Mathematical Notes, Vol. 65, No. 6, 1999

BRIEF COMMUNICATIONS

Fine Homotopy Equivalence and Injection

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KEY WORDS: fine homotopy injection of metric spaces, fine homotopy equivalence of metric spaces, hereditary shape equivalence of metric spaces, maps close with respect to a cover, maps homotopic with respect to a cover.

This paper is concerned with the problem of the coincidence between the classes of fine homotopy injections and equivalences of metric spaces, which was stated by Ancel in [1]. Such maps and relations between them are of importance when we deal with the problems of raising dimensions of spaces and the preservation of the property of being an absolute neighborhood extensor under cell-like maps (CE-maps). The classical result obtained by Kozlowski in [2] gives a fairly precise description of the situation in which these problems have positive solutions.

Theorem 1. *If* $f\colon X \to Y$ *is a* CE*-map of compact metric spaces and* $X \in \mathrm{ANE}$*, then the following conditions are equivalent:*

(1) f *is a hereditary shape equivalence;*
(2) f *is a fine homotopy injection;*
(3) f *is a fine homotopy equivalence;*
(4) $Y \in \mathrm{ANE}$.

Under each of these conditions, $\dim Y \le \dim X$.

Let us also mention an important result of Ancel [1], who proved that, if the set N_f of nondegenerate points of a CE-map f is an infinite-dimensional G_δ-set in Y, then f satisfies all the assumptions of Theorem 1 and therefore, $\dim Y \le \dim X$.

It was proved that, when any of conditions (1)–(4) in the statement of Theorem 1 is violated, there exists a counterexample. In [3], Dranishnikov constructed an example of a CE-map f of a finite-dimensional compact AE space onto an infinite-dimensional one; therefore, this map is not a hereditary shape equivalence.

The classes of maps under consideration form a decreasing sequence: the class of hereditary shape equivalences contains the class of fine homotopy injections, which, in its turn, contains the class of fine homotopy equivalences. The first two classes are known to be different; the coincidence between the classes of fine homotopy injections and equivalences of metric spaces is known for X and Y with sufficiently good extensor properties, e.g., if X or Y is an absolute neighborhood extensor. This paper answers Ancel's question affirmatively for the case in which both X and Y have bad connectedness structure.

Theorem 2. *Let* $f\colon X \to Y$ *be a map of metric spaces. Suppose that, for any cover* $\sigma \in \operatorname{cov} Y$*, there exists a map* $g\colon Y \to X$ *such that*

(a) Id_Y *is* σ*-close to the composition* $Y \xrightarrow{g} X \xrightarrow{f} Y$;
(b) Id_X *is* $f^{-1}(\sigma)$*-homotopic to the composition* $X \xrightarrow{f} Y \xrightarrow{g} X$;
(c) $f \simeq f \circ g \circ f [\operatorname{rel} \sigma]$.

Then the map f *is a fine homotopy equivalence provided that* X *is complete with respect to some metric* d.

Translated from *Matematicheskie Zametki*, Vol. 65, No. 6, pp. 921–924, June, 1999.
Original article submitted February 12, 1998.

 0001-4346/99/6556-0770 $22.00

Corollary 1. *Suppose that a map $f\colon X \to Y$ of metric spaces is a fine homotopy injection and X is complete. Then f is a fine homotopy equivalence.*

Corollary 2. *Suppose that a map $f\colon X \to Y$ of metric spaces is a hereditary shape equivalence and X is an* ANE*-space. Then f is a fine homotopy equivalence and Y is an* ANE *space.*

Problem. Is Corollary 1 valid if the completeness of X is not assumed?

Preliminaries. The set of all open covers of a space X is denoted by $\operatorname{cov} X$; $\omega \in \operatorname{cov} X$ is an open cover of X. We write $\operatorname{cal}(\omega)$ or $\operatorname{mesh}(\omega)$ to denote $\sup\{\operatorname{diam} U \mid U \in \omega\}$. The *star* of $\omega \in \operatorname{cov} X$ about a set $A \subset X$ (or the *neighborhood* of A relative $\omega \in \operatorname{cov} X$) is the set $\bigcup\{U \mid U \in \omega,\ U \cap A \neq \varnothing\}$; we denote it by $\mathrm{N}(A,\omega)$ or $\mathrm{St}(A,\omega)$. The *star enlargement of a cover ω with respect to another cover ω'* is $\mathrm{St}(\omega,\omega') = \{\mathrm{St}(U,\omega') \mid U \in \omega\}$. We denote multiple star enlargements $\mathrm{St}(\omega_1, \mathrm{St}(\omega_2, \dots, \omega_n)\dots)$ by $\omega_n \circ \dots \circ \omega_2 \circ \omega_1$ or, if all ω_i coincide, by $(\omega_1)^k$. The *body of a system ω* of open sets is the set $\bigcup\{U \mid U \in \omega\}$ denoted by $\cup\omega$.

As usual, $\omega \succ \omega_1$ means that ω is a refinement of ω_1. For maps $f, g\colon X \to Y$ and a cover $\omega \in \operatorname{cov} Y$, we write $(f,g) \prec \omega$ to denote that f and g are ω-close. Maps f and g are called *ω-homotopic* ($f \simeq g[\operatorname{rel}\omega]$ for short) if there exists a homotopy $F\colon X \times [0,1] \to Y$ between f and g such that $\{F(x,[0,1]) \mid x \in X\} \succ \omega$.

Definition 1. A map $f\colon X \to Y$ of metric spaces is called a *hereditary shape equivalence* if, for any closed subset $A \subset Y$, the map $f\colon f^{-1}A \to A$ is a shape equivalence.

Definition 2. A map $f\colon X \to Y$ of metric spaces is called a *fine homotopy injection* if, for any cover $\sigma \in \operatorname{cov} Y$, there exists a map $g\colon Y \to X$ such that Id_X is $f^{-1}(\sigma)$-homotopic to the composition $X \xrightarrow{f} Y \xrightarrow{g} X$; if, in addition, Id_Y is σ-homotopic to the composition $Y \xrightarrow{g} X \xrightarrow{f} Y$, then f is a *fine homotopy equivalence.*

The proof of the following lemma is a technical matter, and we omit it.

Lemma 1. *Suppose that $\sigma_i \in \operatorname{cov} Y$ with $i \geq 0$ are locally finite covers of Y and $(\sigma_{i+1})^{k+1} \succ \sigma_i$. Then, for all l such that $1 \leq l \leq N$, we have $\sigma_l^k \circ \sigma_{l+1}^k \circ \sigma_{l+2}^k \circ \dots \circ \sigma_N^k \succ \sigma_l^{k+1}$.*

Proof of Theorem 2. Let us take an arbitrary cover $\omega \in Y$ and construct a map $h\colon Y \to X$ such that Id_Y is ω-homotopic to the composition $Y \xrightarrow{h} X \xrightarrow{f} Y$ and Id_X is $f^{-1}(\omega)$-homotopic to the composition $X \xrightarrow{f} Y \xrightarrow{h} X$. We construct h in the form of an *infinite composition of maps.*

For this purpose, we choose a cover $\sigma_1 \in \operatorname{cov} Y$ for which $(\sigma_1)^3 \succ \omega$ and $\operatorname{cal}(\sigma_1) < 2^{-1}$. By assumption, we can find a map $g_1\colon Y \to X$ such that

$$(\mathrm{Id}_Y, f \circ g_1) \prec \sigma_1, \qquad \mathrm{Id}_X \simeq g_1 \circ f[\operatorname{rel} f^{-1}(\sigma_1)],$$

and therefore, $f \simeq f \circ g_1 \circ f[\operatorname{rel}\sigma_1]$.

Take a cover $\sigma_2 \in \operatorname{cov} Y$ such that

$$(\sigma_2)^3 \succ \sigma_1, \quad \operatorname{cal}(\sigma_2) < 2^{-2}, \quad \operatorname{cal}\bigl(f \circ g_1(\sigma_2)\bigr) < 2^{-2}, \quad \operatorname{cal}\bigl(g_1(\sigma_2)\bigr) < 2^{-2}, \quad g_1(\sigma_2) \succ f^{-1}(\sigma_1).$$

By assumption, there exists a map $g_2\colon Y \to X$ for which

$$(\mathrm{Id}_Y, f \circ g_2) \prec \sigma_2, \qquad \mathrm{Id}_Y \simeq g_2 \circ f[\operatorname{rel} f^{-1}(\sigma_2)],$$

and therefore, $f \simeq f \circ g_2 \circ f[\operatorname{rel}\sigma_2]$.

The maps g_1 and $g_1 \circ (f \circ g_2)$ are 2^{-2}-close, because $(\mathrm{Id}_Y, f \circ g_2) \prec \sigma_2$ and $\operatorname{cal}\bigl(g_1(\sigma_2)\bigr) < 2^{-2}$.

We have constructed a basis for the induction. To be more persuasive, let us give a detailed description of one more step. Take a cover $\sigma_3 \in \operatorname{cov} X$ such that

$$(\sigma_3)^3 \succ \sigma_2, \qquad \operatorname{cal}(\sigma_3) < 2^{-3}, \qquad \operatorname{cal}\bigl(f \circ g_2(\sigma_3)\bigr) + \operatorname{cal}\bigl(f \circ g_1 \circ (f \circ g_2)(\sigma_3)\bigr) < 2^{-3},$$
$$\operatorname{cal}\bigl(g_2(\sigma_3)\bigr) + \operatorname{cal}\bigl(g_1 \circ (f \circ g_2)(\sigma_3)\bigr) < 2^{-3}, \qquad \bigl(g_1 \circ (f \circ g_2)\bigr)(\sigma_3) \succ f^{-1}(\sigma_2).$$

By assumption, there exists a map $g_3\colon Y \to X$ for which

$$(\mathrm{Id}_Y, f\circ g_3) \prec \sigma_3, \qquad \mathrm{Id}_X \simeq g_3\circ f[\mathrm{rel}\, f^{-1}(\sigma_3)],$$

and therefore, $f \simeq f\circ g_3 \circ f[\mathrm{rel}\, f^{-1}(\sigma_3)]$. Note that

$$\big(g_2, g_2\circ(f\circ g_3)\big) < 2^{-3}, \quad \big(g_1\circ(f\circ g_2) \quad g_1\circ(f\circ g_2)\circ(f\circ g_3)\big) < 2^{-3},$$

because $(\mathrm{Id}_Y, f\circ\sigma_3) \prec \sigma_3$ and $\mathrm{cal}\big(g_2(\sigma_3)\big) + \mathrm{cal}\big(g_1\circ(f\circ g_2)(\sigma_3)\big) < 2^{-3}$.

Thus we successively construct maps $g_i\colon Y\to X$ and covers $\sigma_i \in \mathrm{cov}\, Y$, where $i = 1, 2, \dots$, such that

(1) $\sigma^3_{n+1} \succ \sigma_n$;
(2) $\mathrm{cal}(\sigma_{n+1}) < 2^{-(n+1)}$, $\mathrm{cal}(f\circ g_n)(\sigma_{n+1}) < 2^{-(n+1)}$, and $\mathrm{cal}(g_n)(\sigma_{n+1}) < 2^{-(n+1)}$;
(3) $\mathrm{cal}\big(g_m\circ(f\circ g_{m+1})\circ\cdots\circ(f\circ g_n)\big)(\sigma_{n+1}) < 2^{-(n+1)}$ for all $m < n$;
(4) $\mathrm{cal}\big((f\circ g_m)\circ(f\circ g_{m+1})\circ\cdots\circ(f\circ g_n)\big)(\sigma_{n+1}) < 2^{-(n+1)}$ for all $m < n$;
(5) $g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_n)(\sigma_{n+1}) \succ f^{-1}(\sigma_n)$;
(6) $(\mathrm{Id}_Y, f\circ g_n) \prec \sigma_n$;
(7) $\mathrm{Id}_X \simeq g_n\circ f[\mathrm{rel}\, f^{-1}(\sigma_n)]$ and $f \simeq f\circ g_n\circ f[\mathrm{rel}\,\sigma_n]$;
(8) $\big(g_n, g_n\circ(f\circ g_{n+1})\big) < 2^{-(n+1)}$, $\big(g_m\circ(f\circ g_{m+1})\circ\cdots\circ(f\circ g_{n+1}), g_m\circ(f\circ g_{m+1})\circ\cdots\circ(f\circ g_n)\big) < 2^{-(n+1)}$ for all $m < n$.

Take $m = 1, 2, \dots$ and consider the sequence $\{g_m\circ(f\circ g_{m+1})\circ\cdots\circ(f\circ g_n)\}_{n=m+1}^{\infty}$. This is a Cauchy sequence by (8), and it converges to a map $h_m\colon Y\to X$, because (X, d) is complete. By (6) and (4), the maps $(f\circ g_m)\circ\cdots\circ(f\circ g_{m+k})$ and $(f\circ g_m)\circ\cdots\circ(f\circ g_{m+k-1})$ are $2^{-(m+k)}$-close. The definition of h_m implies the estimate $(f\circ h_m, f\circ g_m) < 2^{-(m+1)} + 2^{-(m+2)} + \cdots = 2^{-m}$. Since $(f\circ g_m, \mathrm{Id}_X) \prec \sigma_m$ and $\mathrm{cal}(\sigma_m) < 2^{-m}$, we have $(f\circ h_m, \mathrm{Id}_X) < 2^{-(m+1)}$; thus $\lim(f\circ h_m) = \mathrm{Id}_X$. On the other hand, the obvious equality $h_m = (g_m\circ f)\circ h_{m+1}$ valid for $m \ge 1$ and relations (7) imply that

$$f\circ h_m = (f\circ g_m\circ f)\circ h_{m+1} \simeq f\circ h_{m+1}[\mathrm{rel}\,\sigma_m].$$

Therefore, $f\circ h_1 \simeq \lim(f\circ h_m) = \mathrm{Id}_X[\mathrm{rel}(\sigma_1\circ\sigma_2\circ\cdots \succ \omega)]$.

To show that $h_1\circ f = \lim_{n\to\infty}\big(g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_n)\circ f\big)$ is $f^{-1}(\omega)$-homotopic to the identity map Id_X, we apply the following relations, which are implied by (5) and (7):

$$\begin{aligned} g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_{n+1})\circ f &= g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_{n+1}\circ f) \\ &\simeq g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_n)\circ f\big[\mathrm{rel}\big(g_1\circ(f\circ g_2)\circ\cdots\circ(f\circ g_n)(\sigma_{n+1}) \succ f^{-1}(\sigma_n)\big)\big]. \end{aligned}$$

Thus $h_1\circ f$ is homotopic to $g_1\circ f$ with respect to the cover $f^{-1}(\sigma_1\circ\sigma_2\circ\sigma_3\circ\cdots)$. But (7) implies that $g_1\circ f \simeq \mathrm{Id}_X[\mathrm{rel}\, f^{-1}(\sigma_1)]$. Therefore, $h_1\circ f \simeq \mathrm{Id}_X\big[\mathrm{rel}\big(f^{-1}(\sigma_1^2\circ\sigma_2\circ\sigma_3\circ\cdots) \succ f^{-1}(\omega)\big)\big]$. □

The first author acknowledges the support of the INTAS foundation under grant No. 96-0712. The second author acknowledges the support of the Ministry of Science and Technologies of Slovenija under grant No. J1-7039-95.

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