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On Murayama's theorem on extensor properties of G-spaces of given orbit types

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ABSTRACT

We develop a method of extending actions of compact transformation groups which is then applied to the problem of preservation of equivariant extensor property by passing to a subspace of given orbit types.

1. Introduction

The problem of topological characterization of simplicial complexes motivated Borsuk to introduce an important class of spaces, namely absolute neighborhood retracts (ANR-spaces) which turned out to be a wider class than simplicial complexes, but intimately close to them with respect to other properties. As it was shown by Dugundji [10], each ANR-space is characterized by the property that it admits an arbitrarily fine domination by simplicial complexes.

In equivariant topology the role of simplicial complexes is played by G-CW-complexes, and the role of absolute neighborhood retracts is played by G-ANE-spaces. In the equivariant case Dugundji's characterization mentioned above consists of the following plausible statement:

Conjecture 1.1. *Let G be a compact group. Then any metric G-space $\mathbb{X} \in G$-ANE admits an arbitrarily fine domination by G-CW-complexes, i.e. for each cover $\omega \in \operatorname{cov} \mathbb{X}$ there exist a G-CW-complex $\mathbb{Y}$ and G-maps $\mathbb{X} \xrightarrow{f} \mathbb{Y} \xrightarrow{g} \mathbb{X}$ such that $g \circ f$ and $\mathrm{Id}_{\mathbb{X}}$ can be joined by an ω-G-homotopy.*

This has so far been settled only for two special cases: for compact metric G-ANE-spaces [5], and for metric G-ANE-spaces with an action of a zero-dimensional compact group [4]. In general this is still a conjecture.

We consider one more question concerning the closeness of G-CW-complexes and G-ANE-spaces. It is well known that if $\mathbb{Y}$ is a G-CW-complex then for each closed family $\mathcal{C} \subset \mathrm{Orb}_G$ of orbit types, the G-subspace $\mathbb{Y}_{\mathcal{C}}$ of points of orbit type $\mathcal{C}$ is also a G-CW-complex [9]. Murayama [13] proved that for the family of orbit type $\mathcal{C} = \{(K) \mid (K) \geqslant (H)\}$ the complete analogy with G-CW-complexes is preserved.

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Theorem 1.2. *If G is a compact Abelian Lie group and $\mathbb{X}$ is a metric G-ANE-space then $\mathbb{X}^{(H)} = G \cdot \mathbb{X}^H \in G$-ANE for each closed subgroup $H < G$.*

If a compact Lie group G is nonabelian, then $\mathbb{X}^H$ admits the action of the normalizer $\mathrm{N}(H)$ of H and cannot in general be endowed with the action of G. Therefore there exists a point $x \in \mathbb{X}^G \subset \mathbb{X}^H$ without G_x-slices in $\mathbb{X}^H$. Since exactly this argument was the key in the proof of [13, Proposition 8.7], the case of such a group cannot be considered to be settled.

We show that Conjecture 1.1 implies the validity of Theorem 1.2 for arbitrary compact groups. Let $\omega \in \operatorname{cov}\mathbb{X}$, and let $\mathbb{Y}$ be a G-CW-complex and $\mathbb{X} \xrightarrow{f} \mathbb{Y} \xrightarrow{g} \mathbb{X}$ G-maps such that $g \circ f \simeq_G \mathrm{Id}_{\mathbb{X}} \operatorname{rel}[\omega]$. Since $\mathbb{Y}^{(H)}$ is a G-CW-complex and therefore $\mathbb{Y}^{(H)} \in G$-ANE [13, Theorem 12.5], $\mathbb{X}^{(H)}$ admits an arbitrarily fine domination by G-ANE-complexes. By [13, Theorem 9.2] it follows that $\mathbb{X}^{(H)} \in G$-ANE.

Based on the remark made above it thus follows that Theorem 1.2 is proved for compact metric G-ANE-spaces and for metric G-ANE-spaces with an action of a zero-dimensional compact group. In the present paper we develop a new approach based on a reduction of the problem to that of extending the action of groups which is also of independent interest. As a result we obtain the following

Theorem 1.3. *Let G be a compact Lie group, $\mathbb{X}$ a G-ANE-space and $\mathcal{C} \subset \mathrm{Orb}_G$ a saturated family of orbit types. Then $\mathbb{X}_{\mathcal{C}} \subset \mathbb{X}$ is G-ANE.*

Since for each $H < G$ the family of orbit types $\{(K) \mid (K) \geqslant (H)\}$ evidently satisfies the hypotheses of this theorem, the strengthening of Murayama's theorem is valid for arbitrary action of compact Lie groups. We conjecture that the further generalization of Theorem 1.3 for compact group action on a metric space is also valid, provided that $\mathcal{C} \subset \mathrm{Orb}_G$ is a saturated family in which the intersection $\mathcal{C} \cap \mathcal{E}$ with the family $\mathcal{E}$ of extensor orbit types is cofinal in $\mathcal{E}$.[1] The following theorem, proved in [7, Theorem 9], asserts in favor of this conjecture: if $\mathbb{X}$ is a metric G-ANE-space, then each G-subspace $\mathbb{Y} \subset \mathbb{X}$ containing the bundle $\mathbb{X}_{\mathcal{E}}$ of extensor orbit types is a G-ANE.

We cannot omit the saturation condition from Theorem 1.3 (since there exists a 2-dimensional compact counterexample), but we do have a pleasant (and important) exception for (Σ, d)-universal (in the sense of Palais [14, p. 59]) G-spaces. Until recently the solution of Palais problem on existence of universal G-spaces was known only for finite collection $\Sigma \subset \mathrm{Orb}_G$ of orbit types and finite dimension $d < \infty$ [14, 2.6]; for finite dimension d [3]. The final solution of Palais problem (without any restrictions on dimension d and collection Σ) was obtained in [6]: the equivariant Hilbert space $\mathbb{L}_2$ is an (Orb_G, ∞)-universal G-space. The following result is a cornerstone of the theory of such universal G-spaces for which we prefer alternative term – an *isovariant absolute extensor*, Isov-AE.

Theorem 1.4. *Let G be a compact Lie group, $\mathcal{C} \subset \mathrm{Orb}_G$ a family of orbit types and $\mathbb{X}$ an isovariant absolute extensor. Then the bundle $\mathbb{X}_{\mathcal{C}} \subset \mathbb{X}$ of orbit type $\mathcal{C}$ is G-ANE.*

The consequences of this theorem and another results of the theory of isovariant absolute extensors will be presented in the subsequent publications of the first author. The proofs of Theorems 1.3 and 1.4 will be based on the *problem of extending the action of groups* which was first posed by Shchepin in view of its connection with the problem of extending equivariant maps (see [7]). The diagram $\mathcal{D} = \{\mathbb{X} \xrightarrow{p} X \xhookrightarrow{i} Y\}$, in which the G-space $\mathbb{X}$ has the orbit type $\mathcal{C} \subset \mathrm{Orb}_G$, $p : \mathbb{X} \to X$ is an orbit projection and i is a closed topological embedding of the orbit space X into a space Y, will be called *$\mathcal{C}$-admissible*. We say that the *problem of extending the action is solvable for the $\mathcal{C}$-admissible diagram $\mathcal{D}$*, provided that there exists an equivariant embedding $j : \mathbb{X} \hookrightarrow \mathbb{Y}$ into a G-space $\mathbb{Y}$ of orbit type $\mathcal{C}$ (called a *$\mathcal{C}$-solution* of the problem of extending the action for given diagram) covering i, i.e. the embedding $\tilde{j} : X \hookrightarrow p(\mathbb{Y})$ of orbit spaces induced by j coincides with i. Note that this definition implies that the embedding j is closed and $p(\mathbb{Y}) = Y$. If the family of orbit types $\mathcal{C}$ coincides with Orb_G, then the notation $\mathcal{C}$ is omitted.

We say that the problem of extending the action (denoted briefly by PEA) is *solvable for the class $\mathcal{F}$ of spaces*, if for each admissible diagram $\mathcal{D}$ in which $\mathbb{X}, X$ and Y belong to $\mathcal{F}$ there exists a solution of the PEA for which $\mathbb{Y} \in \mathcal{F}$. For compact group G, the PEA is solvable for the class of stratifiable spaces (see [7]). Here for the class of metric spaces we supplement this result with an information on the G-orbit type of a solution of the PEA.

Theorem 1.5. *Let G be a compact Lie group, $\mathcal{C} \subset \mathrm{Orb}_G$ a family of orbit types with $(G) \in \mathcal{C}$ and $\mathcal{D} = (\mathbb{X} \xrightarrow{p} X \xhookrightarrow{i} Y)$ a metric $\mathcal{C}$-admissible diagram. Then for each solution $s : \mathbb{X} \hookrightarrow \mathbb{Y}$ of the* PEA *for $\mathcal{D}$ there exists a metric $\mathcal{C}$-solution $s_1 : \mathbb{X} \hookrightarrow \mathbb{Y}_1$ of the* PEA *for $\mathcal{D}$ majorized by s, $s \geqslant s_1$.*[2]

We show in Section 3 that Theorem 1.5 implies the validity of Theorems 1.3 and 1.4.

[1] Recall that the orbit type (H) is called *extensor*, i.e. G/H is a metric G-ANE-space (related definitions are in Section 2).

[2] Recall that *s majorizes s_1* if there exists a G-map $h : \mathbb{Y} \to \mathbb{Y}_1$ such that $h \circ s = s_1$, $h \restriction_{\mathbb{X}} = \mathrm{Id}_{\mathbb{X}}$ and $\tilde{h} = \mathrm{Id}_Y$.

2. Preliminary facts and results

In what follows we shall assume all spaces (resp. maps) to be metric (resp. continuous), if they do not arise as a result of some constructions. For $A \subset X$ we use standard notations: $\mathrm{Cl}\, A$ – for the closure; $\mathrm{Int}\, A$ – for the interior. We use the notation $f\restriction_A$ for the restriction of map $f: X \to Y$ on $A \subset X$, or simply $f\restriction$, provided it is clear a set to which we are referring. Since f is an extension of $f\restriction_A$, we denote this as $f = \mathrm{ext}(f\restriction_A)$.

In what follows G will be a compact group. An *action of G on a space X* is a homomorphism $T: G \to \mathrm{Aut}\, X$ of G into the group $\mathrm{Aut}\, X$ of all autohomeomorphisms of X such that the map $G \times X \to X$ given by $(g, x) \mapsto T(g)(x) = g \cdot x$ is continuous. A space X with a fixed action of G is called a *G-space*.

For any point $x \in X$, $G_x = \{g \in G \mid g \cdot x = x\}$ is a closed subgroup of G called the *isotropy subgroup* of x; $G(x) = \{g \cdot x \mid g \in G\} \subset X$ is called the *orbit* of $x \in X$. The set of all orbits is denoted by X/G and the natural map $p = p_X : X \to X/G$, given by $p(x) = G(x)$, is called the *orbit projection*. We call the set X/G of all orbits equipped with the quotient topology induced by p the *orbit space of X* (see [8] for more details on compact transformation groups). In what follows we shall denote G-spaces and their orbit spaces as follows: $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \ldots$ for G-spaces, and $X, Y, Z, \ldots$ for their orbit spaces.

The map $f: \mathbb{X} \to \mathbb{Y}$ of G-spaces is called *equivariant* or a *G-map*, if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in \mathbb{X}$. Each G-map $f: \mathbb{X} \to \mathbb{Y}$ induces a map $\tilde{f}: X \to Y$ of orbit spaces by the formula $\tilde{f}(G(x)) = G(f(x))$. We call an equivariant homeomorphism an *equimorphism*. The equivariant map $f: \mathbb{X} \to \mathbb{Y}$ is said to be *isovariant* if $G_x = G_{f(x)}$ for all $x \in \mathbb{X}$.

The isovariant map $f: \mathbb{X} \to \mathbb{Y}$ is said to be an *isogeny* if its induced map $\tilde{f}: X \to Y$ is a homeomorphism. More generally, the equivariant map f is said to be an *equigeny* if $\tilde{f}$ is a homeomorphism.[3] By [11, 3.7.10] each equigeny is perfect.

Observe that all G-spaces and G-maps generate a category denoted by G-TOP or EQUIV-TOP, provided that no confusion occurs. If "$***$" is any notion from non-equivariant topology, then "G-$***$" or "Equiv-$***$" means the corresponding equivariant analogue.

The subset $A \subset \mathbb{X}$ is called *invariant* or a *G-subset*, if $G \cdot \mathbb{A} = \mathbb{A}$. For each closed subgroup $H < G$ (in what follows this sign will be used for closed subgroups; a normal closed subgroup is denoted as $H \lhd G$) we introduce the following sets: $\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\}$ (which is called an *H-fixed set*) and $\mathbb{X}_H = \{x \in \mathbb{X} \mid G_x = H\}$. It is clear that $\mathbb{X}^{(H)} = \bigcup\{\mathbb{X}_K \mid K < G$ and $H' < K$ for some conjugated subgroup $H' \sim H\}$ coincides with $G \cdot \mathbb{X}^H$ and $\mathbb{X}_{(H)} \rightleftharpoons \bigcup\{\mathbb{X}_K \mid K < G$ conjugates with $H\}$ coincides with $G \cdot \mathbb{X}_H$.

Let Conj_G be the set of all conjugated classes of closed subgroups of G and Orb_G a collection of all homogeneous spaces up to equimorphisms. We endow these sets with the following partial orders: $(K) \leqslant (H) \Leftrightarrow K$ is contained in $H' \in (H)$; $G/K \geqslant G/H \Leftrightarrow$ there exists an equivariant map $f: G/K \to G/H$. It is evident that the bijection $(H) \in \mathrm{Conj}_G \mapsto G/H \in \mathrm{Orb}_G$ inverses this order. In view of this we identify these sets, provided that no confusion occurs, and we shall use the unified term – the *set of G-orbit types* and the unified notation – Orb_G.

We denote the family $\{(G_x) \mid x \in \mathbb{X}\} \subset \mathrm{Orb}_G$ of *orbit types* of $\mathbb{X}$ by $\mathrm{type}(\mathbb{X})$. If $\mathcal{C} \subset \mathrm{Orb}_G$, then $\mathbb{X}_\mathcal{C} \rightleftharpoons \{x \mid (G_x) \in \mathcal{C}\} \subset \mathbb{X}$ – *the bundle of orbit types* $\mathcal{C}$; $\mathbb{X}^\mathcal{C} \rightleftharpoons \{x \mid (G_x) \geqslant (H)$ for some $(H) \in \mathcal{C}\} \subset \mathbb{X}$ (here and throughout the paper the sign $\rightleftharpoons$ is used for the introduction of the new objects placed to the left of it). The G-space $\mathbb{X}$ *has an orbit type* $\mathcal{C}$ if $\mathrm{type}(\mathbb{X}) \subset \mathcal{C}$ or $\mathbb{X} = \mathbb{X}_\mathcal{C}$. The family $\mathcal{C} \subset \mathrm{Orb}_G$ is said to be *closed* if $(H) \in \mathcal{C}$ as soon as $(H) \geqslant (K)$ for some $(K) \in \mathcal{C}$; the family $\mathcal{C} \subset \mathrm{Orb}_G$ is called *saturated* if $(K) \geqslant (L) \geqslant (H)$ for some $(K), (H) \in \mathcal{C}$ implies $(L) \in \mathcal{C}$. Also we say that the subfamily $\mathcal{F}$ of $\mathcal{C} \subset \mathrm{Orb}_G$ is cofinal if for each $(H) \in \mathcal{C}$ there exists $(K) \in \mathcal{F}$ with $(H) \leqslant (K)$.

If $K < G$ and $\mathcal{C} \subset \mathrm{Orb}_G$, we set $\mathcal{C}\restriction_K \rightleftharpoons \{(H) \in \mathrm{Orb}_K \mid H < K$ and $G/H \in \mathcal{C}\}$ and $\mathcal{C}' = \mathcal{C} \cup \{(G)\}$. It is clear that: $\mathbb{X}_\mathcal{C}$ is open in $\mathbb{X}_{\mathcal{C}'}$; if $(K) \in \mathcal{C}$, then $\mathcal{C}\restriction_K$ coincides with $(\mathcal{C}\restriction_K)'$. It is clear also that if $\mathcal{C} \subset \mathrm{Orb}_G$ is closed family and $\mathcal{C}\restriction_K \neq \emptyset$, then $\mathcal{C}\restriction_K$ is closed in Orb_K.

We now introduce several concepts related to extension of equivariant and isovariant maps partially defined in metric G-spaces. A space $\mathbb{X}$ is called an *equivariant absolute neighborhood extensor*, $\mathbb{X} \in$ G-ANE, if each G-map $\varphi: \mathbb{A} \to \mathbb{X}$ defined on a closed G-subset $\mathbb{A} \subset \mathbb{Z}$ of metric G-space $\mathbb{Z}$ and called the *partial G-map* can be G-extended onto a G-neighborhood $\mathbb{U} \subset \mathbb{Z}$ of $\mathbb{A}$, $\hat{\varphi}: \mathbb{U} \to \mathbb{X}$, $\hat{\varphi}\restriction_\mathbb{A} = \varphi$. A space $\mathbb{X}$ is called an *isovariant absolute neighborhood extensor*, $\mathbb{X} \in$ Isov-ANE, if each partial isovariant map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi} \mathbb{X}$ can be isovariantly extended onto a G-neighborhood $\mathbb{U} \subset \mathbb{Z}$ of $\mathbb{A}$. If φ can be extended in $\mathbb{U} = \mathbb{Z}$, then $\mathbb{X}$ is called an *equivariant absolute extensor* ($\mathbb{X} \in$ G-AE) in the equivariant case or an *isovariant absolute extensor* ($\mathbb{X} \in$ Isov-AE) in the isovariant case. If the acting group G is trivial, then these notions are transformed into the notions of absolute [neighborhood] extensors – A[N]E.

The following examples of G-AE-spaces are well known: each Banach G-space (see [12, p. 117] and [1, p. 155]); each linear normed G-space for compact Lie group G (see [13, p. 488]). We shall depend heavily on the Slice theorem [8] which we prefer to formulate as follows: $G/H \in G$-ANE for each closed subgroup H of compact Lie group G.

Definition 2.1. A closed subgroup $H < G$ of compact group G is called an extensor subgroup if one of the following equivalent properties holds:

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[3] We remark that *equigeny* is a new term and it is close to *isogeny*, a term used by Palais in [14, p. 12].

(1) G/H is finite-dimensional and locally connected;
(2) there exists a normal subgroup $P \lhd G$ such that $P < H$ and G/P is a compact Lie group;
(3) G/H is a topological manifold; or
(4) G/H is a metric G-ANE-space.

The equivalence of the first three properties was proved in [15]; for the proof of (2) ≡ (4) see [2]. The last property justifies the name of the term. The following theorem on approximate slice of a G-space was proved in [2].

Theorem 2.2. *Let a compact group G act on a G-space $\mathbb{X}$. Then for each neighborhood $\mathcal{O}(x)$ of $x \in \mathbb{X}$ there exist a neighborhood $\mathcal{V} = \mathcal{V}(e)$ of the unit $e \in G$, an extensor subgroup $K < G$, $G_x < K$, and a slice map $\alpha : \mathbb{U} \to G/K$ where $\mathbb{U}$ is an invariant neighborhood of x such that $x \in \alpha^{-1}(\mathcal{V} \cdot [K]) \subset \mathcal{O}(x)$.*

The orbit type (H) is called *extensor* if $H < G$ is an extensor subgroup. The collection of all *extensor orbit types* is denoted by $\mathcal{E}$. We say that G-subspace $\mathbb{Y} \subset \mathbb{X}$ is *G-dense* if $\mathbb{Y}^H \subset \mathbb{X}^H$ is dense for each subgroup $H < G$. It was proved in [2] with the help of Theorem 2.2 that

(5) $\mathbb{X}_{\mathcal{E}} \subset \mathbb{X}$ is G-dense if and only if $\mathbb{X}$ is an equivariant neighborhood extensor for metric G-spaces with zero-dimensional orbit spaces, $\mathbb{X} \in G$-ANE(0); and
(6) a linear normed G-space $\mathbb{L}$ is a G-AE if and only if $\mathbb{L}_{\mathcal{E}} \subset \mathbb{L}$ is G-dense (equivariant Dugundji's theorem).

There exists an example of a linear normed G-space $\mathbb{L} \notin G$-AE for which $\mathbb{L}_{\mathcal{E}} \subset \mathbb{L}$ is dense (but not G-dense).

The proof of the following Palais Metatheorem [14] is based on the stabilization of nested sequence of compact Lie groups.

Proposition 2.3. *Let $\mathcal{P}(H)$ be a property which depends on compact Lie group H. Suppose that $\mathcal{P}(H)$ is true, provided $\mathcal{P}(K)$ is true for each compact Lie group K isomorphic to a proper subgroup of H. If $\mathcal{P}(H)$ is true for trivial group $H = \{e\}$, then $\mathcal{P}(H)$ is true for all compact Lie groups H.*

If there is no danger of ambiguity, we shall omit definitions of some notions, which arise in a natural manner. As a rule, this remark concerns also the assertions analogous to the proved ones.

3. Reduction of Theorems 1.3 and 1.4 to Theorem 1.5

Theorem 1.5 will be convenient for our aims in the following detailed form:

Proposition 3.1. *Let G be a compact Lie group, $\mathcal{C} \subset \mathrm{Orb}_G$ a family of orbit types with $(G) \in \mathcal{C}$. If a metric G-space $\mathbb{Y}$ contains a G-subspace $\mathbb{X}$ of orbit type $\mathcal{C}$ as a closed subset, then $\mathbb{X}$ is contained in a metric G-space $\mathbb{Y}_1$ of orbit type $\mathcal{C}$ as a closed subset, and there exists a G-map $h : \mathbb{Y} \to \mathbb{Y}_1$ such that $h \restriction_{\mathbb{X}} = \mathrm{Id}_{\mathbb{X}}$.*

Now the **proof of Theorem 1.3** easily follows from the following lemmata.

Lemma 3.2. *Let G be a compact Lie group, $\mathbb{X} \in G$-A[N]E and $\mathcal{C} \subset \mathrm{Orb}_G$ a closed family. Then $\mathbb{X}_{\mathcal{C}} \subset \mathbb{X}$ is G-A[N]E.*

Proof. We consider a closed G-embedding of $\mathbb{X}_{\mathcal{C}}$ into a metric G-space $\mathbb{Y}$. In view of Proposition 3.1, there exist a closed G-embedding $\mathbb{X}_{\mathcal{C}} \hookrightarrow \mathbb{Y}_1$ into a metric G-space $\mathbb{Y}_1$ of orbit type $\mathcal{C}$ and a G-map $h : \mathbb{Y} \to \mathbb{Y}_1$ such that $h \restriction_{\mathbb{X}_{\mathcal{C}}} = \mathrm{Id}_{\mathbb{X}_{\mathcal{C}}}$. If $\mathbb{X} \in G$-AE, then there exists a G-map $r : \mathbb{Y}_1 \to \mathbb{X}$, $r \restriction_{\mathbb{X}_{\mathcal{C}}} = \mathrm{Id}$. Since $\mathcal{C}$ is a closed family and $\mathrm{type}(\mathbb{Y}_1) \subset \mathcal{C}$, $\mathrm{type}(r(\mathbb{Y}_1)) \subset \mathcal{C}$. Therefore $\mathrm{Im}\, r \subset \mathbb{X}_{\mathcal{C}}$ and $r \circ h$ is the desired G-retraction. The case of G-ANE-space is proved analogously. □

Lemma 3.3. *Let G be a compact Lie group and $\mathcal{C} \subset \mathrm{Orb}_G$ a saturated family. Then $\mathbb{X}_{\mathcal{C}}$ is open in $\mathbb{X}_{\mathcal{F}}$ where $\mathcal{F} \rightleftharpoons \cup\{\mathcal{C}^{(K)} \mid (K) \in \mathcal{C}\}$ (the so-called closed hull of $\mathcal{C}$).*

Proof. Let $(G_x) \in \mathcal{C}$. There exists a neighborhood $\mathcal{O}(x) \subset \mathbb{X}_{\mathcal{F}}$ such that $(G_y) \leqslant (H)$ for each $y \in \mathcal{O}(x)$. Since $\mathcal{F}$ is the closed hull of $\mathcal{C}$, there exists $(K) \in \mathcal{C}$, $(K) \leqslant (G_y)$. Since $\mathcal{C}$ is saturated, $(K) \leqslant (G_y) \leqslant (H)$ implies that $(G_y) \in \mathcal{C}$. □

For the proof of **Theorem 1.4** we consider a closed G-embedding $\mathbb{X}_{\mathcal{C}}$ into a metric G-space $\mathbb{Y}$. First we assume that $(G) \in \mathcal{C}$. By Proposition 3.1 there exists a closed G-embedding $\mathbb{X}_{\mathcal{C}}$ into a metric G-space $\mathbb{Y}_1$ of orbit type $\mathcal{C}$ and a G-map $h : \mathbb{Y} \to \mathbb{Y}_1$ such that $h \restriction_{\mathbb{X}_{\mathcal{C}}} = \mathrm{Id}_{\mathbb{X}_{\mathcal{C}}}$. Since $\mathbb{X}$ is an isovariant absolute extensor, there exists a G-map $r : \mathbb{Y}_1 \to \mathbb{X}$ isovariant on the complement such that $r \restriction_{\mathbb{X}_{\mathcal{C}}} = \mathrm{Id}_{\mathbb{X}_{\mathcal{C}}}$. Since $r(\mathbb{Y}_1) \subset \mathbb{X}_{\mathcal{C}}$, $r \circ h$ is the desired G-retraction.

In the general case, $\mathbb{X}_{\mathcal{C}'} \in$ G-AE where $\mathcal{C}' = \mathcal{C} \cup \{(G)\}$. Since $\mathbb{X}_{\mathcal{C}}$ is open in $\mathbb{X}_{\mathcal{C}'}$, the proof is completed.

4. Proof of Theorem 1.5

Let G be a compact Lie group, $\mathcal{C} \subset \mathrm{Orb}_G$ a family of orbit types with $(G) \in \mathcal{C}$ and $h : \mathbb{X} \to \mathbb{X}'$ an equigeny with $\mathrm{type}(\mathbb{X}') \subset \mathcal{C}$. We interest for G-embedding $\mathbb{X} \hookrightarrow \mathbb{Y}$ whether there exist a G-embedding $\mathbb{X}' \hookrightarrow \mathbb{Y}'$ with $\mathrm{type}(\mathbb{Y}') \subset \mathcal{C}$ and an equigeny $H : \mathbb{Y} \to \mathbb{Y}'$ extending h. We say that H *$\mathcal{C}$-solves the problem of extending of an equigeny for $\mathcal{C}$-admissible diagram* $\mathcal{D} = \{\mathbb{Y} \hookleftarrow \mathbb{X} \xrightarrow{h} \mathbb{X}'\}$. Since $\mathrm{Id} : \mathbb{X} \to \mathbb{X}$ is an equigeny, Theorem 1.5 is reduced to more general assertion:

Theorem 4.1. *Let $\mathcal{D}$ be a $\mathcal{C}$-admissible diagram and suppose that $\mathbb{X}$ is closed in $\mathbb{Y}$. Then the problem of extending an equigeny for $\mathcal{D}$ is $\mathcal{C}$-solved.*

Lemma 4.2. *If $\mathbb{X}$ is open in $\mathbb{Y}$, then the problem of extending of equigeny for $\mathcal{D}$ is $\mathcal{C}$-solved.*

Proof. Let $\mathbb{Y}' \rightleftharpoons \mathbb{Y}/\sim$ be the quotient space generated by the equivalence $y \sim x \in \mathbb{X}$ iff $y \in \mathbb{X}$ and $h(x) = h(y)$; $y \sim y_1 \in \mathbb{Y} \setminus \mathbb{X}$ iff $y \in \mathbb{Y} \setminus \mathbb{X}$ and $G(y) = G(y_1)$. Then the desired equigeny $H : \mathbb{Y} \to \mathbb{Y}' = \mathbb{Y}/\sim$ extending h is the quotient map. □

The proof of the following result is based on Lemma 4.2 and follows parallel to [7, Lemma 8]. It reduces the proof of Theorem 4.1 to the case when the space $\mathbb{X}$ has no fixed points.

Lemma 4.3. *The validity of Theorem* 4.1 *for all $\mathcal{C}$-admissible diagrams with $\mathbb{X}^G = \emptyset$ implies its validity for all $\mathcal{C}$-admissible diagrams.*

Proof. Let $\mathbb{F} \rightleftharpoons \mathbb{X}^G$. By hypotheses the equigeny $h \restriction : \mathbb{X} \setminus \mathbb{F} \to \mathbb{X}' \setminus \mathbb{F}$ can be extended up to an equigeny $\eta : \mathbb{Y} \setminus \mathbb{F} \to \mathbb{Z}'$. Next we apply Lemma 4.2 to η and the open embedding $\mathbb{Y} \setminus \mathbb{F} \hookrightarrow \mathbb{Y}$. □

5. Proof of Theorem 4.1

In view of Lemma 4.3 we can assume that the G-space $\mathbb{X}$ has no G-fixed points, $\mathbb{X}^G = \emptyset$. When such is the case it is sufficient by Lemma 4.2 for some G-neighborhood $\mathbb{Z}$, $\mathbb{X} \subset \mathbb{Z} \subset \mathbb{Y}$, to construct a G-embedding $\mathbb{X}' \hookrightarrow \mathbb{Z}'$ with $\mathrm{type}(\mathbb{Z}') \subset \mathcal{C}$ and an equigeny $H_1 : \mathbb{Z} \to \mathbb{Z}'$ extending h.

In what follows the argument will be carried out by induction on compact Lie group G based on Palais Metatheorem 2.3. If $|G| = 1$, then the situation under consideration is trivial. Now we suppose that for each proper subgroup $K < G$ Theorem 4.1 has been proved and let us show its validity in case of the G-action. First we consider a special case:

Lemma 5.1. *If $\mathbb{X}'$ admits a nontrivial slice map $\psi : \mathbb{X}' \to G/K$ with $(K) \in \mathcal{C}$, then Theorem* 4.1 *is valid for the $\mathcal{C}$-admissible diagram $\mathcal{D} = \{\mathbb{Y} \hookleftarrow \mathbb{X} \xrightarrow{h} \mathbb{X}'\}$.*

Before the proof we recall the notion of a twisted product. Let us consider a compact group G, a metric H-space $\mathbb{S}$ where $H < G$ and the diagonal action of H on the product $G \times \mathbb{S}$ defined as $h \cdot (g, y) \rightleftharpoons (g \cdot h^{-1}, h \cdot y)$. By $[g, y]$ we denote the element $H \cdot (g, y) = \{(g \cdot h^{-1}, h \cdot y) \mid h \in H\}$ of the orbit space $(G \times \mathbb{S})/H$. It turns out that the formula $g_1 \cdot [g, y] = [g_1 \cdot g, y]$ where $g, g_1 \in G$, $y \in \mathbb{S}$ correctly define the continuous action of G on the orbit space $(G \times \mathbb{S})/H$ called a *twisted product* (and denoted as $G \times_H \mathbb{S}$).

The notion of the twisted product arise naturally in studies of a G-space admitting the slice map, say, $\varphi : \mathbb{X} \to G/H$, $H < G$. Then $\mathbb{X}$ can be identified with the twisted product $G \times_H \mathbb{S}$ where $\mathbb{S} = \varphi^{-1}([H])$ is an H-slice: $[g, s] \in G \times_H \mathbb{S} \mapsto x = g \cdot s \in \mathbb{X}$ (see this and another properties of twisted products in [8]).

Proof of Lemma 5.1. Since $G/K \in G$-ANE, there exists a G-extension $\tilde{\varphi} : \mathbb{U} \to G/K$ of $\varphi \rightleftharpoons \psi \circ h : \mathbb{X} \to G/K$ defined on some G-neighborhood $\mathbb{U}$, $\mathbb{X} \subset \mathbb{U} \subset \mathbb{Y}$. It is clear that $\tilde{\varphi}^{-1}[K] \supset \varphi^{-1}[K]$ and $\psi^{-1}[K]$ are K-spaces for proper compact subgroup $K < G$ with

$$\mathrm{type}\big(\psi^{-1}[K]\big) \subset \mathcal{C} \restriction_K \rightleftharpoons \big\{(H) \bigm| H < K \text{ and } G/H \in \mathcal{C}\big\} \subset \mathrm{Orb}_K.$$

Since $(K) \in \mathcal{C} \restriction_K$ and $h \restriction : \varphi^{-1}[K] \to \psi^{-1}[K]$ is an equigeny, there exists by inductive hypothesis an equigeny $H' : \tilde{\varphi}^{-1}[K] \to \mathbb{W}' \supset \psi^{-1}[K]$ extending $h \restriction$.

Let us consider the following commutative square diagram:

$$\begin{array}{ccc} G \times_K \varphi^{-1}[K] = \mathbb{X} & \hookrightarrow & G \times_K \tilde{\varphi}^{-1}[K] = \mathbb{Y} \\ \downarrow {\scriptstyle \mathrm{Id} \times_K h\restriction = h} & & \downarrow {\scriptstyle H \rightleftharpoons \mathrm{Id} \times_K H'} \\ G \times_K \psi^{-1}[K] = \mathbb{X}' & \hookrightarrow & \mathbb{Y}' \rightleftharpoons G \times_K \mathbb{W}'. \end{array}$$

Since $H = \mathrm{Id} \times H' : \mathbb{Y} = G \times_K \tilde{\varphi}^{-1}[K] \to \mathbb{Y}' = G \times_K \mathbb{W}'$ is an equigeny, $\mathrm{type}(\mathbb{Y}') \subset \mathcal{C} \restriction_K$ and $\mathcal{C} \restriction_K$ lies naturally into $\mathcal{C}$, the proof of the lemma is completed. □

We continue the proof of Theorem 4.1. The following result is an easy consequence of the Slice theorem and hereditary paracompactness of $\mathbb{Y}$.

Lemma 5.2. *There exist a closed G-neighborhood $\mathbb{E}$, $\mathbb{X} \subset \mathbb{E} \subset \mathbb{Y}$, and its local-finite G-cover $\sigma \in \mathrm{cov}\,\mathbb{E}$ consisting of closed G-subsets $\{\mathbb{E}_\gamma \subset \mathbb{E}\}_{\gamma \in \Gamma}$ such that for each $\gamma \in \Gamma$, $\mathbb{E}_\gamma \cap \mathbb{X} \neq \emptyset$ and the G-space $\mathbb{V}_\gamma \rightleftharpoons h(\mathbb{E}_\gamma)$ admits a nontrivial slice map $\alpha : \mathbb{V}_\gamma \to G/K$ for some $(K) \in \mathcal{C}$.*

Because of our aim – to extend h up to an equigeny defined on a G-neighborhood $\mathbb{Z}$, $\mathbb{X} \hookrightarrow \mathbb{Z} \subset \mathbb{Y}$, in what follows we can certainly assume that $\mathbb{E} = \mathbb{Y}$.

To have a possibility to argue by a new *transfinite* induction, we well order the set Γ indexing the elements of family $\{\mathbb{E}_\gamma\}$. Without loss of generality we can assume that Γ has the maximal element ω which is not limit. We put $\mathbb{Q}_\gamma \rightleftharpoons \mathbb{X} \cup \bigcup\{\mathbb{E}_{\gamma'} \mid \gamma' < \gamma\}$ for each limit ordinal γ, otherwise we set $\mathbb{Q}_\gamma \rightleftharpoons \mathbb{X} \cup \bigcup\{\mathbb{E}_{\gamma'} \mid \gamma' \leqslant \gamma\}$.

It is obvious that $\mathbb{Y} = \mathbb{Q}_\omega$ is the body of the increasing system of closed subsets $\{\mathbb{Q}_\gamma\}$, moreover $\mathbb{Q}_{\gamma'} \cup \mathbb{E}_\gamma = \mathbb{Q}_\gamma$ for $\gamma = \gamma' + 1$. As $\sigma \in \mathrm{cov}\,\mathbb{Y}$ is local finite, the following property of $\mathbb{Q}_\gamma$ holds (see [16, Section 2 of Introduction]):

(1) if the ordinal γ is limit, then $U \subset \mathbb{Q}_\gamma$ is open if and only if $(\mathbb{Q}_{\gamma'}) \cap U$ is open in $\mathbb{Q}_{\gamma'}$ for all $\gamma' < \gamma$ (or equivalently, $\mathbb{Q}_\gamma$ coincides with the limit $\varinjlim\{\mathbb{Q}_{\gamma'} \mid \gamma' < \gamma\}$ of the direct spectrum).

Before proceeding further, we generate some notations. For each $\gamma \in \Gamma$ we choose a closed neighborhood $\mathbb{P}_\gamma$, $\mathbb{X} \subset \mathbb{P}_\gamma \subset \mathbb{Q}_\gamma$, such that

(2) $\mathbb{P}_{\gamma'} \subset \mathbb{P}_\gamma$ for all $\gamma' < \gamma$;
(3) $\mathbb{P}_{\gamma+1} \setminus \mathbb{P}_\gamma \subset \mathbb{E}_\gamma$.

In particular, the closed G-neighborhood $\mathbb{Z} \rightleftharpoons \mathbb{P}_\omega$ of $\mathbb{X}$ in $\mathbb{Y}$ is the limit $\varinjlim\{\mathbb{P}_{\gamma'} \mid \gamma' < \omega\}$ of the direct spectrum. Since of (3) and local finiteness of $\{\mathbb{E}_\gamma\}$ we have

(4) for each limit ordinal γ, $\mathbb{P}_\gamma$ coincides with the direct limit $\varinjlim\{\mathbb{P}_{\gamma'} \mid \gamma' < \gamma\}$.

By transfinite induction we specify closed neighborhoods $\{\mathbb{P}_\gamma\}$ and construct for each $\gamma \in \Gamma$ a closed G-embedding $\mathbb{X}' \hookrightarrow \mathbb{P}'_\gamma$ and an equigeny $H_\gamma : \mathbb{P}_\gamma \to \mathbb{P}'_\gamma$ extending h such that

$(5)_\gamma$ $\mathbb{P}'_{\gamma_1} \subset \mathbb{P}'_\gamma$ and $H_\gamma \restriction_{\mathbb{P}_{\gamma_1}} = H_{\gamma_1}$ for all $\gamma_1 < \gamma$.

We set $\mathbb{Z}' \rightleftharpoons \mathbb{P}'_\omega$. It follows by (5) that for limit ordinal γ, $\mathbb{P}'_\gamma$ is the limit $\varinjlim\{\mathbb{P}'_{\gamma_1} \mid \gamma_1 < \gamma\}$ of the direct spectrum (in particular, $\mathbb{Z}' = \varinjlim\{\mathbb{P}'_\gamma \mid \gamma < \omega\}$) and the continuous map $H = \varinjlim\{H_\gamma\} : \varinjlim\{\mathbb{P}_\gamma \mid \gamma < \omega\} \to \mathbb{Z}' = \varinjlim\{\mathbb{P}'_\gamma \mid \gamma < \omega\}$ of limits of the direct spectra is the equigeny. Since $\mathbb{Z} \subset \mathbb{Y}$ is the limit $\varinjlim\{\mathbb{P}_\gamma \mid \gamma < \omega\}$ of the direct spectrum, H is the required equigeny extending h, that leads to the completion of the **proof of Theorem 4.1**.

Let a topological space (D, τ_D) represent as a union $A \cup B$ of its subspaces. We consider a *weak topology τ_w on D, generated by A and B*: $U \in \tau_w$ if and only if $A \cap U \subset A$ and $B \cap U \subset B$ are open. The subspaces *A and B generate the topology of D* if the weak topology τ_w coincides with τ_D. It is known [16] that

(6) the subspaces A and B generate the topology of D in the case that A and B are closed in D.

The **base** of the inductive argument is easily established with the help of Lemma 5.1. Let $\gamma_0 \in \Gamma$ be a minimal ordinal.

Lemma 5.3. *There exist a closed G-embedding $\mathbb{X}' \hookrightarrow \mathbb{P}'_{\gamma_0}$ and an equigeny $H_{\gamma_0} : \mathbb{P}_{\gamma_0} \to \mathbb{P}'_{\gamma_0}$ extending h.*

Proof. Since the G-space $\mathbb{V}_{\gamma_0} = h(\mathbb{E}_{\gamma_0} \cap \mathbb{X})$ admits a nontrivial slice map, Lemma 5.1 implies the existence of a closed G-embedding $\mathbb{V}_{\gamma_0} \hookrightarrow \mathbb{E}'_{\gamma_0}$ and an equigeny $\eta : \mathbb{E}_{\gamma_0} \to \mathbb{E}'_{\gamma_0}$ extending h.

Let $\mathbb{P}_{\gamma_0} \rightleftharpoons \mathbb{X} \cup \mathbb{E}_{\gamma_0} \subset \mathbb{Y}$. Since $\mathbb{X}$ and $\mathbb{E}_{\gamma_0}$ are simultaneously closed in $\mathbb{P}_{\gamma_0}$, the topology of $\mathbb{P}_{\gamma_0}$ coincides with a weak topology generated by $\mathbb{X}$ and $\mathbb{E}_{\gamma_0}$. Let us consider a union $\mathbb{X}' \cup_{\mathbb{V}_{\gamma_0}} \mathbb{E}'_{\gamma_0}$ with a weak topology which we denote by $\mathbb{P}'_{\gamma_0}$. It is now well understood that the G-map $H_{\gamma_0} \rightleftharpoons \mathrm{Id} \cup \eta : \mathbb{P}_{\gamma_0} = \mathbb{X} \cup \mathbb{E}_{\gamma_0} \to \mathbb{P}'_{\gamma_0} = \mathbb{X}' \cup \mathbb{E}_{\gamma_0}$ defined as Id on $\mathbb{X}$ and η on $\mathbb{E}_{\gamma_0}$ is required. □

The **inductive step** consists of the following proposition. Let $\gamma \in \Gamma$ be successive for $\gamma_1 \in \Gamma$, i.e. $\gamma = \gamma_1 + 1$.

Lemma 5.4. *There exist a closed G-embedding $\mathbb{X}' \hookrightarrow \mathbb{P}'_\gamma$ and an equigeny $H_\gamma : \mathbb{P}_\gamma \to \mathbb{P}'_\gamma$ extending h such that the condition $(5)_\gamma$ holds.*

Proof. By Lemma 5.2 $\mathbb{V}_\gamma$ admits a nontrivial slice map $\varphi : \mathbb{V}_\gamma \to G/K$, $(K) \in \mathcal{C}$. The Slice theorem ($\equiv G/K \in G$-ANE) implies that

(7) the partial G-map $\mathbb{P}'_{\gamma_1} \hookleftarrow \mathbb{V}_\gamma \xrightarrow{\varphi} G/K$ can be extended up to a slice map $\psi : \mathbb{U}' \to G/K$ defined into a closed neighborhood $\mathbb{U}'$, $\mathbb{V}_\gamma \subset \mathbb{U}' \subset \mathbb{P}'_{\gamma_1}$.

Let $\varphi \leftrightharpoons \psi \circ h_{\gamma_1} : \mathbb{U} \to G/K$ be a slice map defined into $\mathbb{U} \leftrightharpoons H^{-1}_{\gamma_1}(\mathbb{U}')$, $\hat{\mathbb{U}} \subset \mathbb{P}_{\gamma_1} \cup \mathbb{E}_\gamma$ be a closed neighborhood of $\mathbb{E}_\gamma \cap \mathbb{X}$ such that $\hat{\mathbb{U}} \cap \mathbb{P}_{\gamma_1} = \mathbb{U}$.

By Lemma 5.1 there exist a closed G-embedding $\mathbb{U}' \hookrightarrow \hat{\mathbb{U}}'$ and an equigeny $\eta : \hat{\mathbb{U}} \to \hat{\mathbb{U}}'$ extending H_{γ_1}.

It is easy to check that the desired G-space $\mathbb{P}'_\gamma$ is the union $\mathbb{P}'_{\gamma_1} \cup_{\mathbb{U}'} \hat{\mathbb{U}}'$ endowed with a weak topology. It is evidently that for $\mathbb{P}_\gamma \leftrightharpoons \mathbb{P}_{\gamma_1} \cup \hat{\mathbb{U}}$ we have $\mathbb{P}_\gamma \setminus \mathbb{P}_{\gamma_1} \subset \mathbb{E}_\gamma$.

The desired G-map $H_\gamma : \mathbb{P}_\gamma = \mathbb{P}_{\gamma_1} \cup \hat{\mathbb{U}} \to \mathbb{P}'_\gamma = \mathbb{P}'_{\gamma_1} \cup_{\mathbb{U}'} \hat{\mathbb{U}}'$ coincides with H_{γ_1} on $\mathbb{P}_{\gamma_1}$ and coincides with η on $\hat{\mathbb{U}}$. Since the topology of $\mathbb{P}_\gamma$ is generated by $\mathbb{P}'_{\gamma_1}$ and $\hat{\mathbb{U}}$, H_γ is the required equigeny extending H_{γ_1}. □

Now let $\gamma \in \Gamma$ be a limit ordinal. We consider the increasing family of constructed G-subspaces $\{\mathbb{P}'_{\gamma'} \subset \mathbb{P}'_{\gamma''}\}_{\gamma' \leqslant \gamma'' < \gamma}$ and the family of equigenies $\{H_{\gamma'} : \mathbb{P}_{\gamma'} \to \mathbb{P}'_{\gamma'}\}_{\gamma' < \gamma}$. Furthermore, we set $\mathbb{P}'_\gamma$ taken as $\varinjlim\{\mathbb{P}'_{\gamma'} \mid \gamma' < \gamma\}$, and $H_\gamma : \mathbb{P}_\gamma \to \mathbb{P}'_\gamma$ taken as $H_{\gamma'}$ on $\mathbb{P}_{\gamma'}$ for all $\gamma' < \gamma$. In view of (3) and local finiteness of $\{\mathbb{E}_\gamma\}$, $\mathbb{P}_\gamma = \varinjlim\{\mathbb{P}_{\gamma'} \mid \gamma < \gamma\}$ is a closed neighborhood of $\mathbb{X}$ in $\mathbb{Q}_\gamma$ and, therefore, $H_\gamma : \mathbb{P}_\gamma \to \mathbb{P}'_\gamma$ is a continuous equigeny. Since $\mathbb{X} \hookrightarrow \mathbb{P}'_\gamma$ is a closed embedding, $\mathbb{P}_{\gamma'} \subset \mathbb{P}_\gamma$ and $H_{\gamma'} = H_\gamma \restriction_{\mathbb{P}_{\gamma'}}$ for all $\gamma' < \gamma$, the proof of Theorem 4.1 is completed.

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References

[1] H. Abels, A universal proper G-space, Math. Z. 159 (1978) 143–158.
[2] S.M. Ageev, An equivariant Dugundji theorem, Uspekhi Mat. Nauk 45 (5) (1990) 179–180; English transl. in: Russian Math. Surveys 45 (5) (1990) 219–220.
[3] S.M. Ageev, Classification of G-spaces, Izv. Ross. Akad. Nauk Ser. Mat. 56 (6) (1992) 1345–1357; English transl. in: Russian Acad. Sci. Izv. Math. 41 (3) (1993) 581–591.
[4] S.M. Ageev, On equivariant homotopy type, Latv. Univ. Zinat. Raksti 576 (1992) 37–44.
[5] S.M. Ageev, Characterization of G-spaces, Doctoral thesis, Moscow State University, Moscow, 1996 (in Russian).
[6] S.M. Ageev, Isovariant absolute extensors, Sb. Math., submitted for publication.
[7] S.M. Ageev, D. Repovš, On extension of group actions, Mat. Sb. 201 (2) (2010) 3–29; English transl. in: Sb. Math. 201 (2) (2010) 159–182.
[8] G.E. Bredon, Introduction to Compact Transformation Groups, Pure Appl. Math., vol. 46, Academic Press, New York, 1972.
[9] T. tom Dieck, Transformation Groups, Walter de Gruyter, Berlin, 1987.
[10] J. Dugundji, Absolute neighborhood retracts and local connectedness in arbitrary metric spaces, Compos. Math. 13 (1958) 229–246.
[11] R. Engelking, General Topology, PWN, Warsaw, 1977.
[12] T. Matumoto, Equivariant K-theory and Fredholm operator, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 18 (1971) 109–125.
[13] M. Murayama, On G-ANRs and their G-homotopy types, Osaka J. Math. 20 (3) (1983) 479–512.
[14] R. Palais, The Classification of G-Spaces, Mem. Amer. Math. Soc., vol. 36, Amer. Math. Soc., Providence, RI, 1960.
[15] L.S. Pontryagin, Topological Groups, Gordon and Breach, New York, 1966.
[16] E.H. Spanier, Algebraic Topology, McGraw–Hill, New York, 1966.