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ON BANACH-MAZUR COMPACTA

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Abstract

We study Banach-Mazur compacta $Q(n)$, that is, the sets of all isometry classes of n-dimensional Banach spaces topologized by the Banach-Mazur metric. Our main result is that $Q(2)$ is homeomorphic to the compactification of a Hilbert cube manifold by a point, for we prove that $Q_{\mathcal{E}}(2) = Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold. As a corollary it follows that $Q(2)$ is not homogeneous.

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1. Introduction

This paper studies topological properties of Banach-Mazur compacta $Q(n)$, that is, the sets of all isometry classes of n-dimensional Banach spaces topologized by the Banach-Mazur metric. Recently, substantial progress was made concerning these spaces. It was proved in [14] that $Q(2)$ is an absolute extensor (defined below). Later this result was generalized to all $n \geq 2$ (see [5]). The long-standing problem about topological equivalence of $Q(n)$ and the Hilbert cube I^∞ was finally solved negatively for $n = 2$ in [4].

THEOREM 1.1. *$Q(2)$ and I^∞ are not homeomorphic.*

For any space X to be homeomorphic to the Hilbert cube I^∞, the following necessary conditions must be satisfied for every point $x \in X$:

(a) $X \setminus \{x\}$ must be homotopically trivial; and
(b) $X \setminus \{x\}$ must be a Hilbert cube manifold.

The key idea of the proof of Theorem 1.1 was to show that $Q(2)$ fails to possess the property (a) at the Euclidean point {Eucl.}, which corresponds to the isometry class of the Euclidean space. On the other hand, the main result of this paper, Theorem 1.2 stated below, implies that the complement $Q(2) \setminus \{x\}$ of every other point $x \in Q(2)$ turns out to be homotopically trivial. Furthermore, Theorem 1.2 demonstrates that as far as the property (b) is concerned, everything turns out to be exactly the opposite: $Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold, while the complement $Q(2) \setminus \{x\}$ of every other point $x \in Q(2)$ is not.

THEOREM 1.2. *$Q_{\mathcal{E}}(2) = Q(2) \setminus \{\text{Eucl.}\}$ is a Hilbert cube manifold.*

As a corollary we prove that $Q(2)$ is not homogeneous (recall that a space X is said to be *homogeneous* if for every pair of points $x_1, x_2 \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x_1) = x_2$).

COROLLARY 1.3. *$Q(2)$ is not a homogeneous space.*

PROOF OF COROLLARY 1.3. By [4], {Eucl.} is not a Z-set in $Q(2)$. On the other hand, it follows by our Theorem 1.2 above that for every point $x \in Q(2) \setminus \{\text{Eucl.}\}$, $\{x\}$ is a Z-set in $Q(2) \setminus \{\text{Eucl.}\}$, hence also a Z-set in $Q(2)$. Therefore $(Q(2), \{\text{Eucl.}\}) \ncong (Q(2), \{x\})$. □

2. Preliminaries

We identify the set $\text{BAN}(n)$ of all n-dimensional Banach spaces with the set of all norms in $\mathbb{R}^n$. The *Banach-Mazur* distance $\rho(X, Y)$ between spaces $X = \{\mathbb{R}^n, \|\cdot\|_X\}$ and $Y = \{\mathbb{R}^n, \|\cdot\|_Y\} \in \text{BAN}(n)$ is defined as follows:

$$\rho(X, Y) = \inf\left\{\|T\| \cdot \|T^{-1}\| \mid T : X \to Y \text{ is an isomorphism}\right\},$$

where $\|T\|$, $\|T^{-1}\|$ are norms of the operators T and T^{-1}, respectively. It is well-known that for every $X, Y, Z \in \text{BAN}(n)$, the following properties hold:

(1) $\rho(X, Z) \le \rho(X, Y)\rho(Y, Z)$;
(2) $\rho(X, Y) = \rho(Y, X) \ge 1$; and
(3) $\rho(X, Y) = 1$ if and only if X and Y are *isometric,* $X \cong Y$, that is, there exists an isomorphism $T : X \to Y$ which preserves the norm $\|x\|_X = \|T(x)\|_Y$ for every $x \in X$.

It follows that the function $\ln \rho(X, Y)$ is a pseudometric on the space $\text{BAN}(n)$, which in the decomposition space $Q(n) = \text{BAN}(n)/\cong$ becomes the metric $d([X], [Y]) = \ln \rho(X, Y)$, where

$$X \cong Y \iff \rho(X, Y) = 1 \iff \ln \rho(X, Y) = 0.$$

The resulting metric space $(Q(n), d)$ of all isometry classes of n-dimensional Banach spaces is called the *Banach-Mazur compactum.*

This compactum allows for a different, more suitable presentation as a decomposition of the space $C(n)$ of all compact convex symmetric (rel 0) bodies in $\mathbb{R}^n$. If one measures the distance between subsets of $\mathbb{R}^n$ by the Hausdorff metric $\rho_H(A, B)$ and defines the linear combination $\sum_{i=0}^{n} \lambda_i A_i$ by means of the Minkowski operation, then $(C(n), \rho_H)$ becomes a locally compact convex space.

Moreover, $C(n)$ can be equipped with an action of the general linear group $\mathrm{GL}(n) \times C(n) \to C(n)$, given by $T \cdot V = T(V)$, where $T : \mathbb{R}^n \to \mathbb{R}^n \in \mathrm{GL}(n)$ and $V \in C(n)$, which agrees with the convex structure on $C(n)$. We show that the orbit space $C(n)/\mathrm{GL}(n)$ is naturally homeomorphic to the Banach-Mazur compactum.

Indeed, for an arbitrary body $V \in C(n)$, the Minkowski functional $p_V(x) = \inf\{t^{-1} \mid tx \in V\}$ defines a norm on $\mathbb{R}^n$ and consequently, induces a continuous bijection $M : C(n) \to \mathrm{BAN}(n)$ defined by $M(V) = (\mathbb{R}^n, p_V)$. Since it is well-known that Banach spaces $M(V)$ and $M(W)$ are isomorphic if and only if $V = T \cdot W$ for some $T \in \mathrm{GL}(n)$, it follows that M induces a continuous bijection of the quotient spaces

$$\tilde{M} : C(n)/\mathrm{GL}(n) \to Q(n) = \mathrm{BAN}(n)/\cong,$$

which is a homeomorphism.

Hereafter, we shall consider only locally compact Lie groups (for example $\mathrm{GL}(n)$), metric spaces and continuous maps, unless otherwise specified. An *action* of G on a space X is a homeomorphism $T : G \to \mathrm{Aut}\, X$ of the group G into the group $\mathrm{Aut}\, X$ of all autohomeomorphisms of X such that the map $G \times X \to X$, given by $(g, x) \mapsto T(g)(x) = gx$, is continuous. A space X with a fixed action of G is called a *G-space.*

For any point $x \in X$, the *isotropy subgroup* of x, or the *stabilizer* of x, is defined as $G_x = \{g \in G \mid gx = x\}$, and the *orbit* of x as $G(x) = \{gx \mid g \in G\}$. The space of all orbits is denoted by X/G and the natural map $\pi : X \to X/G$, given by $\pi(x) = G(x)$, is called the *orbit projection.* The orbit space X/G is equipped with the quotient topology, induced by π.

Actions of noncompact groups G do not agree very well with the orbit structure of X: the orbit of a point x can be dense in X, the orbit space X/G can be non-Hausdorff, two orbits with the same stabilizer can be nonhomeomorphic, *etc.* Palais [22] singled out a class of G-spaces with the action of a locally compact group which do not have such deficiencies—he called such spaces *proper.*

DEFINITION 2.1. (a) Given subsets $A, B \subset X$ consider the following subset of the group G:

$$((A, B)) = \{g \in G \mid gA \cap B \neq \emptyset\}.$$

Then A is said to be *thin* with respect to B, if $((A, B))$ is precompact, that is, it lies in a compact subset of G. Since $((A, B)) = ((B, A))^{-1}$, it follows that B is also thin with respect to A.

(b) $A \subset X$ is said to be *small* if for every point $x \in X$, there exists a neighbourhood $O(x) \subset X$ of x, which is thin with respect to A.

(c) A G-space X is said to be *proper* if it possesses a basis, consisting of small neighbourhoods.

In general, the orbit projection $\pi : X \to X/G$ of a proper G-space X fails to be a closed map. This forces us to seek those closed subsets $F \subset X$ of X for which the restriction $\pi|_F : F \to X/G$ is closed.

DEFINITION 2.2. A closed subset $F \subset Z$ of a G-space Z is said to be *fundamental* if F is small in Z and intersects every orbit, that is, $F \cap G(z) \neq \emptyset$ for every $z \in Z$.

PROPOSITION 2.3. *Suppose that a G-space Z is proper and that the orbit space Z/G is metrizable. Then*

(d) *there exists a fundamental subset $F \subset Z$; and*

(e) *for every fundamental subset $F \subset Z$, the restriction $\pi|_F : F \to Z/G$ is a proper map.*

DEFINITION 2.4. An *exact slice* at the point $x \in X$ is a G-map $\varphi : U \to G(x)$ of some G-neighbourhood $U \subset X$ (that is, $G \cdot U = U$) of the orbit $G(x)$, such that $\varphi(x) = x$. The preimage $\varphi^{-1}(x)$ of the point x is also called a *slice* or a *G_x-kernel.*

The principal results concerning slices belong to Abels [1] and Palais [22].

THEOREM 2.5 (Palais). *A proper completely regular G-space X has a slice at every point x.*

THEOREM 2.6 (Abels). *Let X be a proper G-space with a paracompact orbit space and K a maximal compact subgroup of G. Then there exists a G-map $f : X \to G/K$ (a so-called* global K-slice). *Conversely, if there exists a global K-slice, then X is a proper G-space.*

In the sequel, we shall work in the class $\mathcal{G}$ of all metric proper G-spaces, whose orbit space is also metric. The following properties of the class $\mathcal{G}$ are well known (see [22]).

PROPOSITION 2.7. *Let $X \in \mathcal{G}$ and let Y be a separable metric G-space. Then the following properties hold*:

(f) *The orbit $G(x)$ is closed in X, the stabilizer G_x is compact and the natural map $G/G_x \to G(x)$, given by $g \cdot G_x \mapsto gx$, is a homeomorphism.*
(g) *X can be equipped with an invariant metric, that is, $d(gx, gx') = d(x, x')$, for every $g \in G$ and $X \times Y \in \mathcal{G}$.*
(h) *$G/L \in \mathcal{G}$, for every compact subgroup $L < G$.*

Next, we introduce several notions connected with the property of absolute extendability of maps. A space X is called an *absolute neighbourhood extensor*, $X \in ANE$, if every map $\varphi : A \to X$, defined on a closed subset $A \subset Z$ of a metric space Z, and called a *partial map*, can be extended over some neighbourhood $U \subset Z$ of A, $\hat{\varphi} : U \to X$, $\hat{\varphi}|_A = \varphi$. If we can always take $U = Z$, then X is called an *absolute extensor*, $X \in AE$. We note that in the case when X is a metric space, the concepts of the absolute (neighbourhood) retract and the absolute (neighbourhood) extensor coincide.

If $X \in A[N]E$, Z is a G-space from the class $\mathcal{G}$ and φ is a G-map (which in this case means that φ is constant along every orbit), then the extension $\hat{\varphi}$ can also be chosen to be a G-map. This follows from the closedness of A/G in Z/G (which, in turn follows by the closedness of A in Z). In connection with this example we introduce some more general concepts.

DEFINITION 2.8. A G-space X is called an *equivariant absolute neighbourhood extensor*, $X \in G\text{-}ANE$, if every partial G-map $Z \hookleftarrow A \xrightarrow{\varphi} X$, where Z is a G-space from the class $\mathcal{G}$, can be extended to a G-map $\hat{\varphi} : U \to X$, defined on some G-neighbourhood $U \subset Z$ of A. If we can always take $U = Z$ then X is called an *equivariant absolute extensor*, $X \in G\text{-}AE$.

DEFINITION 2.9. A G-space X is called an *approximate G-$A[N]E$-space*, $X \in G\text{-}AA[N]E$, if for every G-space Z from the class $\mathcal{G}$, every fundamental subset F of Z, and every covering $\omega \in \operatorname{cov}(X)$, the following holds: For every partial G-map $Z \hookleftarrow A \xrightarrow{\varphi} X$ there is an 'approximate' G-extension $\tilde{\varphi} : Z \to X$ [respectively $\tilde{\varphi} : U \to X$, where $U \subset Z$ is a G-neighbourhood of A] such that the restrictions $\varphi|_{A\cap F}$ and $\tilde{\varphi}|_{A\cap F}$ are ω-close, that is, $(\varphi|_{A\cap\varphi}, \tilde{\varphi}|_{A\cap F}) \prec \omega$ (see [13]).

3. Equivariant extensors for locally compact Lie groups

For our purposes, the most important example of a proper $\mathrm{GL}(n)$-space is the space $C(n)$ of all convex bodies.

PROPOSITION 3.1. For every n, $C(n)$ is a proper $\mathrm{GL}(n)$-space.

PROOF. It suffices to prove that the following closed set

$$C(r, R) = \{V \in C(n) \mid B^n(r) \subset V \subset B^n(R)\},$$

where $B^n(a)$ denotes the closed ball with center at 0 and of radius a, is thin for every $0 < r < R < \infty$, that is, that the set

$$\mathscr{R} = ((C(r, R), C(r, R)) = \{g \in \mathrm{GL}(n) \mid gC(r, R) \cap C(r, R) \neq \emptyset\}$$

is precompact.

Suppose not. Then for some sequence $g_n = \|g_{ij}^n\| \in \mathscr{R}$ and some indices (i_0, j_0), one of following cases occur

(α) $g_{i_0 j_0}^n \to \infty$; or
(β) $\det \|g_{ij}^n\| \to 0$.

Suppose that $g_n V_n \in C(r, R)$ for some $V_n \in C(r, R)$. Since the point A, for which only the j_0-th coordinate is equal to r, while all others are 0, lies in V_n, it follows that $g_n A \in B^n(R)$. But the i_0-th coordinate of $g_n A$ is equal to $g_{i_0 j_0}^n r$ and $g_{i_0 j_0}^n$ does not converge to ∞. On the other hand,

$$0 < \mathrm{vol}\, B^n(r) \leq \mathrm{vol}(g_n V_n)$$

and

$$\mathrm{vol}(g_n V_n) = \det \|g_{ij}^n\| \,\mathrm{vol}\, V_n \leq \det \|g_{ij}^n\| \,\mathrm{vol}\, B^n(R).$$

Therefore, $\det \|g_{ij}^n\|$ does not converge to 0. □

The orthogonal group $O(n)$ is a maximal compact subgroup of $\mathrm{GL}(n)$. By Theorem 2.6 there exists a global $O(n)$-slice $f : C(n) \to \mathrm{GL}(n)/O(n)$.

PROPOSITION 3.2. *Let X be a proper G-ANE-space. Then*

(γ) *For every G-neighbourhood U of the orbit $G(x)$, there exist a G-neighbourhood V and a G-map $H : V \times [0, 1] \to U$ such that $H_0 = \mathrm{Id}$, $\mathrm{Im}(H_1) \subset G(x)$, and $H_t|_{G(x)} = \mathrm{Id}$ for all $t \in I$.*

PROOF. Consider in the proper G-space $X \times [0, 1]$ the partial G-map

$$X \times [0, 1] \hookleftarrow X \times \{0\} \cup G(x) \times [0, 1] \cup U_1 \times \{1\} \xrightarrow{\varphi} X$$

such that $\varphi|_{X \times \{0\}} = \mathrm{Id}$, $\varphi|_{G(x) \times [0,1]} = \mathrm{Id}$, and $\varphi|_{U_1 \times \{1\}}$ is the existing retraction (provided by Theorem 2.6) $r : U_1 \to G(x)$ of some G-neighbourhood $U_1 \subset U$.

Let $\tilde{\varphi} : W \to X$ be any extension of φ onto the G-neighbourhood W, which contains a G-neighbourhood of the type $V \times I \supset G(x)$. We get the desired map H by restricting $\hat{\varphi}$ onto $V \times I$. □

The following theorem of Abels [2, 4.4] allows us to reduce the studying of non-compact group actions to compact ones.

THEOREM 3.3. *For every $X \in \mathcal{G}$, $X \in G$-$A[N]E$ if and only if $X \in L$-$A[N]E$ for every compact subgroup $L < G$.*

THEOREM 3.4. *For every n, $C(n)$ is a* $\mathrm{GL}(n)$-*AE space.*

By Theorem 3.3, $C(n) \in \mathrm{GL}(n)$-*AE* if and only if $C(n) \in L$-*AE*, for every compact subgroup $L < \mathrm{GL}(n)$. Another theorem of Abels [2, 4.2] asserts that every locally convex complete topological vector G-space is G-*AE*, for every compact group G. Let us apply the argument from this paper to prove that $C(n) \in L$-*AE*.

Since $C(n)$ is convex (with respect to the Minkowski linear combination of convex bodies), Dugunji's theorem implies that $C(n) \in AE$. Therefore every partial L-map $Z \hookleftarrow A \xrightarrow{f} C(n)$ can be continuously extended over Z, $F : Z \to C(n)$. Now define

$$\hat{F}(z) = \int_L g^{-1} \cdot F(gz) \partial\mu,$$

where $\partial\mu$ is the normalized Haar measure and $\int_L$ means the integral of the set-valued mapping [9]:

$$\Phi_z : L \rightsquigarrow \mathbb{R}^n, \Phi_z(g) = g^{-1} \cdot F(gz) \subset \mathbb{R}^n.$$

On account of the continuous dependence $\Phi_z(g)$ on z and g, the convexity and the closeness of its images, $\hat{F}$ is a continuous map with closed convex values [9]. It is easy to see that $\hat{F}$ is an L-map from Z into $C(n)$ and that $\hat{F} \mid A = f$. □

Let (X, d) be a metric G-space of diameter 1 from $\mathcal{G}$. Then we can introduce a metric on the cone $\operatorname{Con} X = X \times [0, 1]/X \times \{0\}$ as follows:

$$\rho((x, t), (x', t')) = \sqrt{t^2 + (t')^2 - 2tt' \cos\gamma}, \quad \text{where } \cos\gamma = \frac{2 - d^2(x, x')}{2}.$$

It is easy to see that $(\operatorname{Con} X, \rho)$ is a metric G-space (the group G acts along X) and the natural embedding $X \mapsto X \times \{1\} \hookrightarrow \operatorname{Con} X$ is an isometry, while $\operatorname{Con} X$ is not a proper space.

PROPOSITION 3.5. *If a metric G-space X is a G-ANE space, then* $\operatorname{Con} X$ *is a G-AE space.*

PROOF. Suppose that a proper G-space $Z \in \mathcal{G}$ and a partial G-map $Z \hookleftarrow A \xrightarrow{\varphi} \operatorname{Con} X$ are given. Let $A_0 = \varphi^{-1}(*) \subset A$, where $(*)$ is the vertex of $\operatorname{Con} X$. Then for every $a \in A \setminus A_0$, $\varphi(a)$ can be represented in the form $(\varphi_1(a), \varphi_2(a))$, where $\varphi_1 : A \setminus A_0 \to X$ is a continuous G-map and $\varphi_2 : A \to [0,1]$ is a continuous function, constant on the orbits and such that $\varphi_2(A \setminus A_0) \subset (0,1]$ and $\varphi_2(A_0) = 0$.

Since $X \in G\text{-}ANE$, the map $\varphi_1(a)$ can be extended to a G-map $\psi : U \to X$, defined on an open subset U of Z/G, $Z \setminus A_0 \supset U \supset A \setminus A_0$. Since the orbit space Z/G is metrizable, there exists a continuous function $\xi : Z \to [0,1]$, constant on orbits, such that $\xi|_A = \varphi_2$ and $\xi|_{Z\setminus U} = 0$ by the Urysohn theorem. The desired extension $\hat{\varphi} : Z \to \operatorname{Con} X$ of the G-map φ is then defined by the formula:

$$\hat{\varphi} = \begin{cases} (\psi(z), \xi(z)) & z \in U; \\ (*) & z \notin U. \end{cases}$$

□

PROPOSITION 3.6. *Let H be a compact subgroup of the locally compact Lie group G. Then G/H is a G-ANE-space.*

PROOF. Every compact subgroup $H < G$ smoothly acts on the differentiable manifold G/H. By [21, 1.6.6], $G \in H\text{-}ANE$. By Theorem 3.3, $G \in G\text{-}ANE$. □

It is convenient to reduce the studying of the equivariant extensors to the corresponding easier problem for approximate equivariant extensors. For example, if some class $\mathscr{B}$ of G-spaces is invariant under the product on the semiopen segment $J = [0,1)$, then $\mathscr{B}$ is contained in the class G-$A[N]E$ if and only if $\mathscr{B}$ is contained in the class of the approximate G-$A[N]E$.

THEOREM 3.7. *Suppose that the product $X \times J$ of a metric G-space X and $J = [0,1)$ is a G-AANE-space. Then X is a G-ANE-space.*

For the trivial group G this is a well-known fact, which follows from [12] and [18].

PROOF OF THEOREM 3.7. First, we consider any (not necessarily locally finite) covering $\omega \in \operatorname{cov}(X \times J)$ *adjoining* to the subset $X \times \{1\}$ of $X \times [0,1]$. The latter means by definition that:

(δ) For every neighbourhood $U(x,1)$ of the point $(x,1) \in X \times \{1\}$ in $X \times [0,1]$, there exists a smaller neighbourhood $V(x,1)$ such that $W \subset U(x,1)$, for every $W \in \omega$ such that $W \cap V(x,1) \neq \emptyset$.

Let F be a fundamental set of Z (see Proposition 2.3). Then $F \times J$ is a fundamental set of $Z \times J$. After these preliminaries, we begin the extending of the partial G-map $Z \hookleftarrow A \xrightarrow{\varphi} X$. Recall that $X \times J \in G\text{-}AA[N]E$ and construct for the other partial G-map

$$Z \times J \hookleftarrow A \times J \xrightarrow{\psi = \varphi \times \mathrm{Id}_J} X \times J$$

a G-map $\tilde{\psi} : Z \times J \to X \times J$ [respectively $\tilde{\psi} : U \to X \times J$] such that

$$\left(\varphi|_{(A\cap F)\times J}, \tilde{\psi}|_{(A\cap F\times J)}\right) \prec \omega.$$

We give all details of the proof only for the case when $X \times J \in G$-AAE. The case when $X \times J \in G$-$AANE$ is dealt with similarly. Extending $\tilde{\psi}$ over $A \times \{1\}$ by the formula $\tilde{\psi}(a, 1) = (\varphi(a), 1)$, we obtain a G-map (which we denote by the same letter) $\tilde{\psi} : Z \times J \cup A \times [0, 1] \to X \times J$, the restrictions of which onto the closed G-set $A \times [0, 1]$ and the open G-set $Z \times J$ are continuous. Now we apply the following lemma.

LEMMA 3.8. *Suppose that a G-map $f : H \cup E \to Y$ is defined on the union $H \cup E$ of a closed G-space $H \in \mathcal{G}$ and open G-subset E of a proper G-space $T \in \mathcal{G}$, such that $f|_H$ and $f|_E$ are continuous. Then there exists a closed G-subspace $K \subset T$ such that $H \subset K \subset H \cup E$, $H \cap U \subset \operatorname{Int}(K)$ and $f|_K$ is a continuous G-map.*

Apply Lemma 3.8 for $T = Z \times [0, 1]$, $H = A \times [0, 1]$, $E = Z \times J$ and $f = \tilde{\psi}$. We get a closed G-subset L of $Z \times [0, 1]$ such that $A \times [0, 1] \subset L$, $A \times [0, 1) \subset \operatorname{Int} L$ and $\tilde{\psi}|_L$ is a continuous G-map.

Next, we construct a decreasing sequence $L = U_1 \supset \operatorname{Cl} U_2 \supset \cdots$ of open G-neighbourhoods of the set A and a monotone sequence of numbers $0 = t_1 < t_2 < \cdots$, such that $\lim_{i\to\infty} t_i = 1$ and $U_k \times [0, t_k] \subset L$.

Let $\xi : Z \to [0, 1]$ be a continuous real-valued function, constant on the orbits and such that $\xi(U_1 \setminus U_2) = 0$, $\xi(U_i \setminus U_{i+1}) \subset [t_{i-1}, t_i]$ for every $i \geq 2$, and $\xi(A) = 1$. Clearly, the graph $\mathrm{GR} = \{(z, \xi(z)) \mid z \in Z\}$ of f lies in L and the restriction of $\tilde{\psi}$ onto GR is a continuous G-map. The desired extension is now given by the formula:

$$\hat{\varphi}(z) \text{ is the projection of } \tilde{\psi}(z, \xi(z)) \in X \times [0, 1] \text{ onto } X.$$

Using $\operatorname{diam} W_n \to 0$, $W_n \in \omega$, whenever $\operatorname{dist}((x, 1), W_n) \to 0$ for some point $(x, 1) \in X \times \{1\}$, it is easy to check the continuity of $\hat{\varphi}$. □

4. Orbit spaces of equivariant absolute extensors

This section is dedicated to a proof of the following result.

THEOREM 4.1. *Let G be a locally compact Lie group and X a proper G-$A[N]E$ from $\mathcal{G}$. Then the orbit space X/G is an absolute [neighbourhood] extensor.*

Since $C(n)$ is a proper $\mathrm{GL}(n)$-space from $\mathcal{G}$ which is an equivariant absolute extensor, we obtain as an immediate corollary of Theorem 4.1 that for every closed subgroup $H < \mathrm{GL}(n)$, the orbit space $C(n)/H$ belongs to the class of absolute extensors.

We begin with the following embedding theorem.

PROPOSITION 4.2. *Let $X \in \mathcal{G}$. Then there exist a countable number of finite-dimensional G-ANE-spaces R_{nm} $(n, m \in \mathbb{Z}^+)$, from the class $\mathcal{G}$, and a topological G-embedding $i : X \hookrightarrow \prod_{n,m}^{\infty} \mathrm{Con}\, R_{n,m}$.*

Let X be equipped by the invariant metric (see Proposition 2.7 (g)). For every point x and every $\varepsilon > 0$, we fix a G-map $\varphi_{x\varepsilon} : X \to \mathrm{Con}(G(x))$ satisfying the properties of the following proposition.

PROPOSITION 4.3. *Let $X \in \mathcal{G}$. Then for every point $x \in X$ and every $\varepsilon > 0$, there exists a G-map $\varphi : X \to \mathrm{Con}(G(x))$ with $\varphi(x) = x$, such that*

(5) $\mathrm{diam}\, \varphi^{-1}((V \cdot x) \times (0, 1]) < \varepsilon$, *for some neighbourhood V of the stabilizer G_x in G.*

PROOF. Let $r : U(x) \to G(x)$ be a G-retraction. We may assume that not only does the G_x-kernel $r^{-1}(x)$ have diameter less than ε, but also $\mathrm{diam}(V \cdot r^{-1}(x)) < \varepsilon$, for some neighbourhood V of the compact stabilizer G_x. This is possible by Theorem 2.5 and the following lemma.

LEMMA 4.4. *For every neighbourhood $O(x) \subset X$, there exists a smaller neighbourhood $O_1(x)$ such that*

(6) $G_x \cap \mathrm{cl}\{g \mid g\, O_1(x) \setminus O(x) \neq \emptyset\} = \emptyset$; *and*
(7) $G \cdot O_1(x) \cap r^{-1}(x) \subset O(x)$.

The desired G-map of X is then given by the formula:

$$\varphi(x') = \begin{cases} (r(x'), \xi(x')) & x' \in U(x); \\ (*) & x' \notin U(x). \end{cases}$$

Here, the function $\xi : X \to [0, 1]$ is constant on orbits, $\xi(x) = 1$ and $\xi(X \setminus U(x)) = 0$. □

Since by hypothesis X/G is metrizable, there exists a σ-disjoint basis $\mathcal{B} = \{W_\mu\}_{\mu \in M}$ of open subsets, such that $\mathcal{B} = \bigsqcup \mathcal{B}_n$, where $\mathcal{B}_n = \{W_\mu\}_{\mu \in M_n \subset M}$ is a disjoint family and $\coprod_{n=1}^{\infty} M_n = M$.

DEFINITION 4.5. A pair $\nu = (\mu_1, \mu_2) \in M \times M$ of indices is said to be *canonical*, if

(8) $W_{\mu_1} \Subset W_{\mu_2}$ (that is, $\overline{W}_{\mu_1} \subset W_{\mu_2}$); and
there exist $x \in X$ and $\varepsilon > 0$ such that:

(9) $x \in \pi^{-1} W_{\mu_1} \subset V_{x\varepsilon}$ and $U_{x\varepsilon} \subset \pi^{-1} W_{\mu_1}$, where

$$V_{x\varepsilon} = \varphi_{x\varepsilon}^{-1}(G_x \times (1/2, 1]) \quad \text{and} \quad U_{x\varepsilon} = \varphi_{x\varepsilon}^{-1}(G_x \times (0, 1]),$$

and $\pi : X \to X/G$ is the orbit projection.

We denote the set of all canonical pairs by $K \subset M \times M$.

PROPOSITION 4.6. *There exists a correspondence* $\nu \in K \longmapsto (x_\nu, \varepsilon_\nu) \in X \times \mathbb{R}^+$ *such that* (x_ν, ε_ν) *satisfies* (9) *and*

(10) *For every closed subset* $F \subset X$ *and* $x \notin F$ *there exists a canonical pair* $\nu \in K$ *with* $\varphi_{x_\nu \varepsilon_\nu}(x) \notin \varphi_{x_\nu \varepsilon_\nu}(F)$ *(that is,* $\varphi_{x_\nu \varepsilon_\nu}$ *separates the point x from the closed subset F).*

PROOF. Let

$$i(\nu) = \inf\{\varepsilon > 0 \mid (x, \varepsilon) \text{ satisfies (9) for some point } x \in X\}.$$

It is evident that $i(\nu) > 0$. Therefore, every $\nu \in K$ yields a pair (x_ν, ε_ν) possessing (9) and such that

(11) $\varepsilon_\nu < 2i(\nu)$.

Let $4a = \rho(x, F)$. Since $\mathscr{B}$ is a basis, there exist $\nu = (\mu_1, \mu_2) \in K$ and $\varepsilon < a$ such that

$$x \in \pi^{-1} W_{\mu_1} \subset V_{x\varepsilon} \subset U_{x\varepsilon} \subset \pi^{-1} W_{\mu_1}.$$

It follows from (11) that $\varepsilon_\nu < 2a$.

Let us prove that ν is a desired pair. Suppose that a neighbourhood V of G_{x_ν} satisfies the hypotheses of Proposition 4.3:

$$\operatorname{diam} \varphi_{x_\nu \varepsilon_\nu}^{-1}((V x_\nu) \times (0, 1]) < \varepsilon\nu < 2a.$$

Since $x \in V_{x_\nu \varepsilon_\nu}$, it follows that $\varphi_{x_\nu \varepsilon_\nu}(x) = (g x_\nu, t)$, $t > 1/2$.

Pick a neighbourhood $W = g V g^{-1}$ of $e \in G$. Then

$$\varphi_{x_\nu \varepsilon_\nu}(x) \in (W \cdot g x_\mu) \times (1/2, 1]$$

and

$$\begin{aligned} A &= \varphi_{x_\nu \varepsilon_\nu}^{-1}(W \cdot g x_\mu \times (1/2, 1]) \\ &\subset \varphi_{x_\nu \varepsilon_\nu}^{-1}(g \cdot V \cdot x_\mu \times (1/2, 1]) = g \cdot \varphi_{x_\nu \varepsilon_\nu}^{-1}(V \cdot x_\mu \times (1/2, 1]). \end{aligned}$$

By the invariance of the metric, the latter set has diameter smaller than $2a$, hence the diameter of the open neighbourhood A of x is also less than $2a$. As a consequence, it follows that $A \cap F = \emptyset$ and $\varphi_{x_\nu \varepsilon_\nu}(x) \notin \varphi_{x_\nu \varepsilon_\nu}(F)$. □

PROOF OF PROPOSITION 4.2. Let us introduce a countable family of spaces:

$$R_{nm} = \coprod \left\{ G(x_\nu) \mid \nu = (\mu_1, \mu_2) \in K,\ \mu_1 \in \mathscr{B}_n, \mu_2 \in \mathscr{B}_m \right\}.$$

Since $G(x_\nu) \in G\text{-}ANE$, R_{nm} is also a G-ANE. Since $\mathscr{B}_m$ is a disjoint family and

$$\varphi_{x_\nu \varepsilon_\nu}|_{(X\setminus\pi^{-1}W_{\mu_2})} = (*) \in \operatorname{Con}(G(x_\nu)),$$

we obtain that

$$\psi_{nm} : X \to \operatorname{Con} R_{nm}, \qquad \psi_{nm}|_{\pi^{-1}W_{\mu_2}} = \varphi_{x_\nu\varepsilon_\nu}, \qquad \psi_{nm}|_{X\setminus\cup\pi^{-1}W_{\mu_2}} = (*)$$

is a well-defined G-map. Since $\{\psi_{nm}\}$ separates points from closed subsets, the diagonal product

$$\Delta\psi_{nm} : X \to \prod_{n,m} \operatorname{Con} R_{nm}$$

is a topological G-embedding. □

PROPOSITION 4.7. *Suppose that a G-space H is the limit of the inverse spectrum* $\left\{H_1 \overset{q_1}{\leftarrow} H_2 \overset{q_2}{\leftarrow} H_3 \leftarrow \cdots\right\}$ *of G-spaces H_i and G-maps q_i, and that*

(12) *The stabilizer G_h of any point $h \in H_i \setminus H_i^G$ is compact.*

Then the orbit spaces H/G and $\varprojlim\left\{H_1/G \overset{\tilde{q}_1}{\leftarrow} H_2/G \overset{\tilde{q}_2}{\leftarrow} H_3/G \leftarrow \cdots\right\}$ *are homeomorphic.*

PROOF. The homeomorphism $\varphi : H/G \to \varprojlim\{H_i/G, \tilde{q}_i\}$ is given by the formula:

$$\varphi([h]) = ([h_1], [h_2], \dots), \text{ where } h = (h_i) \in H.$$

It is easy to verify that φ is continuous and surjective. We shall thus only verify that φ is injective. Assume that $[h] \neq [e]$, where $h = (h_i)$, $e = (e_i) \in H$ and let us show that then $\varphi([h]) \neq \varphi([e])$. It suffices to prove the following lemma. □

LEMMA 4.8. *There exists an integer i such that $e_i \notin G(h_i)$.*

PROOF. If $e, h \in H^G$, then $e_i \neq h_i = G \cdot h_i$, for some i. So we may assume that $h \notin H^G$, that is, $G_h = \cap G_{h_i} \neq G$. By (12) and inclusion $G_{h_{i+1}} \subset G_{h_i}$, almost all G_{h_i}'s differ from G and almost all G_{h_i} are compact.

Suppose to the contrary, that $e_i = g_i h_i$, $g_i \in G$ for every i. It is easy to show that then

$$e_k = g_k h_k = g_{k+1} h_k = \cdots = g_l h_k$$

for every $k \leq l$. Therefore, $g_l \in g_k \cdot G_{h_k}$, for every $k \leq l$.

Since the stabilizer G_{h_m} is compact for some m, it follows that the sequence $\{g_l\}_{l \geq m} \subset g_m \cdot G_{h_m}$ converges to $g_0 \in g_m \cdot G_{h_m}$. Analogously, one can show that $g_0 \in g_p \cdot G_{h_p}$, for all $p \geq m$. Consequently, $g_0 h_p = g_p h_p = e_p$, for all $p \geq m$, that is, $e = g_0 h$. Contradiction. □

PROOF OF THEOREM 4.1. Using the hypotheses, let us fix a topological G-embedding (Proposition 4.2):

$$i : X \hookrightarrow \prod_{n,m} \operatorname{Con} R_{nm} = D$$

and a closed topological embedding $j : X/G \hookrightarrow L$ of the orbit space X/G into a linear normed space L. It is obvious that

$$i \times (j \circ \pi) = e : X \hookrightarrow L \times D$$

is a closed topological G-embedding. Since the image $e(X)$ does not contain points with a noncompact stabilizer, $e(X)$ does not intersect the closed set $L \times \{*\}$, where $\{*\}$ is the product of the vertices of the cone-factors of D. Therefore, $e(X)$ lies in the proper open G-space $U' = L \times (D \setminus \{*\})$.

Since $L \times D \in$ G-*AE*, it follows that $U' \in$ G-*ANE*. Since $X \in$ G-*ANE*, there exists a G-retraction $r : U \to X$ of some G-neighbourhood U, $e(X) \subset U \subset U'$. Hence, $\tilde{r} : U/G \to X/G$ is a retraction and the inclusion $X/G \in ANE$ is reduced to another inclusion $U/G \in ANE$.

If we now prove that $D/G \in AE$, then $(L \times D)/G = L \times (D/G) \in AE$, and therefore, $U/G \in ANE$ as an open subset of the orbit space. To complete the proof of the theorem, it thus remains to verify that $D/G \in AE$.

Let us introduce the following notations: $D_p = \prod_{n+m \leq r} \operatorname{Con} R_{nm}$ and $q_r : D_{r+1} \to D_r$ is a projection. Since R_{nm} is metrized by a complete invariant metric, it follows that $\operatorname{Con} R_{nm}$ and D_r are also metrized by a complete invariant metric. Thus, the orbit space D_r/G is also metrized by a complete metric. It follows from $D_r \in$ G-AE and Proposition 3.2 that $D_r/G \in \mathrm{LC} \cap \mathrm{C}$. Due to its countable-dimensionality and the Haver theorem [15] we obtain that $D_r/G \in AE$.

Since $\operatorname{Con} R_{r+1} \in AE$, the projection q_m is a fiberwise G-contractible map, that is, there exist fiberwise G-maps $s : D_r \to D_{r+1}$, $q_r \circ s = \mathrm{Id}$ and $H : D_{r+1} \times [0, 1] \to D_{r+1}$, $q_r \circ H = q_r$, such that $H_0 = \mathrm{Id}$ and $\mathrm{Im}(H_1) = \mathrm{Im}(s)$. Passing to the orbit spaces we obtain fiberwise contractible maps $\tilde{q}_r : D_{r+1}/G \to D_r/G$, that is, $\tilde{q}_r$ is a fine homotopy equivalence. Since all the conditions of Curtis's theorem [11] are satisfied, we conclude that $\varprojlim\{D_i/G, q_i\}$ is an AE. But by Proposition 4.7 this inverse limit coincides with the orbit space D/G. □

5. Proof of Theorem 1.2

By Theorem 2.6 and Proposition 3.1, there exists a $\mathrm{GL}(n)$-retraction $r : C(n) \to \mathrm{GL}(n)/O(n) = \mathfrak{E}$, which is nevertheless unacceptable for us because of its nonconstructibility. Another geometric $\mathrm{GL}(n)$-retraction, generated by the Löwner ellipsoid, will be more convenient.

THEOREM 5.1 (see [17]). *For every convex body $V \in C(n)$, there exists a unique ellipsoid $E_V \in C(n)$, which contains V and has the minimal Euclidean volume.*

The $\mathrm{GL}(n)$-invariance of E_V (that is, $E_{AV} = AE_V$ for all $A \in \mathrm{GL}(N)$) then follows by minimality of the volume. A continuous dependence E_V on V with respect to the Hausdorff metric was proved in [5]. Therefore, $\mathscr{L} : C(n) \to \mathfrak{E}$, $\mathscr{L}(V) = E_V$, is a $\mathrm{GL}(n)$-retraction of $C(n)$ onto the *ellipsoid orbit* $\mathfrak{E}$ ($\mathscr{L}$ is called the *Löwner retraction*).

Since the symmetry group Sym_{B^n} of B^n is $O(n)$, the $O(n)$-slice $L(n) = \mathscr{L}^{-1}(B^n)$ is an $O(n)$-space. In other words, $L(n)$ consists of all bodies $V \in C(n)$ whose minimal Löwner ellipsoid coincides with B^n. The orbit space $Q(n) = C(n)/\mathrm{GL}(n)$ is homeomorphic to $L(n)/O(n)$. Therefore, by Theorem 4.1,

$$L(n)/O(n) = Q(n) \in AE \quad \text{and} \quad Q_{\mathscr{E}} = Q(n) \setminus \{\text{Eucl.}\} = L_{\mathscr{E}}(n)/O(n) \in ANE,$$

where $L_{\mathscr{E}} = L(n) \setminus \{B^n\}$, and so Theorem 1.2 is reduced to the following:

THEOREM 5.2. *$Q_{\mathscr{E}}(2) = L_{\mathscr{E}}(2)/O(2)$ is a Hilbert cube manifold.*

We prove Theorem 5.2 in three main steps which are carefully outlined below.

Step 1. Reduction of Theorem 5.2 to Proposition 5.3 and the Toruńczyk characterization for Q-manifolds

PROPOSITION 5.3. *For every integer $n \geq 2$ and every $\delta > 0$, there exist $O(n)$-maps $f_i : L_{\mathscr{E}}(n) \to L_{\mathscr{E}}(n)$, $i \in \{1, 2\}$, such that*

(1) *f_i and $\mathrm{Id}_{L_{\mathscr{E}}(n)}$ are δ-close; and*
(2) *if $n = 2$ then $\mathrm{Im}\, f_1 \cap \mathrm{Im}\, f_2 = \emptyset$.*

PROOF OF THEOREM 5.2. According to the Toruńczyk characterization criterion [19], in order to prove Theorem 5.2, it suffices to check that for every $\varepsilon > 0$ and for all pairs of maps $\varphi_i : I^\infty \to Q_{\mathscr{E}}(n)$, $i \in \{1, 2\}$, there are continuous maps $g_i : I^\infty \to Q_{\mathscr{E}}(n)$, ε-close to φ_i, $i \in \{1, 2\}$, such that if $n = 2$ then $\mathrm{Im}\, g_1 \cap \mathrm{Im}\, g_2 = \emptyset$.

Since $F = \cup \operatorname{Im} \varphi_i$ and $F_1 = \pi^{-1}(F)$ are compact (here $\pi : L_{\mathcal{E}}(n) \to L_{\mathcal{E}}(n)/O(n)$ is the orbit projection), there exists $\delta > 0$ such that $\operatorname{dist}(\pi(a), \pi(b)) < \varepsilon$, for every $a, b \in F_1$, with $\operatorname{dist}(a, b) < \delta$.

By Proposition 5.3 for every $n \geq 2$, there are $O(n)$-maps $f_i : L_{\mathcal{E}}(n) \to L_{\mathcal{E}}(n)$, $i \in \{1, 2\}$, satisfying (1) for $\delta > 0$ and (2) for $n = 2$. The induced maps $\tilde{f}_i$ of the orbit spaces, $i \in \{1, 2\}$, have the following properties for $n = 2$:

$$\rho\left(\tilde{f}_i|_F, \operatorname{Id}_F\right) < \varepsilon \quad \text{and} \quad \cap \operatorname{Im} \tilde{f}_i = \emptyset.$$

Finally, the desired maps $g_i : I^\infty \to Q_{\mathcal{E}}(2)$, $i \in \{1, 2\}$, are defined by the formula $g_i = \tilde{f}_i \circ \varphi_i$. □

Step 2. Construction of f_1

Let us consider so-called *contact map* $\alpha : L(n) \to \exp(S^{n-1})$, defined by $\alpha(V) = V \cap S^{n-1}$. The following lemma, whose routine verification is omitted, records several basic properties of α.

LEMMA 5.4. (3) *α preserves the action of $O(n)$, $\alpha(A \cdot V) = A \cdot \alpha(V)$, for every $A \in O(n)$;*

(4) *$\alpha(V) \neq \emptyset$, for every $V \in L(n)$;*

(5) *$\alpha(V)$ is a central symmetric subset of S^{n-1}; and*

(6) *$\alpha(V) = S^{n-1}$ if and only if $V = B^n$.*

LEMMA 5.5. (7) *Let $V \subseteq W \subseteq B^n$, where $V \in L(n)$ and $W \in C(n)$. Then $W \in L(n)$.*

(8) *For every subset $A \subseteq B^n$, $\alpha(\operatorname{Conv}(A)) = \operatorname{Conv}(A) \cap S^{n-1} = A \cap S^{n-1}$.*

PROOF. (7) The minimal Löwner ellipsoid for W and V coincides with B^n. Hence $W \in L(n)$.

In order to prove (8), it suffices to observe that every point $s \in \operatorname{Conv}(A) \cap S^{n-1}$ is an extreme point of B^n and therefore is also an extreme point of $\operatorname{Conv}(A) \subseteq B^n$. But all extreme points of $\operatorname{Conv}(A)$ are contained in A. Therefore $s \in A$. □

Unfortunately, the contact map α is discontinuous. The following reasoning compensates for this unpleasant moment. Let us denote by $\widehat{x0y}$ the nonoriented angle between the rays $[0x)$ and $[0y)$, where $x, y \in B^n$ and $x, y \neq 0$. Next, we introduce a version of the closed ε-neighbourhood of a set, which will be convenient for us. Let $\varepsilon > 0$ and $V \in L(n)$. By V_ε we denote

$$V \cup \left\{x \in B^n \setminus \{0\} \mid \text{ there exists } y \in V \text{ with } \|x\| = \|y\| \text{ and } \widehat{x0y} \leq \varepsilon\right\}.$$

It is clear that V_ε preserves the action of $O(n) : (g \cdot V)_\varepsilon = g \cdot V_\varepsilon$, for every $g \in O(n)$, $V \in L_\varepsilon(n)$. The compactness of V implies that V_ε is compact; the inequality $\|x - y\| < \widehat{x0y}$, for every $\|x\| = \|y\|$, implies that

(9) $V \subseteq V_\varepsilon \subseteq \overline{N}(V;\varepsilon)$, where $\overline{N}(V;\varepsilon)$ is a closed ε-neighbourhood of V in B^n.

We need V_ε to be continuously dependent on V and ε.

PROPOSITION 5.6. *Let* $\varepsilon_k \to \varepsilon > 0$ *and* $V_k \in L(n) \to V$. *Then* $(V_k)_{\varepsilon_k} \to V_\varepsilon$.

PROOF. Let $R_k = (V_k)_{\varepsilon_k}$ and $R = V_\varepsilon$. Suppose that the assertion of the proposition is false, that is, that $R_k \not\to R$. Then there exist $\alpha > 0$ and a sequence $k_i \to \infty$ such that

(10) $x_0 \notin N(R_{k_i};\alpha)$, for some $x_0 \in R$; or
(11) there exists $x_i \in R_{k_i}$, $i \geq 1$, with $x_i \notin N(R;\alpha)$.

In the first case, $\widehat{x_0 0 y_0} \leq \varepsilon$, for some $y_0 \in V$, with $\|y_0\| = \|x_0\|$. Since $V_k \to V$, there exists a sequence $y_k \in V_k \to y_0$. It is easy to see that there exists a sequence $x_k \in B^n \to x_0$, $\widehat{x_k 0 y_k} \leq \varepsilon_k$, $\|x_k\| = \|y_k\|$. It means that $x_k \in (V_k)_{\varepsilon_k} = R_k$, for every k and the limit point x_0 of $\{x_k\}$ belongs to $N(R_{k_i};\alpha)$, for some k_i. This contradicts (10).

In the second case, there exists a sequence $\{y_i \in V_{k_i}\}$ such that $\|y_i\| = \|x_i\|$ and $\widehat{y_i 0 x_i} \leq \varepsilon_{k_i}$. By compactness of B^n, we can suppose that there exist the limits $y_i \to y \in V$ and $x_i \to x \in B^n$. Then $\|y\| = \|x\|$ and $\widehat{x0y} \leq \varepsilon$. Therefore, $x \in V_\varepsilon = R$. This contradicts the fact that $x_i \notin N(R;\alpha)$. □

Consider the following set-valued map:

$$F : L_\varepsilon(n) \rightsquigarrow \mathbb{R}^+, \quad F(V) \stackrel{\text{def}}{=} \{t > 0 \mid B^n \setminus N(V;t) \neq \emptyset\},$$

where $N(V;t)$ is the open t-neighbourhood of V in B^n.

Since $N(V;t)$ is a continuous set-valued map from $L_\varepsilon(n) \times \mathbb{R}^+$ into B^n (in the Hausdorff metric) and $B^n \setminus V \neq \emptyset$, the map F is lower semicontinuous and has domain $L_\varepsilon(n)$. Let us consider the function $f : \mathrm{Graph}(F) \to \mathbb{R}^+$ given by $f(V,t) = t$ and defined on the graph F. Then the function $g : L_\varepsilon(n) \to \mathbb{R}^+$, defined by

$$g(V) = \sup\{t > 0 \mid B^n \setminus N(V;t) \neq \emptyset\} = \sup\{f(V,t) \mid (V,t) \in \mathrm{Graph}(F)\}$$

is well defined and lower semi-continuous [9, page 48] (in set-valued analysis g is called a *marginal function* [24]).

By the Dowker theorem [13], there exists a continuous function $\gamma : L_\varepsilon(n) \to \mathbb{R}^+$ with $\gamma(V) < \delta \cdot g(V)$, $V \in L_\varepsilon(n)$. By Proposition 5.6, it is clear that $V_{\gamma(V)}$ continuously depends on $V \in L_\varepsilon(n)$. The desired continuous $O(n)$-map $f_1 : L_\varepsilon(n) \to L_\varepsilon(n)$ is defined by setting $f_1(V) = \mathrm{Conv}(V_{\gamma(V)})$. By (9), f_1 and $\mathrm{Id}_{L_\varepsilon(n)}$ are δ-close.

Let $\operatorname{dist}(v, w)$ be the spherical distance between $v, w \in S^{n-1}$ and $\overline{N}_{\mathrm{sph}}(A; R)$ be the closed R-neighbourhood of the subset $A \subset S^{n-1}$ with respect to the spherical distance. By Lemma 5.5 (8),

$$\alpha \circ f_1(V) = \operatorname{Conv}(V_{\gamma(V)}) \cap S^{n-1} = V_{\gamma(V)} \cap S^{n-1} = \overline{N}_{\mathrm{sph}}(V; \gamma(V)).$$

The last equality means the boundary of $f_1(V)$ to contain an open (nonempty) subset S^{n-1}, for every $V \in L_{\mathscr{E}}(n)$. The mapping f_2 will be constructed without such property and therefore $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \emptyset$.

Step 3. Construction of f_2

THEOREM 5.7. *For every $\sigma > 0$, there exists an $O(n)$-mapping $F : L_{\mathscr{E}}(n) \to C(n)$ such that*

(12) $\rho(F, \operatorname{Id}_{L_{\mathscr{E}}(n)}) < \sigma$; *and*

(13) *for every $V \in L_{\mathscr{E}}(n)$, $F(V) = \operatorname{Conv}(\sum_{i=1}^m \lambda_i D_i)$, where D_i is an H_i-orbit, H_i is a proper subgroup of $O(n)$ and $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i \geq 0$.*

In connection with this theorem we formulate a geometric conjecture, which is trivially true in dimension 2. If Conjecture 5.8 is valid then our proof of Theorem 1.2 immediately generalizes to arbitrary $n \geq 2$.

CONJECTURE 5.8. *The body $\sum_{i=1}^m \lambda_i D_i$ (hence also $\operatorname{Conv}(\sum_{i=1}^m \lambda_i D_i)$) in the formula (13) 'essentially differs' from the ball, that is, its boundary does not contain open subsets of the sphere.*

PROOF. By the Palais theorem (Theorem 2.5) any orbit $O(n)V$, $V \in L_{\mathscr{E}}(n)$, allows an $O(n)$-retraction $r'_V : \mathscr{U}_V \to O(n)V$, $r'_V(V) = V$. Here we can assume that:

(14) $\rho_H(W, r'_V(W)) < \sigma/2$, for all $W \in \mathscr{U}_V$.

LEMMA 5.9. *For every $\theta > 0$ there exists a finite set $K \subset \operatorname{Bd} V$ such that:*

(i) *$W = \operatorname{Conv}(\operatorname{St}_V K)$ and V have equal stabilizers*; *and*

(ii) *$\rho_H(V, W) < \theta$.*

PROOF. It follows from the existence of slices that for some numbers $\theta > \theta_1 > 0$ from $\rho_H(V, V') < \theta$ and $\operatorname{St}_{V'} \supseteq \operatorname{St}_V$, it always follows that $\operatorname{St}_{V'} = \operatorname{St}_V$. Consider a discrete subset $K \subset \operatorname{Bd} V$ such that $\rho_H(V, \operatorname{Conv} K) < \theta_1$. Then

$$V \supseteq \operatorname{Conv}(\operatorname{St}_V K) = W \supseteq \operatorname{Conv} K$$

and therefore $\rho_H(V, W) < \theta_1$. Next, it follows from $\operatorname{St}_W = \operatorname{St}_{\operatorname{Conv}(\operatorname{St}_V K)} = \operatorname{St}_{\operatorname{St}_V K} \supseteq \operatorname{St}_V$ and $\rho(V, W) < \theta_1$ that $\operatorname{St}_W = \operatorname{St}_V$. □

For every $V \in L_{\mathscr{E}}(n)$, fix $V' = \mathrm{Conv}(HK_V) \in C(n)$ such that $H = \mathrm{St}_V$, $K_V \subset \mathrm{Bd}\, V$, $|K_V| < \infty$ and $\rho_H(V, V') < \sigma/2$. Let us introduce the composition

$$r = h_V \circ r' : \mathscr{U}_V \to O(n)V \to O(n)V',$$

where $h_V(gV) = gV'$ is an $O(n)$-homeomorphism.

If we get V' sufficiently close to V then we obtain the following:

(15) $\mathrm{dist}(W, r_V W) < \sigma$, for every $W \in \mathscr{U}_V$.

We inscribe a locally finite cover $\{T_\mu\}$ into the open cover $\{\mathscr{U}_V/O(n)\}$ of the orbit space $L_{\mathscr{E}}(n)/O(n) = Q_{\mathscr{E}}(n)$. Let $T_\mu \subset \pi(U_{V_\mu})$.

We now define the desired $O(n)$-map $F : L_{\mathscr{E}}(n) \to C(n)$ as follows:

$$F(W) = \sum_\mu \gamma_\mu(\pi W) \cdot r_{V_\mu}(W), \qquad W \in L_{\mathscr{E}}(n),$$

where $\{\gamma_\mu(\cdot)\}$ is a continuous partition of unity, subordinate to the cover $\{T_\mu\}$.

We verify the conditions (12) and (13) of Theorem 5.7. Let $T_1, \ldots, T_m \in \{T_\mu\}$ be all the elements which contain πW and let $T_i \subset \pi(U_{V_i})$. It follows by (4) that $\rho_H(W, r_{V_i}(W)) < \sigma$, for all i. Then by convexity of the ball of radius σ at $C(n)$ we have that $\mathrm{dist}(W, FW) < \sigma$. Thus (12) has been verified.

Condition (13) follows, since $H\, \mathrm{Conv}\, K$ is a union of a finite number of H-orbits for every proper subgroup $H < O(n)$ and finite K. □

It is well known [2] that there exists a $O(n)$-retraction $R : C(n) \to L(n)$ which takes $C_{\mathscr{E}}(n)$ into $L_{\mathscr{E}}(n)$. But we need the following precise result which follows from geometric considerations:

THEOREM 5.10. *There exists a continuous $O(n)$-retraction $R : C(n) \to L(n)$, such that V and $R(V)$ are affinely equivalent, for every $V \in C(n)$.*

PROOF. Let $L(V)$ be the Löwner ellipsoid, circumscribed around V, $g \in \mathrm{GL}(n)$, $g(L(V)) = B^n$. As is well known, g can be represented as $g = g_2 \circ g_1$, where $g_2 \in O(n)$ and g_1 is self-adjoint. Here $R(V) = g_1(V)$. □

Since $L(n)$ is compact, for every $\delta > 0$ there exists $\sigma > 0$, $\sigma < \delta/2$, such that for every $V \in L(n)$ and every $W \in C(n)$,

$$\rho_H(V, W) < \sigma \Rightarrow \rho_H(W, R(W)) < \delta/2.$$

By Theorem 5.7 there is a mapping $F : L_{\mathscr{E}}(n) \to C(n)$ such that $\rho(F, \mathrm{Id}_{L_{\mathscr{E}}(n)}) < \sigma$. The desired map f_2 is $R \circ F$.

Indeed,

$$\rho_H(V, f_2V) = \rho_H(V, R \circ F(V)) < \rho_H(V, F(V)) + \rho_H(F(V), R(FV)) < \sigma + \delta/2 < \delta.$$

Since for $n = 2$, the boundary $F(V)$, $V \in L_{\mathcal{E}}(n)$, does not contain an open subset of a sphere, $f_2(V)$ which is affinely equivalent $F(V)$, also does not contain any open subsets of the sphere. Therefore, $\mathrm{Im} f_1 \cap \mathrm{Im} f_2 = \emptyset$. □

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