we obtain an immersed submanifold, which we again denote by S. By the Pontryagin–Thom construction in the Wells form [16, 19, 20], the manifold S represents an element $\kappa(\phi)$ in the stable homotopy group $\pi_{n+1}(\mathbb{S}^n)$, $n > 2$, of spheres. (This group will be denoted by Π_1.) The cobordism class of this element is completely determined by the parity of the number of points of self-intersection on the immersed curve S in the square I^2 of values and parameters.

The additional structure O described at the beginning of the present section permits one to define a canonical partition $\Sigma = \Sigma_+ \cup \Sigma_-$ of the zero-dimensional set Σ of birth-death critical values into two subsets (zero-dimensional manifolds) according to the value of the incidence coefficient of a pair of Morse singularities in a small neighborhood of the component of critical points. The cobordism class of either of the manifolds Σ_+ and Σ_- (which are cobordant as immersed manifolds) determines an element $\rho(\phi)$ in the stable homotopy group $\Pi_0 = \pi_n(\mathbb{S}^n)$, $n \geqslant 1$, of spheres. This element in general depends on the choice of the structure O on the manifold of separatrix disks. The ambiguity in the choice of O does not affect $\rho(\phi)$ treated as an element of the quotient group $\Pi_0/2\Pi_0$. Indeed, each segment on S consisting of Morse singularities is bounded by a pair of singularities in Σ. The change of orientation on the manifold of stable separatrix disks along this segment changes the types of singularities in Σ at its endpoints. Consequently, the element $\rho(\phi)$ (mod 4) is independent of the choice of O. On the other hand, singularities in Σ annihilate in pairs under a cobordism. For a coordinated choice of the structure O on the curve of singular points before (after) the surgery, the signs in the corresponding pair of cusps of S in a neighborhood of the point where a pair of birth-death critical points disappears (appears) are opposite and the structure O can be extended to the cobordism. Hence the element in the quotient group $\Pi_0/2\Pi_0$ determines an invariant on the cobordism class of the Cerf diagram.

Let t be the generator of Π_1 represented by the homotopy class of the suspension over the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$; in terms of the Pontryagin–Thom construction, this generator is represented by the cobordism class of a plane immersed horizontal figure eight curve.

Theorem. *One has*

$$t \circ \rho(\phi) = \kappa(\phi), \tag{4}$$

where the left-hand side is the composition of stable spheroids.

Example. For a path in the function space with Cerf diagram shown in Fig. 1α or 1δ, one has $\kappa = t$. For the families in Fig. 1β or 1γ, the element ρ is of odd order in the group Π_1.

4. Proof of the Theorem via the K_1-Functor

In this section, we assume that $\dim(M) = m \geqslant 5$. Using the parametric version of the Smale lemma (see [10]), one can readily show that the homotopy class $\phi \in \pi_1(P(M \times I), E(M \times I))$ contains a path f_λ, $\lambda \in [0, 1]$, such that the following conditions hold:

1) the family of functions f_λ has only critical points of indices $i + 1$ and i, where $i = [m/2]$;

2) the critical values of index $i + 1$ (respectively, i) lie in the interval $(1/2, 1)$ (respectively, $(0, 1/2)$).

If we replace f_λ by a homotopic path, then the Cerf diagram is replaced by a cobordant diagram. Obviously, both parts of (4) are preserved in this process, since A_3 singularities (swallowtails) do not arise under the cobordism. Let us prove formula (4) assuming that the element ϕ is represented by a family of functions f_λ satisfying conditions 1) and 2). For an arbitrary value $\lambda \in [0, 1]$ except for finitely many singular values, there is a well-defined incidence coefficient matrix $A(\lambda)$ (we assume that the structure $O(\lambda)$ has been chosen) of critical points in the Morse complex of $M \times I$ generated by the function f_λ. The critical points of index $i + 1$ (respectively, i) are numbered starting from 1 in descending (respectively, ascending) order of critical values. Let us adopt the convention that the matrix $A(\lambda)$ is stabilized by the $(0, 0)$th entry $+1$; this will be useful if there are no critical points. Consider the singular values of the parameter λ, which are classified by the following list of singularities.

(a) Singularities of the type of birth-death of a pair of Morse critical points with neighboring indices. Note that by an additional transformation of the stable separatrix disk for the point of index $i + 1$ and the unstable separatrix disk for the point of index i in a neighborhood of the birth or death point one can ensure that the set of singular separatrix trajectories lying on these disks contains a unique trajectory that joins these two critical points with incidence coefficient ± 1. (In the definition of this coefficient, we assume that the structure O has been chosen.) Then the sign of this trajectory, i.e., the incidence coefficient of the pair of singular points, permits one to define the partition of Σ into Σ_+ and Σ_-.

(b) Singularities of the type of addition-subtraction of a handle in the cell decomposition generated by the Morse function.

(c) Singularity of the type of coincidence of two critical values. (According to condition 2, these values have the same index.)

Now consider the singularities corresponding to points of discontinuity of the function $\det(A(\lambda))$, which takes the values ± 1. A discontinuity of this function can be caused either by the coincidence of a pair of critical values (which results in the multiplication of the matrix A on the left or on the right by an elementary transposition matrix depending on the index of these critical values) or by the presence of a birth-death point in Σ_-, which results in the splitting of the one-dimensional scalar matrix -1 from the right bottom diagonal entry. The number of points of discontinuity of $\det(A(\lambda))$ is even, since f_0 and f_1 have no critical points and $A(0) = A(1) = +1$. Hence the parity of the number x of points of self-intersection on the Cerf diagram coincides with the parity of the number y of singularities in Σ_-. Now note that the value of the homomorphism κ constructed from this diagram determines a generator in Π_1 if and only if $x = 1 \pmod 2$. Next, the value $y \pmod 2$ determines an element of $\Pi_0/2\Pi_0$. Thus $x = y \pmod 2$, or, equivalently, $\kappa = t \circ \rho$. The proof of the theorem is complete.

5. Proof of the Theorem with the Use of the *h*-Principle

Throughout the proof, we assume that M is stably parallelized and simply connected. Consider the sequence

$$\pi_1(P(M \times I), E(M \times I)) \xrightarrow{(h_P, h_E)_\sharp} \pi_1(hol\, P(M \times I), E(M \times I)) \xrightarrow{hol\,\kappa \oplus hol\,\rho} \Pi_1 \oplus \Pi_1. \qquad (5)$$

We define the homomorphism $hol\,\kappa \oplus hol\,\rho$ of the group $\pi_1(hol\, P(M \times I), E(M \times I))$ by analogy with the homomorphism $\kappa \oplus \rho$ so as to ensure that $(hol\,\kappa \oplus hol\,\rho) \circ h_P = \kappa \oplus \rho$.

Let us compute the range of $\kappa \oplus \rho$ in the group $\Pi_1 \oplus \Pi_0/2\Pi_0$. Let hf_λ, $\lambda \in [0, 1]$, be the family of formal functions to be studied, holonomic in a small regular neighborhood of its critical points as well as in a neighborhood of the boundary $\partial(M \times I) \cup (M \times \{0\} \cup M \times \{1\})$. In particular, the functions hf_0 and hf_1 are holonomic and have no critical points.

Let us make a surgery of the Cerf diagram of the family hf_λ into one of the simplest diagrams shown in Fig. 1. Note that all *a priori* possible cobordism classes of diagrams are represented in this figure. Moreover, without loss of generality one can extend a cobordism of the diagram to a homotopy of some holonomic function family in a regular neighborhood of the curve of singular points. Next, note that the family of formal functions can also be deformed with the preservation of holonomy conditions in a neighborhood of the curve of singular points. Indeed, a cobordism of the diagram preserving all birth-death points can be extended to a homotopy of the formal family by a standard argument.

Let τ be the homotopy parameter. Consider a disk $D_0^{n+2} \subset M \times I \times I$ centered at a singular point where a pair of birth-death points disappears. Further, we assume that the disk $D_{\tau_0-\varepsilon} = D_0^{n+2} \cap M \times I \times \{\tau_0 - \varepsilon\}$ entirely contains a regular neighborhood of two branches S_1 and S_2 of critical points of the families $hf_\lambda(\tau_0 - \varepsilon)$ mirror symmetric in λ as well as singularities in Σ, one on either branch. The disk $D_{\tau_0+\varepsilon} = D_0^{n+2} \cap M \times I \times \{\tau_0 + \varepsilon\}$ contains a regular neighborhood of the modified branch S of the curve of singular points of the family $hf_\lambda(\tau_0 + \varepsilon)$ and does not contain singularities in Σ.

One can readily verify that a deformation given in a neighborhood of the curve of singular points can be extended to a deformation of the entire formal family for the parameter τ ranging in $[\tau_0 - \varepsilon, \tau_0 + \varepsilon]$. Indeed, for $\tau = \tau_0 - \varepsilon$ let us join the birth and death points lying on the branches S_1 and S_2 by a small segment J. By an additional deformation of the formal family hf_λ with support in D_0 on the interval $\tau_0 - \varepsilon \leqslant \tau \leqslant \tau_0 - \varepsilon/2$, we can ensure that the gradient field of the formal family $hf_\lambda(\tau_0 - \varepsilon/2)$ coincides with the gradient field of the standard holonomic family inside D_0.

After this construction, the continuation of the deformation of the formal family for $\tau_0 - \varepsilon/2 \leqslant \tau \leqslant \tau_0 + \varepsilon$ in D_0 is obvious, and outside D_0 it is constructed in a standard manner.

Thus it suffices to show that the diagrams β and γ in Fig. 1 cannot be realized for any formal family hf_λ. Since the map $h_E \colon E \to hE$ is contractible, it follows that the functions hf_0 and hf_1 viewed as formal functions are homotopic to a constant. Let us study the possibility of realizing the Cerf diagram in the form of the false wrinkle β in Fig. 1. First, we describe a function family in a neighborhood of a false wrinkle in closed from.

Let f_λ be a family of functions with proper boundary conditions whose Cerf diagram is the standard wrinkle α shown in Fig. 1. Let $I_0 \subset I_\lambda$ be an interval in the parameter space of germ deformations, and let $a \in M^m \times I(x)$. We take a deformation of the family f_λ satisfying the following technical condition: in some given ball neighborhood $U_r(a) \subset M^m \times I$ of radius r of the point a, the function f_λ is independent of $\lambda \in I_0$ and is given by the standard quadratic form q of index $i + 1$. To simplify the subsequent notation, we set $r = 2$ and $I_0 = [1/3, 2/3]$.

We redefine the values of the family of germs f_λ in the balls $U_1(a) = U_1$ for these values of the parameter λ; outside the ball, the family f_λ remains unchanged. By doing so, we obtain a family g_λ defined in the interior of $U_1 \times [1/3, 2/3]$ whose restriction to $U_1 \times \{1/3, 2/3\}$ coincides with f_λ.

Let us make auxiliary constructions. By Δ we denote the two-dimensional plane passing through the center a of the ball U_2 and determined by the coordinates x_1 and x_{i+2}. Note that the restrictions of the quadratic parts of the functions f_λ to the plane Δ are not sign definite. Without loss of generality, we can assume that all these restrictions coincide with each other and with q. Consider a one-parameter family of rotations $Z_\lambda\colon U_1 \to U_1$, $\lambda \in I_0$, in the positive direction in the plane Δ such that $Z_{1/3}$ is the identity transformation and $Z_{2/3}$ is the rotation by an angle of π. We define $g_\lambda(b)$, $b \in U_1$, by the formula $g_\lambda(b) = q(Z_\lambda(b))$. The pairs $(g_{2/3}(x), f_{2/3}(x))$ and $(g_{1/3}(x), f_{1/3}(x))$ of germs coincide for each point $x \in U_1$. Hence the family of functions g_λ is defined in a neighborhood of the false wrinkle.

By $V_1 \subset V_2 \subset M \times I \times I$ we denote a pair of neighborhoods of the false wrinkle thus constructed in the product of the source space by the parameter space; these neighborhoods are assumed to be an extension of the pair of neighborhoods $U_1 \times I_0 \subset U_2 \times I_0$ of the arc of the false wrinkle over the interval $a \times I_0$. Let us compute the obstruction to the extension into the solid torus $V_2 \setminus V_1$ of the formal family coinciding with the holonomic family f_λ on the external boundary ∂V_2 and with g_λ on the internal boundary ∂V_1.

The family g_λ is given on ∂V_1 (∂V_2) by a nonzero section of the bundle $T(M \times I)$ and is determined by a map $G\colon \partial U_1 \times I_0 \to \mathbb{S}^m$ ($G\colon \partial U_2 \times I_0 \to \mathbb{S}^m$). Likewise, the family f_λ is determined on ∂V_1 (∂V_2) by a map $F\colon \partial U_1 \times I_0 \to \mathbb{S}^m$ ($F\colon \partial U_2 \times I_0 \to \mathbb{S}^m$).

One can readily see that on ∂V_1 and ∂V_2 the map G is obtained from F by a composition with the family of rotations of the sphere $\mathbb{S}^m$ around the circle $\mathbb{S}^1 \subset \mathbb{S}^m$ by angles from 0 to 2π parametrized by the coordinate on I_0. Note that F and G coincide on the lateral part $(U_1 \setminus U_2) \times \{\partial I_0\}$ of the annulus $K = U_1 \times I_0 \setminus U_2 \times I_0$. The obstruction $o(F, G)$ to the continuation of the map into the interior of K lies in the cohomology group $H^{m+1}(\partial K; \pi_{m+1}(\mathbb{S}^m))$. The value of the obstruction is determined by the value of the generator $\psi \in \pi_1(SO(m))$ under the James–Whitehead homomorphism $J\colon \pi_1(SO(m)) \to \pi_{m+1}(\mathbb{S}^m) = \Pi_1$. It is known that $[J(\psi)] = t$, where $t \in \Pi_1$ is a generator. Hence our problem of continuation of the map into the interior of K has no solution. The problem of continuation of the map into the interior of the solid torus $V_2 \setminus V_1$ is also unsolvable for similar reasons.

Now let us prove that the problem of continuation of the map G from ∂V_1 to a family hg_λ of formal functions on the entire space $M \times I \times I$ with the standard conditions on the boundary of the product of the source space by the parameter space is also unsolvable. To this end, we study the ambiguity in the continuation of a family hg_λ of formal functions on $\partial(M \times I \times I)$ to a family of formal functions on ∂V_2. The diagram S is contained in the ball ∂D_0, and the restriction of hg_λ to ∂D_0 is uniquely, modulo a homotopy, specified by a constant map. Indeed, the manifold $M^m \times I \times I \setminus D_0$ is a stably parallelized manifold realizing a cobordism between ∂D_0 and the external boundary $M^m \times \partial(I \times I)$. The family hg_λ of formal functions on this manifold has no critical points and is represented by a map $M^m \times I \times I \setminus D_0 \to \mathbb{S}^m$. Since $D_0 \setminus V_2$ is also a stably parallelized $(m+2)$-dimensional manifold, we see that the obstruction to the continuation of the map $\partial(D_0 \setminus V_1) \to \mathbb{S}^m$ from the boundary to the entire $D_0 \setminus V_1$ lies in $\pi_{m+1}(\mathbb{S}^m)$. On the one hand, the restrictions $hg_\lambda\colon \partial D_0 \to \mathbb{S}^m$ and $hf_\lambda\colon \partial D_0 \to \mathbb{S}^m$ of the formal families are homotopic. (They are homotopic to a constant map.) Here by hf_λ we denote the formal family determined by the holonomic family f_λ. On the other hand, we have proved that $G\colon \partial V_1 \to \mathbb{S}^m$ differs from $F\colon \partial V_1 \to \mathbb{S}^m$ by a generator of $\pi_{m+1}(\mathbb{S}^m)$. Since the problem of continuation from $\partial(D_0 \setminus V_1)$ to $D_0 \setminus V_1$ is solvable for F, it follows that for G the problem has no solutions. Thus we have proved that a false wrinkle cannot be realized by a Cerf diagram for any family of formal functions. The case of the diagram δ can be analyzed in a similar way. We have computed the range of the homomorphism $h\kappa \oplus h\rho$ and proved the theorem.

6. Discussion

We have studied one-parameter families of functions with generalized Morse singularities and proved that the invariants κ and ρ constructed from the Cerf diagrams of the corresponding families in the stable homotopy groups of spheres are dependent even for families of formal functions. From the algebraic viewpoint, the relation between the values of κ and ρ is related to the functor $K_1(\mathbb{Z})$, i.e., the determinant, and from the viewpoint of homotopy the relation is expressed via the generator of the group Π_1.

For two-parameter families of functions, a similar algebraic relation is connected to the functor $K_2(\mathbb{Z})$. The first author [3] conjectured that some Cerf diagrams of families of functions possess admissible local structure but are excluded for families of holonomic functions (if treated as Cerf diagrams of families of formal functions). It was assumed in the conjecture that a family of functions without critical values on the boundary of the parameter space is formal. We would like to restate the conjecture in connection with the latest results in [8]. It would be meaningful to consider the following problem.

Problem. In the framework of the h-principle due to Eliashberg and Mishachev, prove the result in [3] that the Cerf diagrams of two-parameter families of functions are not realizable. Informally speaking, show that the h-principle in pseudoisotopy theory "remembers" K-theory on the level of spaces of formal functions with moderate singularities.

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