Set-Valued Analysis **6:** 345–362, 1998.

A Selection Theorem for Strongly Regular Multivalued Mappings

SERGEI M. AGEEV
Department of Mathematics and Physics, Brest State Pedagogical Institute, 224665 Brest, Belorussia. e-mail: box@univer.belpak.brest.by

DUŠAN REPOVŠ
Institute for Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, P.O.B. 2964, 1001 Ljubljana, Slovenia. e-mail: dusan.repovs@fmf.uni-lj.si

(Received: 22 January 1997; in final form: 25 November 1998)

Abstract. We prove the following generalization of a theorem of Ferry concerning selections of strongly regular multivalued maps onto the class of paracompact spaces: Let $\Phi\colon X \to (Z, \rho)$ be a map of a paracompact space X into a metric space (Z, ρ) whose values $\Phi(x)$ are complete subspaces of Z and absolute extensors (AE), for every $x \in X$. Suppose that Φ can be represented as $\Phi = \Gamma \circ \varphi$, where $\varphi\colon X \to Y$ is a continuous singlevalued map of X onto some finite-dimensional paracompact space Y and $\Gamma\colon Y \to (Z, \rho)$ is a strongly regular map. Then for every closed subset $A \subset X$ and every selection $r\colon A \to Z$ of the map $\Phi|_A\colon A \to Z$, there exists an extension $\hat{r}\colon X \to Z$ of r such that $\hat{r}$ is a selection of the map Φ. We also prove a local version of this theorem.

Mathematics Subject Classifications (1991): Primary: 54C60, 54C65; Secondary: 54C55, 55C15.

Key words: strongly regular maps, selection of multivalued maps, absolute neighborhood extensors, extensions of selections.

1. Introduction

A classical theorem of Michael [3] asserts that every lower semicontinuous (lsc) map $\Phi\colon X \to Z$ of an $(n+1)$-dimensional paracompact space X into a complete metric space Z has a continuous selection, provided that all values $\Phi(x)$ are n-connected closed subsets and that the collection $\{\Phi(x) \mid x \in X\}$ of all values of Φ is an equi-LC^n family (we quote here the global version from [3]). The problem of finding purely topological conditions for the collection $\{\Phi(x) \mid x \in X\}$ which would guarantee the existence of selections of Φ in the case when X is infinite-dimensional, has been around ever since the theory of selections was originated in 1956. It remains unsolved and there are good reasons for the absence of such a solution [4].

In 1974 Pixley [6] generalized a construction of Borsuk and obtained an example of a lower semicontinuous map of the Hilbert cube into itself, all values of which are cubes (of various dimensions), the collection of all values has the property of uniform local absolute extendability (UE-LAE), but the map has no selection. It is natural in such a situation to strengthen the type of continuity of the map Φ and preserve good topological properties of the collection $\{\Phi(x) \mid x \in X\}$. Such an approach was already suggested by Pixley [6]. Namely, he proposed to either strengthen the UE-LAE property to the uniform Lefschetz property (UE-Lf), or to consider continuous instead of semicontinuous maps. The first approach was carried out for compact metric spaces and continuous maps by Moiseev [5].

In the present paper we make the first step in the other direction. Namely, we strengthen semicontinuity of Φ not only to continuity but to strong regular continuity of the map Φ, which roughly speaking, means that for close points $x, x' \in X$, their values $\Phi(x)$ and $\Phi(x')$ are homotopy equivalent and the homotopies do not significantly change the distance. This property plays a significant role in one of the main results of Ferry [1], to the effect that every strongly regular compact-valued mapping $\Phi\colon\ Y \to Q$ of a finite-dimensional separable metric space Y into the Hilbert cube Q, all of whose images $\Phi(y)$ of points $y \in Y$ are absolute extensors (AE), is a fiberwise retraction of the constant map $\Phi'\colon\ Y \to Q$, which sends every point $y \in Y$ onto the entire Q, $\Phi'(y) = Q$. This means that the restriction of some continuous map $r\colon\ Y \times Q \to Y \times Q$ onto $\{y\} \times Q$ is a retraction of $\{y\} \times \Phi(y)$. Therefore the selection properties of the maps Φ and Φ' are the same.

The main result of the present paper is a generalization of Ferry's theorem to the class of all paracompact spaces:

THEOREM 1.1. *Let $\Phi\colon\ X \to (Z, \rho)$ be a map of a paracompact space X into a metric space (Z, ρ) whose values $\Phi(x)$ are complete subspaces of Z and absolute extensors (AE), for every $x \in X$. Suppose that Φ can be represented as $\Phi = \Gamma \circ \varphi$, where $\varphi\colon\ X \to Y$ is a continuous singlevalued map of X onto some finite-dimensional paracompact space Y and $\Gamma\colon\ Y \to (Z, \rho)$ is a strongly regular map. Then for every closed subset $A \subset X$ and every selection $r\colon\ A \to Z$ of the map $\Phi|_A\colon\ A \to Z$, there exists an extension $\hat{r}\colon\ X \to Z$ of r such that $\hat{r}$ is a selection of Φ.*

We also prove the following local version of Theorem 1.1:

THEOREM 1.2. *Suppose that we replace the AE-condition in the hypotheses of Theorem* 1.1 *by the assumption that the values $\Phi(x)$ are absolute neighborhood extensors* (*ANE*). *Then the selection r can be extended locally, i.e., there exist an open set $U \supset A$ and an extension $\hat{r}\colon\ U \to Z$ such that $\hat{r}$ is a selection of the restriction $\Phi|_U\colon\ U \to Z$.*

Note that if the selection r can be factorized through a closed subset of Y, then the proofs of Theorems 1.1 and 1.2 are easily reduced to the finite-dimensional selection theorem.

A simple argument shows that under the hypotheses of Theorem 1.1, the multivalued map $\Phi\colon\ X \to (Z, \rho)$ is a fiberwise retraction of the constant map $\Phi'\colon\ X \to$

L, $\Phi'(x) = L$, of X onto a linear normed space L which contains (Z, ρ) as a closed subspace.

Our proof is based on the fact (cf. Theorem 3.2) that strong regularity of a multivalued map Φ with ANE-values implies the uniform super Lefschetz property of Φ. This allows for an application in the proof of a well-known method of extending partial δ-realizations of polyhedra to their full ε-realizations. The main technical tool of the proof is the notion of the *supercover*, which represents a family of the covers of one topological space, parametrized by the points of another topological space, with some additional properties.

We shall denote the family of all open covers of the space Z by $\mathrm{cov}(Z)$. Every point $z \in Z$ is contained in several elements of the cover $\{W_\mu\} \in \mathrm{cov}(Z)$. By the Axiom of choice there exists a mapping $z \in Z \mapsto W_\mu = W_{\mu(z)}$, such that $z \in W_\mu$. We shall call z the *center* of $W_{\mu(z)}$. In this way we arrive at the notion of the *centered* covers of Z, i.e., covers $w = \{W\} \in \mathrm{cov}(Z)$ with a fixed map of sets $z \in Z \mapsto W = W(z) \in w$, such that $z \in W(z)$. So a centered cover can be written as $\{W(z) \mid z \in W(z)$, for every $z \in Z\}$ and has the cardinality of the space Z. We shall denote the family of all centered covers of Z by $\mathrm{cov}_0(Z)$.

DEFINITION 1.3. (a) Let X be any set and Z any topological space. A map $\Delta\colon X \to \mathrm{cov}(Z)$, given by $\Delta(x) = \Delta_x$ is called an X-*cover* of the space Z.

(b) Let X and Z be any topological spaces and $w = \{W(x)\} \in \mathrm{cov}_0(X)$. An X-cover $\Delta\colon X \to \mathrm{cov}(Z)$ of the space Z is called an w-*cover* of Z if for every $x \in X$ and every $x' \in W(x)$, the cover Δ_x is a refinement of the cover $\Delta_{x'}$, $\Delta_x > \Delta_{x'}$.

(c) An X-cover of a space Z is called an X-*supercover* (or simply a *supercover* of Z when it is clear what X is) if it is an w-cover of Z, for some $w \in \mathrm{cov}_0(X)$.

Every open cover $\omega \in \mathrm{cov}(X \times Z)$ (where X is any topological space) generates an X-*cover*, defined by the following formula:

(1) $\Delta_x = \{U \subset Z \mid \{x\} \times U$ is the intersection of some element of ω with the fiber $\{x\} \times Z\}$.

In fact we can prove a more general proposition: Every X-cover is precisely the cover of the product $X \times Z$ (not necessarily open) whose elements are open subsets of the fibers $\{x\} \times Z$.

A *trivial* example of an X-*supercover* of a space Z is a map $\Delta\colon X \to \{\delta\}$, where δ is a fixed cover of Z. A less trivial example is the rectangle product $\omega = \{O_\lambda \times W_\mu\} \in \mathrm{cov}(X \times Z)$, satisfying the following condition:

(2) For every $x \in X$, there exists a neighborhood $O(x)$ such that $O(x) \subset O_\lambda$, whenever $x \in O_\lambda$.

It can easily be seen that the formula (2) generates a supercover $\Delta\colon X \to \mathrm{cov}(Z)$ of the space Z. Let us now consider the most significant example of a supercover, revealing the nature of this notion, which shall hereafter be called a *canonical* supercover.

EXAMPLE 1.4. For every Y-cover Π of a space Z and for every centered cover $w \in \mathrm{cov}_0(Y)$ of a paracompact space Y, there exists a *canonical* supercover Φ: $Y \to \mathrm{cov}(Z)$ of Z, with the following property:

(3) For every $y \in Y$, there exist an neighborhood $O(y)$ and an element $W(y_1) \in w$, such that $y \in W(y_1)$ and the cover Φ_{y_2} is a refinement of the cover Π_{y_1}, for every $y_2 \in O(y)$, i.e., $\Phi_{y_2} > \Pi_{y_1}$.

Construction. Let $w = \{W(y) \mid y \in Y\} \in \mathrm{cov}_0(Y)$ and let $w' = \{W'_\lambda \mid \lambda \in \Lambda\} \in \mathrm{cov}(Y)$ be a locally finite refinement of the cover w. Without loss of generality, we can assume that $\mathrm{cl}\, w' = \{\mathrm{cl}\, W'_\lambda \mid \lambda \in \Lambda\} > w$. For every $\lambda \in \Lambda$, fix a point $y(\lambda) \in Y$ such that $\mathrm{cl}\, W'_\lambda \subset W(y(\lambda))$.

Let

$$\Phi_y = \bigwedge \left\{\Pi_{y(\lambda)} \mid \lambda \in \Lambda,\ y \in \mathrm{cl}\, W'_\lambda \subset W(y(\lambda))\right\}.$$

Clearly, for every point y one considers in this equality only the intersection of finitely many covers $\Pi_{y(\lambda)}$. As usually, the intersection $\bigwedge_{i=1}^{n} w_i$ of finitely many covers $w_1, \ldots, w_n$ is the cover which consists of the intersections $\bigcap\{W_i \mid W_i \in w_i, i \leqslant n\}$ of elements of the covers w_i.

It is easy to see that Φ: $Y \to \mathrm{cov}(Z)$ is a supercover of the space Z with the property (3). Let $O(y) \subset W'_\lambda$ be a neighborhood, intersecting only those elements of $\mathrm{cl}\, w'$ which contain the point y. Then $\Phi_y > \Phi_{y_2}$ and $\Phi_{y_2} > \Pi_{y(\lambda)}$, for every $y_2 \in O(y)$. This supercover Δ will be called the *canonical supercover* induced by $y \mapsto \Pi_y$ and the cover $\{W(y)\}$.

Every continuous map φ: $X \to Y$ induces a map which transfers every Y-supercover Δ of Z into an X-supercover Δ^φ of Z via the formula $(\Delta^\varphi)_x = \Delta_{\varphi(x)}$, $x \in X$. For properties of X-supercovers when X is paracompact, see Section 3. In conclusion, we state a result which we shall need in the proofs of Theorems 1.1 and 1.2. It essentially allows for a construction of a genuine selection r' from a Δ-selection r, with a control of the distance of 'approximate' selections r from the 'exact' selections r' of the map Φ. This theorem is an analogue of [3, Theorem 4.1].

THEOREM 1.5. *Let φ: $X \to Y$, Γ: $Y \to (Z, \rho)$ and $\Phi = \Gamma \circ \varphi$ be as in the hypotheses of Theorem* 1.2. *Then for every Y-supercover E: $Y \to \mathrm{cov}(Z)$, there exists a Y-supercover Δ: $Y \to \mathrm{cov}(Z)$ of the space Z, such that for every (Δ^φ)-selection r: $X \to Z$ of the map Φ, there exists an exact selection r': $X \to Z$ of the map Φ which is (E^φ)-near r.*

If, in addition $\Phi(x) \in \mathrm{AE}$, for every $x \in X$, then for the trivial Y-supercover $E(y) = \{Z\}$, the Y-supercover Δ is also trivial and hence, there always exists an exact selection r': $X \to Z$.

In Theorem 1.5, (Δ^φ) and (E^φ) are X-supercovers of Z, induced by the map φ. Also, r is (E^φ)-near r' means that for every $x \in X$, the points $r(x)$ and $r'(x)$ lie in some common element of the cover E_x^φ, and a (Δ^φ)-selection means that for every $x \in X$, the point $r(x)$ lies in some element of the cover Δ_x^φ which intersects the set $\Phi(x)$.

2. Preliminaries

By $\mathrm{mesh}(w)$ we shall denote $\sup\{\mathrm{diam}\, U \mid U \in w\}$. The *star* of the set $A \subset X$ with respect to the cover $w \in \mathrm{cov}(X)$ is the set

$$\mathrm{st}(A, w) = \bigcup\{U \mid U \in w \text{ and } U \cap A \neq \emptyset\}.$$

The *star* of a cover w with respect to another cover w' is the cover

$$\mathrm{St}(w, w') = \{\mathrm{St}(U, w') \mid U \in w\}\,.$$

Multiple stars $\mathrm{St}(w_1, \mathrm{St}(w_2, \dots, (w_n), \dots)$ will be denoted by $w_n \circ \cdots \circ w_2 \circ w_1$ and if w_i are the same, then by $(w_1)^k$. The *body* of a system of open sets w is the set

$$\bigcup w = \bigcup\{U \mid U \in w\}\,.$$

As always, $w > w_1$ will mean that the cover w is a refinement of w_1. If $f, g\colon X \to Y$ are any maps and $w \in \mathrm{cov}(Y)$, then the property that f is w-*close* to g will be denoted by $\rho(f, g) < w$.

The *nerve* of a cover $w = \{U_\beta \mid \beta \in B\}$ is the polyhedron $\mathcal{N}\langle w\rangle$ in the Whitehead weak topology, whose vertices $\langle U_\beta\rangle$ are in one-to-one correspondence with the index set B, and $w = \langle U_{\beta_1}, \dots, U_{\beta_s}\rangle$ is an $(s-1)$-dimensional simplex of $\mathcal{N}\langle w\rangle$ with vertices $\langle U_{\beta_i}\rangle$ if and only if $\bigcap U_{\beta_i} \neq \emptyset$. Furthermore, the k-dimensional *skeleton* of the nerve $\mathcal{N}\langle w\rangle$ is the subpolyhedron $\mathcal{N}\langle w\rangle^{(k)}$ of $\mathcal{N}\langle w\rangle$, consisting of at most k-dimensional simplices. Finally, $\mathcal{N}\langle w\rangle^{(-1)} = \emptyset$.

If the image $f(\sigma)$ of a cover σ under the map $f\colon A \to B$ is a refinement of the cover w, then a simplicial map $\pi(\sigma, w)\colon \mathcal{N}\langle\sigma\rangle \to \mathcal{N}\langle w\rangle$ is defined, taking every vertex $\langle H\rangle \in \mathcal{N}\langle\sigma\rangle$ into a vertex $\langle U\rangle \in \mathcal{N}\langle w\rangle$ such that $f(H) \subset U$. We shall say that the map π is *induced* by the relation $f(\sigma) > w$. A map $\vartheta\colon X \to \mathcal{N}\langle w\rangle$ is said to be *canonical* if the preimage $\vartheta^{-1}(\overset{\circ}{\mathrm{St}}\langle U\rangle)$ of every open star $\overset{\circ}{\mathrm{St}}\langle U\rangle$ of the element $U \in w$, is contained in U. It is well known that for every open cover w of a paracompact space X there exists a canonical map [2].

PROPOSITION 2.1. *Let $\varphi\colon X \to Y$ be a map between paracompact spaces X and Y, and suppose that* $\dim Y \leqslant n$. *Then for every cover $w \in \mathrm{cov}(Y)$, there exists a cover $\tau \in \mathrm{cov}(X)$ with $\varphi(\tau) > w$ and a simplicial map $\pi\colon \mathcal{N}\langle\tau\rangle \to \mathcal{N}\langle w\rangle$, induced by the relation $\varphi(\tau) > w$, such that for every canonical map $\vartheta\colon X \to \mathcal{N}\langle\tau\rangle$ the image $\pi \circ \vartheta(X) \subset \mathcal{N}\langle w\rangle^{(n)}$ (or equivalently, $\vartheta(X) \subset \pi^{-1}(\mathcal{N}\langle w\rangle^{(n)})$).*

Proof. Since by hypothesis, $\dim Y \leqslant n$, one can construct a refinement $w' \in \mathrm{cov}(Y)$ of w of order at most $n+1$. We take for τ the cover $\varphi^{-1}(w')$ and for π the composition

$$\mathcal{N}\langle\tau\rangle \xrightarrow{\pi_1} \mathcal{N}\langle w'\rangle \xrightarrow{\pi_2} \mathcal{N}\langle w\rangle$$

of simplicial maps, induced by the relations $\varphi(\tau) > w'$ and $w' > w$, respectively. Since the nerve $\mathcal{N}\langle w'\rangle$ coincides with the skeleton $\mathcal{N}\langle w'\rangle^{(n)}$ it follows that

$$(\pi \circ \vartheta)(X) = \pi_2(\pi_1(\vartheta(X))) \subset \pi_2(\mathcal{N}\langle w'\rangle^{(n)}) \subset \mathcal{N}\langle w\rangle^{(n)}. \qquad \square$$

We shall assume throughout the paper that all single-valued maps are continuous, unless they arise as a result of some special constructions – in which case we shall separately check whether the property of continuity holds or not.

Next, we introduce some notions from the multivalued analysis. A multivalued map $\Phi\colon X \to Z$ is said to be *closed-valued* (resp. *compact-valued*, *complete-valued*) if the image $\Phi(x)$ of every point $x \in X$ is a closed (resp. compact, complete) subset of Z. A multivalued map $\Phi\colon X \to Z$ is said to be *surjective* if $\Phi(X) = \bigcup\{\Phi(x) \mid x \in X\}$ coincides with Z.

A singlevalued map $r\colon X \to Z$ is said to be a *selection* of a map $\Phi\colon X \to Z$ if $r(x) \in \Phi(x)$, for every point $x \in X$. A multivalued map $\Phi\colon X \to Z$ is said to be *continuous* if for every point $x_0 \in X$ and every cover $\varepsilon \in \operatorname{cov}(Z_0)$ of the set $Z_0 = \Phi(X)$, there exists a neighborhood $U(x_0) \subset X$, such that for every pair of points $a, b \in U(x_0)$, it follows that $\Phi(a) \subset \operatorname{St}(\Phi(b), \varepsilon)$.

A multivalued map $\Phi\colon X \to Z$ is said to be *strongly regular* if for every point $x_0 \in X$ and every cover $\varepsilon \in \operatorname{cov}(Z_0)$ of $Z_0 = \Phi(X)$, there exists a neighborhood $U(x_0) \subset X$ such that for every point $x' \in U(x_0)$, there exist maps $g\colon \Phi(x_0) \to \Phi(x')$, $f\colon \Phi(x') \to \Phi(x_0)$ and homotopies $h_t\colon \Phi(x_0) \to \Phi(x_0)$, $k_t\colon \Phi(x') \to \Phi(x')$ with the following properties:

(i) $\rho(g, \mathrm{Id}_{\Phi(x_0)}) < \varepsilon$ and $\rho(f, \mathrm{Id}_{\Phi(x')}) < \varepsilon$;
(ii) For every $t \in [0, 1]$, $\rho(h_t, \mathrm{Id}_{\Phi(x_0)}) < \varepsilon$ and $\rho(k_t, \mathrm{Id}_{\Phi(x')}) < \varepsilon$; and
(iii) $h_0 = f \circ g$, $k_0 = g \circ f$, $h_1 = \mathrm{Id}_{\Phi(x_0)}$ and $k_1 = \mathrm{Id}_{\Phi(x')}$.

Note that every strongly regular map is continuous. It is also clear that $\Phi\colon X \to Z$ is strongly regular (resp. continuous) if and only if $\Phi\colon X \to \Phi(X)$ is strongly regular (resp. continuous). For the statements and proofs of more important facts on strongly regular maps, we need to introduce some concepts connected with the name of Lefschetz.

DEFINITION 2.2. Let α be a system of open subsets of a space Z and let $\mathcal{N}_0$ be a subpolyhedron of the polyhedron $\mathcal{N}$, containing all vertices. A *partial α-realization* of the polyhedron $\mathcal{N}$ is a map $\mathcal{N}_0 \xrightarrow{f} Z$ such that for every simplex $\Delta \in \mathcal{N}$, the set $f(\Delta \cap \mathcal{N}_0)$ is contained in some element $V \in \alpha$.

DEFINITION 2.3. A family $\mathcal{G} = \{Z_\alpha\}$ of closed subsets of a metric space Z is said to have the *uniform Lefschetz* property (*equi-Lf*), provided that for every cover $\delta \in \operatorname{cov}(\bigcup \mathcal{G})$ of the body of the family $\mathcal{G}$, there exists a cover $\gamma \in \operatorname{cov}(\bigcup \mathcal{G})$ such that for every set Z_α, every partial γ-realization $\mathcal{N} \subset \mathcal{N}_0 \xrightarrow{\xi} Z_\alpha$ of any polyhedron $\mathcal{N}$ can be extended to a full δ-realization $\mathcal{N} \xrightarrow{\xi} Z_\alpha$.

Remarks. (1) If the body $\bigcup \mathcal{G}$ is closed in Z, then instead of a cover of the body $\bigcup \mathcal{G}$ in Definition 2.3 one must take a cover of the space Z itself.

(2) The dependence of γ on δ will be denoted by $\gamma = \text{equi-Lf}_{\mathcal{G}}(\delta)$. If the family $\mathcal{G}$ consists of only one element Z_0, then we shall write $\gamma = \mathrm{Lf}_{Z_0}(\delta)$.

DEFINITION 2.4. Topological space Z is called an *absolute* [*neighborhood*] *extensor*, if every continuous map $\varphi\colon A \to Z$, defined on a closed subspace A of a metric space M, can be continuously extended on the whole space M [on some neighborhood of A]. A class of all absolute [neighborhood] extensors is denoted by A[N]E.

It is known that the class of absolute [neighborhood] retracts (A[N]R) is contained in the class of A[N]E and these two classes coincide for metric spaces [2]. Another well-known fact deals with a coincidence of ANE's and spaces with Lefschetz's property: A metric space Z is an ANE if and only if for every $\delta \in \mathrm{cov}(Z)$, there exists $\gamma \in \mathrm{cov}(Z)$ such that $\gamma = \mathrm{Lf}_Z(\delta)$ (see [2]).

DEFINITION 2.5. A closed-valued map $\Phi\colon X \to Z$ is said to be *uniformly Lefschetz* if the family $\{\Phi(x) \mid x \in X\}$ of its values has the equi-Lf property.

It is clear, that the uniform Lefschetz property implies uniform local absolute extendability and the equi-LC^∞ property of the family $\{\Phi(x) \mid x \in X\}$ (cf. [2, 6]).

3. Uniformly Super Lefschetz M-Maps

Note that the concept of a uniform Lefschetz property is very useful. However, it is more restrictive than the notion of *strong* regularity. In order to be able to compare them with respect to their strength and usefulness, we give below a modified version of Definition 2.5 in which covers have been replaced by supercovers.

DEFINITION 3.1. A closed map $\Phi\colon X \to Z$ is said to be *uniformly super Lefschetz* if the family of its values $\{\Phi(x) \mid x \in X\}$ has the following property:

(A) For every X-supercover $E\colon X \to Z$ of the space Z there exists an X-supercover $\Delta\colon X \to Z$ such that for every point $x \in X$ and every partial Δ_x-realization $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x)$ of any polyhedron $\mathcal{N}$, there exists its extension to a full E_x-realization $\mathcal{N} \xrightarrow{\zeta} \Phi(x)$.

Remark. The dependence of the supercover Δ on the supercover E will be denoted by $\Delta = (A)(E)$.

THEOREM 3.2. *If $\Phi\colon X \to Z$ is a strongly regular multivalued map of paracompact space X into a metric space Z, the images $\Phi(x)$ of all points $x \in X$ are ANE's, and the image $\Phi(X)$ of the entire space X is Z, then the multivalued map Φ is uniformly super Lefschetz.*

Remark. In fact, the uniform super Lefschetz property is equivalent to the strong regularity of the M-map Φ. However, we shall not need this fact.

The proof of Theorem 3.2 is based on the following proposition:

PROPOSITION 3.3. *Under the hypotheses of Theorem* 3.2, *for every point $x_0 \in X$ and every cover $\varepsilon \in \operatorname{cov}(Z)$, there exist a cover $\delta \in \operatorname{cov}(Z)$ and a neighborhood $U(x_0) \subset X$ such that:*

(4) *For every point $x' \in U(x_0)$ and every partial δ-realization $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x')$ of any polyhedron $\mathcal{N}$, there exists its extension to a full ε-realization $\mathcal{N} \xrightarrow{\zeta} \Phi(x')$.*

Proof. Let $\varepsilon_1, \varepsilon_2$, and δ be covers such that $(\varepsilon_1)^3 > \varepsilon$, $\varepsilon_2 = \mathrm{Lf}_{\Phi(x_0)}(\varepsilon_1)$, and $(\delta)^3 > \varepsilon_2$. Then the cover δ is the desired one. Let us verify that. Since Φ is a strongly regular map it follows that for every point $x_0 \in X$ and every cover δ, there exists its neighborhood $U(x_0) \subset X$ for which the conditions of strong regularity are satisfied.

Let $x' \in U(x_0)$ and $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x')$ be a partial δ-realization of an arbitrary polyhedron $\mathcal{N}$. Then $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x') \xrightarrow{f} \Phi(x_0)$ is a partial (ε_2)-realization of the polyhedron $\mathcal{N}$. Since $\varepsilon_2 = \mathrm{Lf}_{\Phi(x_0)}(\varepsilon_1)$, there exists a full (ε_1)-realization $\mathcal{N} \xrightarrow{\tilde{\zeta}} \Phi(x')$ of the polyhedron $\mathcal{N}$. Therefore $\zeta = g \circ \tilde{\zeta}$ will be the desired ε-realization $\mathcal{N} \xrightarrow{\zeta} \Phi(x')$ of the polyhedron $\mathcal{N}$. □

Proof of Theorem 3.2. Let $w = \{U(x) \mid x \in X\} \in \operatorname{cov}(X)$ be a cover such that for every point $x' \in U(x)$, E_x is a refinement of $E_{x'}$. Apply Proposition 3.3 for the point x and the cover $E_x \in \operatorname{cov}(Z)$. We obtain a neighborhood $\mathcal{O}(x) \subset U(x)$ and a cover $\delta_x \in \operatorname{cov}(Z)$ which satisfy the property (4) above.

Consider the canonical supercover $\Delta\colon X \mapsto \operatorname{cov}(Z)$, $\Delta(x) = \Delta_x$, induced by $x \mapsto \delta_x$, and the cover $\{\mathcal{O}(x)\}$. It follows by property (3) from Example 1.4 that:

(5) For every $x \in X$, there exists a neighborhood $\mathcal{O}(x')$, $x \in \mathcal{O}(x')$, such that $\Delta_x > \delta_{x'}$.

Let $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x)$ be a partial Δ_x-realization of an arbitrary polyhedron $\mathcal{N}$. It follows from (5) that ξ is a partial $\delta_{x'}$-realization. It follows from (4) that there is a full $E_{x'}$-realization $\mathcal{N} \xrightarrow{\zeta} \Phi(x)$ such that $\zeta|_{\mathcal{M}} = \xi$. Observe that $E_{x'} > E_x$ for points $x \in \mathcal{O}(x') \subset U(x')$. Therefore ζ is an E_x-realization. □

Supercovers have many properties of ordinary covers. Let $\Delta\colon X \to \operatorname{cov}(Z)$ be an X-supercover. We shall say that maps $f, g\colon X \to Z$ are Δ-*close* if for every $x \in X$, the images $f(x)$ and $g(x)$ lie in a common element of the cover Δ_x. The notion of a Δ-selection is introduced in an analogous manner: a map $r\colon X \to Z$ is called a Δ-*selection* of a map $\Phi\colon X \to Z$ if for every point $x \in X$, its image $r(x)$ belongs to the star $\mathrm{St}(\Phi(x), \Delta_x)$ of the image $\Phi(x)$ with respect to the cover Δ_x.

For paracompact spaces the property of extendability of star refinement of supercovers holds:

LEMMA 3.4. *For every n and for every Y-supercover E, there exists a Y-supercover Δ such that $(\Delta_x)^n > E_x$, for every $x \in X$.*

Proof. Let $w = \{\mathcal{O}(x) \mid x \in X\} \in \mathrm{cov}_0(X)$ be a cover such that for every point $x' \in \mathcal{O}(x)$, the cover E_x is a refinement of $E_{x'}$. We associate to the point x a cover $\Pi_x \in \mathrm{cov}(Z)$ so that $(\Pi_x)^n > E_x$. This correspondence and the cover w induce a canonical supercover $\Delta\colon X \mapsto Z$, $\Delta(x) = \Delta_x$, with property (5). Since $x \in \mathcal{O}(x')$, it follows that $(\Delta_x)^n > (\Pi_{x'})^n > E_{x'} > E_x$. □

LEMMA 3.5. *For every n and every X-supercover E, there exist an X-supercover Δ and a cover $\sigma = \{W(x)\} \in \mathrm{cov}_0(X)$ with the property that for every n-tuple of points $a_1, \ldots, a_n \in W(x)$, the multiple star $\Delta_{a_n} \circ \cdots \circ \Delta_{a_1}$ is a refinement of the cover E_x.*

Remark. We shall denote this property of supercovers by $(\Delta)^n > E$. We shall also say that the cover σ *realizes* the given refinability of the supercover.

Proof. Let $\widetilde{E}$ be an X-supercover such that $(\widetilde{E}_x)^n > \Delta_x$, for every $x \in X$. Consider a cover $w = \{\mathcal{O}(x) \mid x \in X\} \in \mathrm{cov}_0(X)$ such that $\widetilde{E}_{x'}$ is a refinement of $\widetilde{E}_x$, for every point $x \in \mathcal{O}(x')$. Consider the canonical supercover $\Delta\colon X \mapsto Z$, $\Delta(x) = \Delta_x$, such that for every $x \in X$, there exists a neighborhood $W(x)$ and a element $\mathcal{O}(x') \in w$, $x \in \mathcal{O}(x')$, with $\Delta_{x''} > \widetilde{E}_{x'}$, for each $x'' \in W(x)$. Let $a_i \in W(x)$. Then $(\Delta_{a_n} \circ \cdots \circ \Delta_{a_1}) > (\widetilde{E}_{x'})^n > E_x$. □

A more detailed study of the concept of uniformly super Lefschetz mappings and refinements of supercovers will only be needed in the proof of Theorem 1.1 (cf. Sections 4–6).

Let X be a paracompact space, $X_1 \supset X_2 \supset \cdots$ a nested sequence of closed subspaces, such that for every k, $X_{k+1} \subset \mathrm{Int}\, X_k$. Let Z be a Banach space with a fixed nested sequence $V_1 \supset V_2 \supset \cdots$ of closed convex subsets such that for every k, $V_{k+1} \subset \mathrm{Int}\, V_k$, and let $V = \bigcap_{k\geqslant 1} V_k$.

DEFINITION 3.6. A closed map $\Phi\colon X \to Z$ is said to be *strongly uniformly super Lefschetz* for the filtrations $\{X_k\}$ and $\{V_k\}$ if $\{\Phi(x) \mid x \in X\}$ has the following property:

(SA) For every k and every X-supercover $E\colon X \to Z$ of Z such that $\Phi(x) \subseteq V$ and $V_k > E_x$, for every $x \in X_k$, there exists an X-supercover $\Delta\colon X \to Z$ such that:

(iv) $V_{k+1} > \Delta_x$, for every $x \in X_{k+1}$; and

(v) For every $x \in X$ and every partial Δ_x-realization $\mathcal{N} \supset \mathcal{M} \xrightarrow{\xi} \Phi(x)$ of any polyhedron $\mathcal{N}$, there exists its extension to a full E_x-realization $\mathcal{N} \xrightarrow{\xi} \Phi(x)$.

Remarks. (1) $V_k > E_x$ means that V_k is contained in some element of E_x; and (2) The dependence of Δ on E will be denoted by $\Delta = (\mathrm{SA})(E)$.

The following claim is quite evident:

LEMMA 3.7. *For every cover $\sigma \in \mathrm{cov}(Z)$ with $V_k > \sigma$ and every integer n, there exists a cover $\gamma \in \mathrm{cov}(Z)$ such that $V_{k+1} > \gamma$ and $\gamma^n > \sigma$.*

Using this lemma, analogous version of 3.2–3.5 can be proved for strongly uniformly super Lefschetz mappings.

THEOREM 3.8. *If Φ: $X \to Z$ is a strongly regular multivalued map of paracompact space X into a metric space Z, the images $\Phi(x)$ of all points $x \in X$ are ANE's, and the image $\Phi(X)$ of the entire space X is Z, then the multivalued map Φ is strongly uniformly super Lefschetz.*

THEOREM 3.9. *Let k and m be arbitrary natural numbers. Then for every integer n and every X-supercover E, such that $V_k > E_x$, for every $x \in X_m$, there exists an X-supercover Δ and a cover $\sigma = \{W(x)\} \in \mathrm{cov}_0(X)$ such that $\Delta_{a_n} \circ \cdots \circ \Delta_{a_1} > E_x$, for every $a_1, \ldots, a_n \in W(x)$, and $V_{k+1} > \Delta_x$, for every $x \in X_m$.*

Remark. We shall denote this property of supercovers by $\Delta^n >_{\mathrm{SA}} E$.

4. Proofs of Theorems 1.1 and 1.2

From the very beginning we shall be proving Theorems 1.1 and 1.2 by means of the following simplification: $Z = L$, $\Phi(X) = \Gamma(Y) = L$, where L is a Banach space. Let us show that this causes no loss of generality. In fact, we take an isometrical embedding of the image $\Phi(X)$ into some Banach space L (see [2]). It is clear that if Γ was a strongly regular map then the map Γ': $Y \coprod \{*\} \to L$, $\Gamma'|_Y = \Gamma$, and $\Gamma'(*) = L$ will also be strongly regular. We define a multivalued map Φ': $X \coprod \{*\} \to L$ by the formula $\Phi' = \Gamma' \circ \varphi'$, where φ': $X \coprod \{*\} \to Y \coprod \{*\}$ is a singlevalued map such that $\varphi'|_X = \varphi$ and $\varphi'(*) = *$. It is clear that if the extension problem can be solved for the partial selection $X' = X \coprod \{*\} \supset A \xrightarrow{r} L$ in the simplified situation then it can also be solved in the original situation.

Theorems 1.1 and 1.2 will be deduced from Theorem 1.5 and the following proposition:

PROPOSITION 4.1. *Let Δ be a Y-supercover. Then every partial selection r: $A \to L$ of the map Φ can be extended onto some neighborhood $\mathcal{O}(A)$ to a (Δ^φ)-selection*

$\hat{r}\colon \mathcal{O}(A) \to L$. Moreover, there exists a closed G_δ-subset $\bar{A}$, $A \subset \bar{A} \subset \mathcal{O}(A)$ such that $\hat{r}|_{\bar{A}}$ is the genuine selection of φ.

Proof. Since the Banach space L is an absolute extensor for paracompact spaces [2], there exists an extension $\hat{r}\colon X \to L$ of the map r. Consider a Y-supercover $\widetilde{\Delta}$ which is a star-refinement of Δ, $(\widetilde{\Delta})^2 > \Delta$, and a cover $w = \{U(x)\} \in \mathrm{cov}_0(X)$ such that $(\Delta^\varphi)_x > (\Delta^\varphi)_{x'}$, for every $x' \in U(x)$. Without losing generality we may assume that $\hat{r}(U(a)) \subset \mathrm{St}(\Phi(a), \widetilde{\Delta}_{\varphi(a)})$, for every $a \in A$ and that $\Phi(x) \subset \mathrm{St}(\Phi(x'), \widetilde{\Delta}_{\varphi(x)})$, for every $x' \in U(x)$ (since Φ is continuous).

For every $x \in U(a) \subset \mathcal{O}(A) \stackrel{\text{def}}{=} \bigcup\{U(a) \mid a \in A\}$, we have that

$$\begin{aligned}\hat{r}(x) \in \mathrm{St}(\Phi(a), \widetilde{\Delta}_{\varphi(a)}) &\subset \mathrm{St}(\Phi(x), (\widetilde{\Delta}_{\varphi(a)})^2) \subset \mathrm{St}(\Phi(x), \Delta_{\varphi(a)})\\ &\subset \mathrm{St}(\Phi(x), \Delta_{\varphi(x)}).\end{aligned}$$

Therefore $\mathcal{O}(A)$ is the desired neighborhood. Since Φ is a continuous M-map, $A_\varepsilon = \{x \in X \mid \hat{r}(x) \in \mathrm{St}(\Phi(x); \varepsilon)\}$, $\varepsilon > 0$, satisfies the following properties:

(vi) $\mathrm{Int}\, A_\varepsilon \supseteq \mathrm{Cl}\, A_{\varepsilon'}$, for every $\varepsilon > \varepsilon'$; and
(vii) $B \stackrel{\text{def}}{=} \bigcap_{n=1}^{\infty} A_{1/n}$ coincides with $\{x \in X \mid \hat{r}(x) \in \Phi(x)\}$.

Let $C \subset \mathcal{O}(A)$ be a closed neighborhood of A. Then $\bar{A} = B \cap C$ is a closed G_δ-subset of X, $A \subset \bar{A} \subset \mathcal{O}(A)$ and $\hat{r}|_{\bar{A}}$ is a genuine selection. □

Deduction of Theorems 1.1 and 1.2 from Theorem 1.5 and Proposition 4.1. Let r be a partial selection of Φ. By Proposition 4.1 we can assume that A is a closed G_δ-subset. Multiply everything by $\mathbb{R}$ to get a partial selection $\tilde{r}\colon \widetilde{A} \to \widetilde{L}$ of a multivalued map $\widetilde{\Phi} = \Phi \times \mathrm{Id}$, where $\widetilde{A} = A \times \mathbb{R}$, $\widetilde{X} = X \times \mathbb{R}$, $\widetilde{Y} = Y \times \mathbb{R}$, $\widetilde{L} = L \times \mathbb{R}$, $\widetilde{\varphi} = \varphi \times \mathrm{Id}$, and $\tilde{r} = r \times \mathrm{Id}$ (note that $\widetilde{X}$ is again paracompact). The map $\widetilde{\Phi}$ is strongly regular and it satisfies all the hypotheses of Theorem 1.5. Therefore all conclusions of Theorem 1.5 are valid for the map $\widetilde{\Phi}$.

The centered family $\varepsilon = \{E(l,t) \mid (l,t) \in \widetilde{L}\}$, defined by $E(l,t) = L \times (-2,2)$, if $|t| < 2$ and $E(l,t)$ is a neighborhood of (l,t) of diameter t^{-2}, if $|t| \geqslant 2$, generates the trivial $\widetilde{Y}$-supercover E. Let us consider the nested sequence $\{V_k = L \times [-1 - \frac{1}{k}, 1 + \frac{1}{k}]\}$ of closed convex subsets of $\widetilde{L}$, $V = \bigcap_{k \geqslant 1} V_k = L \times [-1, 1]$, and the nested sequence

$$\left\{X_k = X \times \left[-\frac{1}{2} - \frac{1}{2k}, \frac{1}{2} + \frac{1}{2k}\right]\right\}$$

of closed subsets of $\widetilde{X}$. It can be easily seen that $\widetilde{\Phi}(\tilde{x}) \subseteq V$ and $V_k > E(\widetilde{\varphi}(\tilde{x}))$, for every $\tilde{x} \in X_k$ and $k \geqslant 1$. By Theorem 3.8, $\widetilde{\Phi}$ is the strong uniform super Lefschetz mapping for the filtrations $\{X_k\}$ and $\{V_k\}$.

The following easily verifiable property of the cover ε is stated separately.

LEMMA 4.2. *Let X be a paracompact space, $f\colon A \times [-\infty, \infty] \to L \times [-\infty, \infty]$, $f(A \times t) \subset L \times t$, for every $t \in [-\infty, \infty]$, a continuous map and $g\colon U \to \widetilde{L}$ a* *continuous map from neighborhood* $U \subset \widetilde{X}$ *of* $\widetilde{A}$ *such that for every* $\tilde{a} \in \widetilde{A}$, $g(\tilde{a})$ *and* $f(\tilde{a})$ *lie in an element of* ε *(i.e.,* $g|_{\widetilde{A}}$ *is* ε*-close to* $f|_{\widetilde{A}}$*). Then there exists a neighborhood* $W \subset U$ *of* $\widetilde{A}$ *in* $\widetilde{X}$ *such that a composed map* $h\colon \widehat{W} = W \cup A \times [-\infty, \infty] \to \widetilde{L}$, $h|_W = g$, $h|_{A\times[-\infty,\infty]} = f$, *defined on* $\widehat{W} \subset X \times [-\infty, \infty]$, *is also continuous.*

Hint. It is enough to consider only the simplest case $|A| = 1$.

Proof of Theorem 1.2. Applying Theorem 1.5 to the trivial $\widetilde{Y}$-supercover E, we obtain a $\widetilde{Y}$-supercover Δ. By Proposition 4.1, we can assume $\widetilde{A}$ to be a G_δ-subset in $\widetilde{X}$ and that a partial selection $\tilde{r}$ can be extended to some Δ^φ-selection $u\colon \mathcal{O}(\widetilde{A}) \to \widetilde{L}$, where $\mathcal{O}(\widetilde{A}) \supset \widetilde{A}$ is a neighborhood of $\widetilde{A}$ in $\widetilde{X}$. By Theorem 1.5, there exists a genuine selection $\hat{u}\colon \mathcal{O}(\widetilde{A}) \to \widetilde{L}$ of the map $\widetilde{\Phi}$, which is ε-close to u. Hence the projection of the point $\hat{u}(x, t) \in \widetilde{L}$ onto L belongs to $\Phi(x)$.

It is easy to see that Lemma 4.2 applies to $f = r \times \mathrm{Id}_{[-\infty,\infty]}$ and $g = \hat{u}\colon \mathcal{O}(\widetilde{A}) \to \widetilde{L}$. Then there exists a neighborhood $W \subset \mathcal{O}(\widetilde{A})$ of $\widetilde{A}$ such that composed map $h\colon \widehat{W} = W \cup A \times [-\infty, \infty] \to \widetilde{L}$, $h|_W = \hat{u}$, $h|_{A\times[-\infty,\infty]} = f$, is continuous. Since $\widetilde{A}$ is a G_δ-set, A is also a G_δ-set. It is also easy to see that the graph $\{(x, \chi(x))\}$ of a continuous function $\chi\colon V \to [0, \infty]$, V is a neighborhood of A in X, which assumes the value ∞ only at points from A, can be inscribed in $\widehat{W}$ if and only if A is a G_δ-subset of X. Finally, the desired selection $\hat{r}\colon V \to L$ of the map Φ is given by $\hat{r}(x)|_A = r$ and $\hat{r}(x)$ is the projection of the point $h(x, \chi(x)) \in \widetilde{L}$ onto L, for $x \in V \setminus A$. Continuity of $\hat{r}$ is thus evident. □

Proof of Theorem 1.1. Here we need the following strengthening of Theorem 1.5 for M-mapping with filtration:

THEOREM 4.3. *Let* $V_1 \supset V_2 \supset \cdots$, $V_{k+1} \subset \operatorname{Int} V_k$, *be a nested sequence of closed convex subsets of a Banach space* L, $V = \bigcap_{k\geqslant 1} V_k$. *Let* $X_1 \supset X_2 \supset \cdots$, $X_{k+1} \subset \operatorname{Int} X_k$, *be a nested sequence of closed subspaces of a paracompact space* X *and let* $\varphi\colon X \to Y$, $\Gamma\colon Y \to L$, $\Phi = \Gamma \circ \varphi$, $\Phi(x) \subseteq V$, *for every* $x \in X_1$, *be as in the hypotheses of Theorem* 1.2. *Then for every* Y*-supercover* $E\colon Y \to \operatorname{cov}(Z)$ *with* $V_k > E_{\varphi(x)}$, $x \in X_k$, *there exist an* Y*-supercover* $\Delta\colon Y \to \operatorname{cov}(Z)$ *of* Z *and an integer* m, *such that* $V_{k+m} > \Delta_{\varphi(x)}$, $x \in X_{k+m}$, *and for every* (Δ^φ)*-selection* $r\colon X \to Z$ *of the map* Φ, *there exists an exact selection* $r'\colon X \to Z$ *of* Φ *which is* (E^φ)*-near* r.

Remark. The proof of Theorem 4.3 is similar to the proof of Theorem 1.5 (see Section 5).

Applying Theorem 4.3 to the trivial $\widetilde{Y}$-supercover E, we obtain a $\widetilde{Y}$-supercover Δ, with $L \times [-1, 1] > \Delta_{\widetilde{\varphi}(\tilde{x})}$, for every $\tilde{x} \in X \times [-1, 1] = \bigcap X_i$. As $\widetilde{L} \times [-1, 1]$ is convex, an extension $\hat{r}\colon \widetilde{X} \to \hat{L}$ of $\tilde{r}$ can be choosen with $\hat{r}(X \times [-1, 1]) \subset L \times [-1, 1]$. Repeating the proof of Proposition 4.1 respectively for $\hat{r}$, we obtain the cover $\omega = \{U(\tilde{x})\} \in \operatorname{cov}_0(\widetilde{X})$ from Proposition 4.1 which can be choosen so that

$U(x,0) = L \times (-\frac{1}{2}, \frac{1}{2})$, for every $x \in X$. Therefore the neighborhood $\mathcal{O}(\widetilde{A})$ of $\widetilde{A}$ contains $L \times (-\frac{1}{2}, \frac{1}{2})$. The rest of the proof of this case coincides with the local one because a neighborhood $W \subset \mathcal{O}(\widetilde{A})$ of $\widetilde{A}$ can be choosen with $L \times (-\frac{1}{2}, \frac{1}{2}) \subset W$ and the domain of the function ξ can be extended over the whole space X. □

5. Proof of Theorems 1.5 and 4.3

Let $\Phi\colon X \overset{\varphi}{\to} Y \overset{\Gamma}{\to} L$ be a multivalued map which satisfies the hypotheses of Theorems 1.2 and 4.3. By Section 4, the proofs of Theorems 1.1 and 1.2 reduce to the proofs of Theorems 1.5 and 4.3. We can explicitly construct the supercover $\{\Delta_y\}$ from Theorem 1.5 (and shall briefly explain what additional constructions have to be done for Theorem 4.3). For this purpose we consider the following sequence of Y-supercovers of the space L:

$$\begin{array}{l} E, \widetilde{E}, E_n, E'_n, \\ \qquad E_{n-1}, E'_{n-1} \dots \\ \qquad\qquad\qquad \dots, E_0, E'_0, \\ \qquad\qquad\qquad\qquad\quad \Delta \end{array} \tag{$*$}$$

which satisfies the following properties:

(6) $E'_i = (A)(E_i)$, for $0 \leqslant i \leqslant n$ ($E'_i = (\mathrm{SA})(E_i)$ in the case of Theorem 4.3);
(7) $(E_i)^3 > E'_{i+1}$ ($E_i^3 >_{\mathrm{SA}} E'_{i+1}$ in the case of Theorem 4.3), for $0 \leqslant i < n$; and
(8) $(\widetilde{E})^3 > E$, $(E_n)^4 > \widetilde{E}$, and $\Delta^8 > E'_0$ ($\widetilde{E}^3 >_{\mathrm{SA}} E$, $E_n^4 >_{\mathrm{SA}} \widetilde{E}$, and $\Delta^8 >_{\mathrm{SA}} E'_0$ in the case of Theorem 4.3).

The supercover Δ will be the desired one. Since the supercover Δ is defined in Theorem 1.5 by means of the supercover E, we shall briefly denote it by $\Delta = (B)(E)$.

Due to the obvious dependence of Δ on E the following can be observed. If $\Gamma(y) \in AE$, for all y, the supercover E is trivial, and E_y consists of only one element, namely the space L, for every y, then in the sequence $(*)$ all supercovers can be taken to be equal to E (including Δ).

We shall reduce the proof of Theorem 1.5 to the following proposition:

PROPOSITION 5.1. *Let E and Δ be Y-supercovers such that $\Delta = (B)(E)$ and supercover $\widetilde{E}$ is the second term of the sequence $(*)$. Then*

(C) *For every $(\Delta)^\varphi$-selection $r\colon X \to L$ and every Y-supercover M there exists an M^φ-selection $r'\colon X \to L$ of the map Φ which is $(\widetilde{E})^\varphi$-close to the $(\Delta)^\varphi$-selection r.*

Remark. We shall denote the dependence of the M^φ-selection r' on all others by $r' = (C)(E, \widetilde{E}, \Delta, r, M)$.

Proof of Theorem 1.5. Let E and Δ be Y-supercovers such that $\Delta = (B)(E)$ and the Y-supercover $\widetilde{E}_1$, $(\widetilde{E}_1)^3 > E$, is the second term of the sequence $(*)$. Analogously, let $\Delta_1 = (B)(\widetilde{E}_1)$ and Y-supercover $\widetilde{E}_2$, $(\widetilde{E}_2)^3 > \widetilde{E}_1$, be the second term of the corresponding sequence $(*)$, etc. Without losing generality we may assume that mesh $\Delta_i < 2^{-i}$ and mesh $\widetilde{E}_i < 2^{-i}$.

Apply Proposition 5.1 to the Δ^φ-selection r in order to get a $(\Delta_1)^\varphi$-selection r_1: $X \to L$, $r_i = (C)(E, \widetilde{E}_1, \Delta, r, \Delta_1)$, which is $(\widetilde{E}_1)^\varphi$-close to r. Analogously, introduce $(\Delta_i)^\varphi$-selections

$$r_i = (C)(\widetilde{E}_{i-1}, \widetilde{E}_i, \Delta_{i-1}, r_{i-1}, \Delta_i), \quad i \in \{2, 3, \ldots\}.$$

As a result we have constructed $(\Delta_i)^\varphi$-selections r_i such that $\rho(r_i, r_{i+1}) < \widetilde{E}_{i+1}$. It is easy to get the estimates:

$$\rho(r_n, r_m) < (\widetilde{E}_{n+1})^\varphi \circ \cdots \circ (\widetilde{E}_m)^\varphi$$

and

$$(\widetilde{E}_{n+1})^\varphi \circ \cdots \circ (\widetilde{E}_m)^\varphi > (\widetilde{E}_n)^\varphi .$$

Therefore the sequence of maps $\{r_i\}$ is fundamental. Since $r_i(x) \in \mathrm{St}(\Phi(x), \Delta_i)$ and $\Phi(x)$ is complete, the fundamental sequence $\{r_i\}$ converges to some continuous map r': $X \to L$, for which $r'(x) \in \Phi(x)$. Since $\rho(r, r') < (\widetilde{E}_1^\varphi)^3$ while $(\widetilde{E}_1)^3 > E$ this completes the proof of Theorem 1.5. $\square$

Before we procede with the proof of Proposition 5.1, we pause for the following important technical result:

PROPOSITION 5.2. *For every Y-supercover A of the space L, there exists a Y-supercover B such that:*

(D) *Every map ψ: $P \to \mathrm{St}(\Gamma(y), B_y)$ of a paracompact space P into a B_y-neighborhood of the fiber $\Gamma(y)$ can be A_y-approximated by a map $\widetilde{\psi}$: $P \to \Gamma(y)$.*

Remark. The dependence of the supercover B on A will be simply denoted by $B = (D)(A)$.

Proof. Let $(A_1)^3 > A$, $A_2 = (A)(A_1)$, and $(B)^3 > A_2$ be Y-supercovers, and let $w = \{W_\lambda\} = \psi^{-1}(B_y)$ be a cover of P. Let $\mathcal{N}\langle w\rangle$ be the nerve of the cover w, $\mathcal{N}\langle w\rangle \xrightarrow{\xi_0} \Gamma(y)$ be a map, given on its 0-dimensional skeleton, which takes the vertex $\langle W_\lambda\rangle$ into a point $t_\lambda \in \Gamma(y)$ such that $t_\lambda \in \psi(W_\lambda)$ and $\psi(W_\lambda)$ is contained in some element of the cover B_y. The map ξ_0 defines a partial A_2-realization of the nerve $\mathcal{N}\langle w\rangle$. Let ξ: $\mathcal{N}\langle w\rangle \to \Gamma(y)$, $\xi|_{\mathcal{N}\langle w\rangle^{(0)}} = \xi_0$, be the existing $(A_1)_y$-realization of the nerve. Then the composition $\widetilde{\psi}$: $P \xrightarrow{\vartheta} \mathcal{N}\langle w\rangle \xrightarrow{\xi} \Gamma(y)$, where ϑ is the canonical map, is the desired A_y-approximation of ψ. $\square$

6. Proof of Proposition 5.1

Let us construct another sequence of Y-supercovers of the space L:

$$\Delta, M, M_n, \widetilde{M}_{n-1}, M_{n-1}, \widetilde{M}_{n-2}, M_{n-2}, \dots, \widetilde{M}_0, M_0 \qquad (**)$$

such that:

(9) $\widetilde{M}_k = (D)(M_{k+1})$ for $0 \leqslant k < n$;
(10) $(M_k)^2 > \widetilde{M}_k$ for $0 \leqslant k < n$; and
(11) $(M_n)^2 > M$, $M > \Delta$.

Without losing generality, we may assume that M, M_k and $\widetilde{M}_k$ are convex Y-supercovers, i.e., the covers M_y, $(M_k)_y$ and $(\widetilde{M}_k)_y$ consist of convex sets, every subsequent supercover in $(*)$ and $(**)$ is a refinement of the preceding one.

Consequently, we construct for every $k \in \{n, n-1, \dots, 0\}$, covers $w_k = \{W_k(y) \mid y \in Y\} \in \operatorname{cov}_0(Y)$ such that:

(α) $(w_k)^3 > w_{k+1}$;
(β) $\Gamma(a) \subset \operatorname{St}(\Gamma(b), (M_0)_y)$, for every pair of points $a, b \in W_k(y) \in w_k$; and
(γ) Every cover w_k realizes all conditions of refinability of Y-supercovers (6)–(11) (see remark after Lemma 3.5).

Let $\dim Y = n$. By Proposition 2.1, there exist a cover $\tau' \in \operatorname{cov}(X)$ and a simplicial map $\pi'\colon \mathcal{N}\langle\tau'\rangle \to \mathcal{N}\langle w_0\rangle$, induced by the refinability of the covers $\varphi(\tau') > w_0$, such that for every canonical map $\vartheta'\colon X \to \mathcal{N}\langle\tau'\rangle$, the image $\pi' \circ \vartheta'(X)$ lies in $\mathcal{N}\langle w_0\rangle^{(n)}$.

We can find a refinement $\tau = \{V(x) \mid x \in X\} \in \operatorname{cov}_0(X)$ of the cover $\tau' \in \operatorname{cov}(X)$ such that the image $r(V(x))$ lies in an element of the cover $(M_0)_{\varphi(x)} \in \operatorname{cov}(L)$. From this and from the hypothesis that r is a Δ^φ-selection it follows that:

(12) $r(V(x)) \subset \operatorname{St}(\Gamma(\varphi(x)), \Delta_{\varphi(x)} \circ (M_0)_{\varphi(x)})$.

Let $\vartheta\colon X \to \mathcal{N}\langle\tau\rangle$ be a canonical map and $\pi''\colon \mathcal{N}\langle\tau\rangle \to \mathcal{N}\langle\tau'\rangle$ a simplicial map, induced by the refinability $\tau > \tau'$. Since $\vartheta' = \pi'' \circ \vartheta\colon X \to \mathcal{N}\langle\tau'\rangle$ is the canonical map, $\pi' \circ \vartheta'(X) = \pi' \circ \pi''(\vartheta X) \subset \mathcal{N}\langle w_0\rangle^{(n)}$. Therefore

$$\vartheta(X) \subset (\pi' \circ \pi'')^{-1}(\mathcal{N}\langle w_0\rangle^{(n)}) = \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(n)}),$$

where $\pi = \pi' \circ \pi''\colon \mathcal{N}\langle\tau\rangle \to \mathcal{N}\langle w_0\rangle$.

We shall construct the desired M^φ-selection r' as the composition

$$X \xrightarrow{\vartheta} \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(n)}) \xrightarrow{\psi} L$$

of maps ϑ and ψ. The map ψ will be constructed by induction on the preimages $\pi^{-1}(\mathcal{N}\langle w_0\rangle^{(k)})$ of the skeleta of the nerve $\mathcal{N}\langle w_0\rangle$:

$$\psi_k\colon \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(k)}) \to L,\ \psi_{k+1} \mid \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(k)}) = \psi_k, \quad k = 0, 1, 2, \dots, n,$$

such that for every $x \in X$, $\vartheta(x) \in \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(k)})$:

$(\delta)_k$ $\psi_k(\vartheta(x))$ and $r(x)$ are $\widetilde{E}_{\varphi(x)}$-close; and

$(\varepsilon)_k$ $\psi_x(\vartheta(x)) \in \mathrm{St}(\Phi(x), M_{\varphi(x)})$.

We first construct the map ψ_0. Let $\langle W_0(y)\rangle$ be a vertex of $\mathcal{N}\langle w_0\rangle^{(0)}$ and $\langle V(x)\rangle$ be a vertex from $\pi^{-1}(\langle W_0(y)\rangle)$. Then $\varphi(V(x)) \subset W_0(y)$. By (β) and (12) we get

(13) $r(V(x)) \subset \mathrm{St}(\Gamma(y), \Delta_{\varphi(x)} \circ (M_0)_{\varphi(x)} \circ (M_0)_y)$.

Since the cover w_0 realizes all refinabilities of the supercovers it follows that:

(14) $\Delta_{\varphi(x)} \circ (M_0)_{\varphi(x)} \circ (M_0)_y > \Delta_{\varphi(x)} \circ \Delta_y^2$.

It follows from the above that:

(15) $r(V(x)) \subset \mathrm{St}(\Gamma(y), \Delta_{\varphi(x)} \circ \Delta_y^2)$ and, consequently,
$\Gamma(y) \cap \mathrm{St}(r(V(x)), \Delta_{\varphi(x)} \circ \Delta_y^2) \neq \emptyset$.

If we define the map ψ_0' on the vertex $\langle V(x)\rangle$ as the point:

(16) $v_x \in \Gamma(y) \cap \mathrm{St}(r(V(x)), \Delta_{\varphi(x)} \circ \Delta_y^2)$, we get a map:

$$\mathcal{M} = \pi^{-1}(\langle W_0(y)\rangle) \supset \pi^{-1}(\langle W_0(y)\rangle)^{(0)} \xrightarrow{\psi_0'} \Gamma(y)$$

which is a partial $(E_0')_y$-realization of polyhedron $\mathcal{M}$. Indeed, if $\sigma = \langle V(x_0), \dots, V(x_k)\rangle \in \mathcal{M}$, then:

(17) $\pi(V(x_i)) \subset W_0(y)$, for every i;
(18) $V = \bigcap_{i=0}^k V(x_i) \neq \emptyset$.

Let $v \in V$; and

(19) $\psi_0'(\bigcup_{i=0}^k \langle V(x_i)\rangle) = \bigcup_{i=0}^k v_{x_i} \subset \mathrm{St}(r(V(x_i)), \Delta_{\varphi(x)} \circ \Delta_y^2) \subset \mathrm{St}(v, \Delta_{\varphi(x)} \circ \Delta_y^2 \circ (M_0)_{\varphi(x)}) \subset \mathrm{St}(v, \Pi)$, where $\Pi = \Delta_{\varphi(x)} \circ \Delta_y^3$.

Hence, $\psi_0'(\sigma \cap \mathcal{M}^{(0)})$ is contained in some element of Π^2 which is, by (8), a refinement of $(E_0')_y$. Since $E_0' = (A)E_0$, there exists an $(E_0)_y$-realization ψ_0: $\mathcal{M} \to \Gamma(y)$ such that $\psi_0|\mathcal{M}^{(0)} = \psi_0'$. By studying the value of ψ_0 on all the vertices $\langle W_0(y)\rangle$, we obtain a map ψ_0: $\pi^{-1}(\mathcal{N}\langle w_0\rangle^{(0)}) \to L$ which satisfies the properties (a)$_k$ and (b)$_k$ stated below and $(\delta)_k$–$(\varepsilon)_k$, for $k = 0$. Difficult verifications of $(\delta)_k$ and $(\varepsilon)_k$, for every ψ_k will be realized in (25)–(26).

Formulation of properties (a)$_k$ *and* (b)$_k$. For every k-simplex

$$\delta = \langle W_0(y_0), \dots, W_0(y_k)\rangle \in \mathcal{N}\langle w_0\rangle^{(k)} \quad \left(\text{i.e., } \bigcap_{i=0}^{k} W_0(y_i) \neq \emptyset\right),$$

there exists a point $y \in Y$ and an element $W_k(y) \in w_k$, $W_k(y) \supset \bigcup_{i=0}^k W_0(y_i)$, with the following properties:

(a)$_k$ $\psi_k(\pi^{-1}(\delta)) \subset \mathrm{St}(\Gamma(y), (M_k)_y)$; and

(b)$_k$ The restriction $\pi^{-1}(\delta) \xrightarrow{\psi_k} L$ of the map ψ_k on $\pi^{-1}(\delta)$ is an $(E_k)_y^2$-realization of the polyhedron $\pi^{-1}(\delta)$.

Suppose that we have already constructed maps ψ_i, $i \leqslant k$, satisfying (a)$_i$ and (b)$_i$, and such that ψ_i and ψ_{i-1} agree on the common part of their domains. Consider an arbitrary $(k+1)$-simplex

$$\delta = \langle W_0(y_0), \ldots, W_0(y_{k+1})\rangle \in \mathcal{N}\langle w_0\rangle^{(k+1)}.$$

Denote by δ_i the k-simplices of the boundary $\partial\delta$. It follows from (a)$_k$ that $\psi_k(\pi^{-1}(\delta_j)) \subset \mathrm{St}(\Gamma(z_j), (M_k)_{z_j})$, where z_j is the center of a neighborhood $W_k(z_j) \in w_k$, such that $W_k(z_j)$ contains all elements $W_0(y_i)$, corresponding to the vertices δ_j. Consequently, $W_k(z_j) \supset \bigcap_{i=0}^{k+1} W_0(y_i) \neq \emptyset$ for all j, and hence $\bigcap_{j=0}^{k+1} W_k(z_j) \neq \emptyset$.

By (α), there exists an element $W_{k+1}(y) \in w_{k+1}$, containing $\bigcup_{j=0}^{k+1} W_k(z_j)$. From (a)$_k$ and (β) we obtain the inclusion:

$$\begin{aligned}(20)\quad \psi_k(\pi^{-1}(\partial\delta)) &= \bigcup \psi_k(\pi^{-1}(\delta_j)) \subset \bigcup \mathrm{St}(\Gamma(z_j), (M_k)_{z_j}) \subset \mathrm{St}(\Gamma(y),\\ &\quad (M_k)_{z_j} \circ (M_0)_y).\end{aligned}$$

Since $M_k \circ M_0 > \widetilde{M}_k$ and $\widetilde{M}_k = (D)(M_{k+1})$, there exists a map $\chi\colon \pi^{-1}(\partial\delta) \to \Gamma(y)$, such that $\rho(\chi, \psi_k) < (M_{k+1})_y$. Since the cover $(M_{k+1})_y$ is convex, the linear homotopy $H\colon \pi^{-1}(\partial\delta) \to L$ from ψ_k to χ is realizable inside $\mathrm{St}(\Gamma(y), (M_{k+1})_y)$.

Therefore, and by (b)$_k$, it follows that the map $\pi^{-1}(\delta) \supset \pi^{-1}(\partial\delta) \xrightarrow{\chi} \Gamma(y)$ is a partial $(M_{k+1})_y \circ (E_k^2)_y$-realization of the polyhedron $\pi^{-1}(\delta)$. Since $(E_k^2) \circ M_{k+1} > (E_k^2) \circ \delta > E'_{k+1}$ and $E'_{k+1} = (A)(E_{k+1})$, there exists an E_{k+1}-realization:

$$\psi'_{k+1}\colon \pi^{-1}(\delta) \to \Gamma(y) \quad \text{such that } \psi'_{k+1}|\partial\delta = \chi.$$

So we have constructed the homotopy $H\colon \mathcal{Q} \times [0,1] \to L$ of polyhedron $\mathcal{Q} = \pi^{-1}(\partial\delta)$ and the map $\psi'_{k+1}\colon \mathcal{P} \times \{1\} \to L$ defined on the upper boundary of the polyhedral cylinder $\mathcal{P} \times [0,1]$, $\mathcal{P} = \pi^{-1}(\delta)$. These maps agree on the common domain $\mathcal{Q} \times \{1\}$. Since $(\mathcal{P}, \mathcal{Q})$ is a polyhedral Borsuk pair (see [2]), the homotopy H can be extended to homotopy $\widehat{H}\colon \mathcal{P} \times [0,1] \to L$ such that:

(21) $\widehat{H}|\mathcal{Q} \times [0,1] = H$;
(22) $\widehat{H}|\mathcal{P} \times \{1\} = \psi'_{k+1}$; and
(23) $\widehat{H}(\sigma \times [0,1]) = \psi_{k+1}(\sigma \times \{1\}) \cup H((\sigma \cap \mathcal{Q}) \times [0,1])$, for every simplex $\sigma \in \mathcal{P}$.

Let us denote the restriction of $\widehat{H}$ onto the lower boundary $\mathcal{P} \times \{0\}$ by

$$\psi_{k+1}^{\delta}\colon \mathcal{P} = \pi^{-1}(\delta) \to L.$$

So there exists a map

$$\psi_{k+1}\colon \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(k+1)}) \to L \quad \text{with } \psi_{k+1}|\pi^{-1}(\delta) = \psi_{k+1}^{\delta}.$$

By (23),

$$\begin{aligned}\psi_{k+1}(\pi^{-1}(\delta)) &= \psi_{k+1}^{\delta}(\pi^{-1}(\delta)) \subset \psi'_{k+1}(\mathcal{P}) \cup H(\mathcal{Q} \times [0,1])\\ &\subset \mathrm{St}(\Gamma(y), (M_{k+1})_y).\end{aligned}$$

Thus the property (a)$_{k+1}$ has been verified.

Now let us verify (b)$_{k+1}$. By (23) we have that

(24) $\psi^{\delta}_{k+1}(\sigma) \subset \psi'_{k+1}(\sigma) \cup H((\sigma \cap \mathcal{Q}) \times [0,1]) \subset \mathrm{St}(\psi'_{k+1}(\sigma), (M_{k+1})_y)$, for every simplex $\sigma \in \pi^{-1}(\delta)$.

But $\psi'_{k+1}(\sigma)$ lies in an element of $(E_{k+1})_y$, and $(E_{k+1})_y \circ (M_{k+1})_y > (E^2_{k+1})_y$. Hence ψ^{δ}_{k+1} is a $(E^2_{k+1})_y$-realization of polyhedron $\pi^{-1}(\delta)$.

In this way we have constructed a map $\psi\colon \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(n)}) \to L$, satisfying (a)$_n$ and (b)$_n$. Let us show that $r' = \psi \circ \vartheta\colon X \to L$ is an M^{φ}-selection of Φ and $\rho(r, r') < (\widetilde{E})^{\varphi}$ (here $X \xrightarrow{\vartheta} \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(n)})$ is the canonical map).

Indeed, let

$$x \in \bigcap_{i=0}^{k} V(x_i), \qquad \vartheta(x) \in \sigma = \langle V(x_0), \ldots, V(x_k)\rangle \subset \pi^{-1}(\mathcal{N}\langle w_0\rangle^{(n)}).$$

By (a)$_n$–(b)$_n$, there exists $W_n(y) \in w_n$ with $W_n(y) \supset \bigcup_{i=0}^{k} \varphi(V(x_i))$ such that $\psi_n(\sigma) \subset \mathrm{St}(\Gamma(y), (M_n)_y)$ and $\psi_n(\sigma) \subset W \in (E_n)^2_y$. From ($\beta$) and $\varphi(x) \in W_n(y)$ we have $\Gamma(y) \subset \mathrm{St}(\Gamma(\varphi(x)), (M_0)_y)$. Therefore:

$$\begin{aligned} r'(x) = \psi_n(\vartheta(x)) \subset \psi_n(\sigma) \subset \mathrm{St}(\Gamma(y), (M_n)_y)) \subset \\ \subset \mathrm{St}(\Gamma(\varphi(x)), (M_0)_y \circ (M_n)_y) \subset \mathrm{St}(\Gamma(\varphi(x)), M_{\varphi(x)}) \end{aligned} \tag{25}$$

and, hence,

$$\psi_n(\sigma) \subset \mathrm{St}(\psi_n(\langle V(x_0)\rangle), (E_n)^2_y) = \mathrm{St}(v_{x_0}, (E_n)^2_y) \subset W \in (E_n)^4_y > \widetilde{E}_y. \quad \square \tag{26}$$

Acknowledgements

The first author was supported in part by the International Science Foundation grant No. NFU000 and the Russian Foundation for Fundamental Research grant No. 93-011-1402. The second author was supported in part by the Ministry of Science and Technology of the Republic of Slovenia grant No. J1-0885-0101-98. We acknowledge comments and suggestions from the referee and P. V. Semenov.

References

1. Ferry, S. C.: Strongly regular mappings with compact ANR fibers are Hurewicz fiberings, *Pacific J. Math.* **75** (1978), 373–382.
2. Hu, S.-T.: *Theory of Retracts*, Wayne State Univ. Press, Detroit, 1965.
3. Michael, E.: Continuous selections II, *Ann. of Math. (2)* **64** (1956), 562–580.
4. Michael, E.: Some refinements of a selection theorem with 0-dimensional domain, *Fund. Math.* **140** (1992), 279–287.
5. Moiseev, E. V.: Selections of mappings with finite-dimensional images, in: *General Topology. Space and Mappings*, Moscow State Univ. Press, Moscow, 1992, pp. 61–64 (in Russian).
6. Pixley, C.: An example concerning continuous selections of infinite dimensional spaces, *Proc. Amer. Math. Soc.* **43** (1974), 237–244.
7. Repovš, D. and Semenov, P. V.: *Continuous Selections of Multivalued Mappings*, Kluwer Acad. Publ., Dordrecht, 1998.