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THE METHOD OF APPROXIMATIVE EXTENSION OF MAPPINGS IN THE THEORY OF EXTENSORS

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Abstract: We develop the method of approximative extension of mappings which enables us not only to simplify the proofs of many available theorems in the theory of extensors but also to obtain a series of new results. Combined with Ancel's theory of fiberwise trivial correspondences, this method leads to considerable progress in the characterization of absolute extensors in terms of local contractivity. We prove the following assertions: Suppose that a space X is represented as the union of countably many closed ANE-subspaces X_i and a countably dimensional subspace D: 1. If each X_i is a strict deformation neighborhood retract of X and $X \in$ LC, then $X \in$ ANE. 2. If $X \in$ LEC then $X \in$ ANE.

Keywords: approximative extension, extensor

§ 1. Introduction

This article is devoted to extensions of continuous mappings $f : A \to X$ from closed subspaces $A \subset Z$ of metric spaces Z to some neighborhoods of A. The spaces X, possessing this extension property for all partial mappings $Z \hookleftarrow A \xrightarrow{f} X$, are called *absolute neighborhood extensors* ($X \in$ ANE). The problem of recognizing ANE-spaces is rather urgent in modern topology and is solved by distinguishing various subclasses $\mathfrak{P}$ of the class of all ANE-spaces which admit a convenient description in some sense: the class of convex subsets in a locally convex vector space (Dugungji's theorem [1]), the class of countably dimensional locally contractible metric spaces (Haver's theorem [2]), the class of metric spaces with a contractible base all of whose finite intersections are also contractible (Toruńczyk's theorem [3]), etc. In all principally important cases, the class $\mathfrak{P}$ is invariant under the product by the half-interval $J = [0, 1)$: $\mathfrak{P} \supset \mathfrak{P} \times J$. In such situation, the problem of *exact* extension of mappings $f : A \to X$ is solved by reducing it to a less burdensome problem of *approximative* extension of a partial mapping.

Definition 1.1. A space X is called an *approximative neighborhood extensor* ($X \in$ A-ANE) if, for every covering $\omega \in \operatorname{cov} X$ and every partial mapping $Z \hookleftarrow A \xrightarrow{f} X$, there exists a mapping $\tilde{f} : U \to X$ on a neighborhood U of A such that $\operatorname{dist}(f, \tilde{f}\restriction_A) \prec \omega$.

The following theorem plays a principal role in this exposition.

Theorem 1.2. *Suppose that a class $\mathfrak{P}$ of metric spaces is closed under the product by J. Then $\mathfrak{P} \subset$ ANE if and only if $\mathfrak{P} \subset$ A-ANE.*

If the condition of the theorem is violated (i.e., the class $\mathfrak{P}$ is not closed under the product by J) then we only have the strict inclusion ANE$\cap\mathfrak{P} \subsetneq$ A-ANE$\cap\mathfrak{P}$. As an example of a compact approximative absolute neighborhood extensor that is not an absolute neighborhood extensor, we can take the one-point compactification of the set of naturals.

This criterion is encountered in implicit form in the articles [3–5], but its fundamental role in the theory of absolute extensors becomes clear only now. The main thesis of this article is that, to simplify the proofs of the majority of available theorems and to obtain new results, we should use the notion of approximative ANE. Indeed, to construct an approximate extension of a mapping is much easier than

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to construct an exact extension. And if the class $\mathfrak{P}$ is closed under the product by a half-interval then the problem of recognizing ANE within $\mathfrak{P}$ is identical to that of recognizing an approximative ANE. Finally, the notion of approximative ANE interacts harmoniously with various notions and constructions of the theory of extensors. In particular, this article demonstrates such harmonious interaction with Ancel's technique [6] developed for fiberwise trivial correspondences (mappings).

To illustrate the above thesis, we give some similar proofs of Haver's and Toruńczyk's theorems and establish some new unexpected result.

Theorem 1.3. *If a countably dimensional space X has an open base $\{W_\gamma\}$ of weakly homotopy trivial sets* (*i.e., the homotopy groups $\pi_i(W_\gamma)$ are trivial for all $\gamma \in \Gamma$ and $i \geq 0$*) *then $X \in$* ANE.

It is reasonable to compare the last theorem with that of Toruńczyk [3]: If we additionally require in Theorem 1.3 that the base $\{W_\gamma\}$ is multiplicative then each (not necessarily countably dimensional) space is an absolute neighborhood extensor.

The concluding part of the article is devoted to studying interrelations between the three most important classes in the theory of extensors: ANE, locally equiconnected (LEC), and locally contractible spaces (LC). Clearly, ANE $\subset$ LEC $\subset$ LC. The problem of coincidence of the first two classes was open for a long time [7, p. 246]. The metric vector space by Cauty [8] gives an example of an LEC-space which is not ANE. Therefore, the problem is urgent of finding a widest class of spaces for which the easier properties LEC and LC imply ANE.

We indicate some available results in this direction. The intersections of the above three classes with countably dimensional spaces coincide (Haver's theorem [2]). The intersections of the first two classes with the spaces admitting a countable increasing filtration of closed ANE-subspaces coincide as well (Nhu and Sakai's theorem [9]), whereas the intersections of the second and third classes differ (Borsuk's example [10]). However, both Haver's theorem and Nhu and Sakai's theorem can be considerably generalized; moreover, their proofs can be simplified essentially.

Theorem 1.4. *Suppose that a space X is represented as the union of countably many closed* ANE-*subspaces X_i and a countably dimensional subspace D. If each X_i is a strict deformation neighborhood retract of X and $X \in$* LC *then $X \in$* ANE.

Note that all assertions involving the condition of countable dimensionality can be strengthened if only we use the notion of C-spaces; moreover, this is done by a literal analogy to the arguments in the article (see [11]).

Theorem 1.5. *Suppose that a space X is represented as the union of countably many closed* ANE-*spaces X_i and a countably dimensional subspace D. If $X \in$* LEC *then $X \in$* ANE.

These theorems are in turn manifestations of some more general fact about LEC-embedded subspaces.

Theorem 1.6. *Suppose that an* LC-*space X is represented as the union of countably many closed subspaces $X_i = \bigcup_{j=1}^{\infty} F_{ij}$ and a countably dimensional subspace D: $X = \bigcup_{i=1}^{\infty} X_i \cup D$. If every embedding $X_i \hookrightarrow X$ is locally equiconnected* (LEC) *and each partial mapping $Z \hookleftarrow A \xrightarrow{\varphi} F_{ij}$ has a neighborhood extension $\hat{\varphi} : U \to X_i$ then $X \in$* ANE.

Since a strict deformation neighborhood locally equiconnected retract is LEC-embedded in the ambient space while every subspace of an LEC-space is LEC-embedded, Theorems 1.4 and 1.5 are particular instances of Theorem 1.6. If $X = \bigcup_{j=1}^{\infty} C_j$ is the countable union of compact sets and if the LEC-space X is an absolute neighborhood extensor for the class of compact spaces, then it is easy to see that all conditions of Theorem 1.6 are satisfied and $X \in$ ANE. This is the main result of [9, Main Theorem]. We also note that Theorems 1.4–1.6 can be easily transformed into some theorems about preservation of the class of ANE-spaces under compactifications with countably dimensional appendix.

Since all conditions of Theorems 1.4–1.6 are preserved under the product by the half-interval J, in view of Theorem 1.2 it suffices to establish the relation $X \in$ A-ANE in them. This makes it possible to use in the proofs of the theorems the fiberwise trivial mappings in the Ancel sense [6] in view of their connection with A-ANE.

Proposition 1.7. *If for every partial mapping $Z \hookleftarrow A \xrightarrow{\varphi} X$ the projection $\pi_\varphi : G_\varphi \to A$ of the graph $G_\varphi \subset Z \times X$ of φ to A is fiberwise trivial within the projection $p \rightleftharpoons \mathrm{pr}_Z : Z \times X \to Z$, then $X \in$ A-ANE.*

We emphasize just away that we need only some small part of the whole Ancel theory which is large and rich in ideas. Moreover, we want to give a lucid exposition of its main concepts which is independent of [6]. Therefore, we especially devote § 5 to the Ancel theory as seen from the viewpoint of the theory of extensors.

§ 2. Preliminaries

All spaces (single-valued mappings) are assumed to be metric (continuous), unless they appear as a result of special constructions.

The set of all open coverings of a space X is denoted by $\operatorname{cov} X$, and $\omega \in \operatorname{cov} X$ stands for some open covering of X. The *star* (or *neighborhood*) of a set $A \subset X$ relative to $\omega \in \operatorname{cov} X$ is the set $\bigcup\{U \mid U \in \omega, U \cap A \neq \varnothing\}$, denoted by $\mathrm{St}(A;\omega)$ or $\mathrm{N}(A;\omega)$. The *star of a covering ω relative to another covering ω'* is the covering $\mathrm{St}(\omega;\omega') = \{\mathrm{St}(U;\omega') \mid U \in \omega\}$. For brevity, the repeated stars $\mathrm{St}(\omega_1;\mathrm{St}(\omega_2;\dots;\omega_n)\dots)$ are denoted by $\omega_n \circ \dots \circ \omega_2 \circ \omega_1$, and if ω_i are equal to one another then we denoted them by $(\omega_1)^k$. The *carrier* of a system ω of open sets is the set $\bigcup\{U \mid U \in \omega\}$, denoted by $\bigcup \omega$.

The record $\omega \succ \omega_1$ means as usual that the covering ω is a refinement of the covering ω_1. It is well known that every covering $\sigma \in \operatorname{cov} X$ of a metric space X admits a *starlike refinement* $\omega \in \operatorname{cov} X$, namely, a covering such that $\omega \circ \omega \succ \sigma$ (Stone's theorem [12]). The following convenient criterion of starlike refinement belongs to folklore and we give it without proof.

Proposition 2.1. *A covering $\sigma = \{S_\lambda \mid \lambda \in \Lambda\}$ is a starlike refinement of a covering $\omega = \{W_\beta \mid \beta \in B\}$ if and only if for every λ there exists $\beta = \beta(\lambda)$ such that $\bigcap_{\lambda\in\Lambda'} S_\lambda \neq \varnothing$ for a subset $\Lambda' \subset \Lambda$ implies*
(1) $\bigcup_{\lambda\in\Lambda'} S_\lambda \subset \bigcap_{\lambda\in\Lambda'} W_{\beta_\lambda}$.

If $f, g : X \to Y$ are mappings and $\omega \in \operatorname{cov} Y$ then the fact of ω-proximity of f and g is designated as $\mathrm{dist}(f,g) \prec \omega$. The restriction of a mapping f to a subset $A \subset X$ is denoted by $f\restriction_A$.

The *nerve* of a covering $\omega = \{U_\beta \mid \beta \in B\}$ is a polyhedron $\mathfrak{N}\langle\omega\rangle$, with weak Whitehead topology, whose vertices $\langle U_\beta\rangle$ are in a one-to-one correspondence with the index set B and $\omega = \langle U_{\beta_0},\dots,U_{\beta_s}\rangle$ is an s-dimensional simplex of $\mathfrak{N}\langle\omega\rangle$ with vertices $\langle U_{\beta_i}\rangle$ if and only if $\bigcap U_{\beta_i} \neq \varnothing$. The *$k$-dimensional skeleton* $\mathfrak{N}\langle\omega\rangle^{(k)}$ is the subpolyhedron of $\mathfrak{N}\langle\omega\rangle$ that comprises at most k-dimensional simplices; $\mathfrak{N}\langle\omega\rangle^{(-1)} = \varnothing$. The *open star* $\overset{\circ}{\mathrm{St}}\langle U_{\beta_0}\rangle$ of a vertex $\langle U_{\beta_0}\rangle$ is the set $\{\sum \alpha_\beta \cdot \langle U_\beta\rangle \in \mathfrak{N}\langle\omega\rangle \mid \alpha_{\beta_0} \neq 0\}$.

A mapping $\theta : X \to \mathfrak{N}\langle\omega\rangle$ is called *canonical* if the preimage $\theta^{-1}(\overset{\circ}{\mathrm{St}}\langle U_\beta\rangle)$ of the open star of each vertex $\langle U_\beta\rangle$ lies in U_β. A canonical mapping is well known [1] to exist for every open covering ω of a paracompact space X.

We introduce a series of notions that are connected with the extension of mappings. A space X is called an *absolute neighborhood extensor*, in writing $X \in$ ANE, if every mapping $\varphi : A \to X$ defined on a closed subset $A \subset Z$ of a metric space Z and called a *partial mapping* extends to some neighborhood $U \subset Z$ of A, $\hat\varphi : U \to X$, $\hat\varphi\restriction_A = \varphi$. If φ extends always to $U = Z$ then X is called an *absolute extensor*, in writing $X \in$ AE. In case $X \in$ A[N]E, we say that the space X possesses the property of A[N]E-*absolute* (*neighborhood*) *extendibility*. Note that the notions of absolute (neighborhood) extensor and absolute (neighborhood) retract coincide whenever X is a metric space [1].

The property $X \in$ ANE is equivalent to extendibility of partial realizations to global realizations [1, p. 156]. In the sequel, we only need the definition of realization of a nerve.

Definition 2.2. Let α be a system of open sets of a space X and let $\mathfrak{N}_0$ be a subpolyhedron of a polyhedron $\mathfrak{N}$ which contains all vertices. A *partial α-realization* of $\mathfrak{N}$ is a mapping $\mathfrak{N}_0 \xrightarrow{\varphi} Z$ such that $\varphi(\Delta \cap \mathfrak{N}_0)$ lies in some element $V \in \alpha$ for each simplex $\Delta \in \mathfrak{N}$. If $\mathfrak{N}_0 = \mathfrak{N}$ then φ is an *α-global realization* of $\mathfrak{N}$.

Every ANE-space is locally contractible ($X \in$ LC) and locally equiconnected ($X \in$ LEC). We give the relevant definitions.

Definition 2.3. A space X possesses the property of *local contractivity* ($X \in$ LC) if for every point x in X and every neighborhood $U(x)$ of x there exists a neighborhood $V(x)$ such that the embedding $V(x) \hookrightarrow U(x)$ and the constant mapping $c : V(x) \to \{x\} \hookrightarrow U(x)$ are homotopic in $U(x)$.

Definition 2.4. An embedding $X' \hookrightarrow X$ is *locally equiconnected* (LEC) if there exist a neighborhood $\mathscr{U}$ of the diagonal $\Delta X'$ in $X' \times X$ and a mapping $\lambda : \mathscr{U} \times I \to X$ such that $\lambda(x', x, 0) = x'$, $\lambda(x', x, 1) = x$, and $\lambda(x', x', t) = x'$ for all $(x', x) \in \mathscr{U}$ and $t \in I$.

A space X is *locally equiconnected* if the identity embedding $X' \hookrightarrow X$ is locally equiconnected [6]. Every ANE-space is well known to be locally equiconnected and in consequence locally contractible. An important example of an LEC-embedding $X' \hookrightarrow X$ is one in which X' is a *strict deformation neighborhood retract* of X.

Definition 2.5. A subspace $X' \subset X$ is a *strict deformation neighborhood retract* of X if there exist a neighborhood U, $X' \subset U \subset X$, and a homotopy $F : U \times I \to X$ such that $F_0 = \mathrm{Id}_U$, $F_t\restriction_{X'} = \mathrm{Id}_{X'}$, $t \in I$, and F_1 is a retraction of U to X'.

Since X' is a retract of U, it follows that X' is closed in U and in consequence closed in X.

Proposition 2.6. *If an* ANE*-space X' lies in X and is a strict deformation neighborhood retract of X then $X' \hookrightarrow X$ is an* LEC*-embedding.*

Proof. Since $X' \in$ ANE, it follows that $X' \in$ LEC and so there exist a neighborhood $\mathscr{V} \subset X' \times X'$ of the diagonal $\Delta X'$ and a mapping $\lambda' : \mathscr{V} \times I \to X'$ such that $\lambda'(x_1, x_2, 0) = x_1$, $\lambda'(x_1, x_2, 1) = x_2$, and $\lambda'(x_1, x_1, t) = x_1$ for all $(x_1, x_2) \in \mathscr{V}$ and $t \in I$. Since X' is a strict deformation neighborhood retract of X, we can easily deduce the existence of a neighborhood $\mathscr{U} \subset X' \times X$ of the diagonal $\Delta X'$ and a mapping $\lambda'' : \mathscr{U} \times I \to X$ such that $\lambda''(x', x, 1) = x$, $(x', \lambda''(x', x, 0)) \in \mathscr{V}$, and $\lambda''(x', x', t) = x'$ for all $(x', x) \in \mathscr{U}$ and $t \in I$. Then the sought mapping $\lambda : \mathscr{U} \times I \to X$, guaranteeing the LEC-embedding of X' into X, is defined by the formula $\lambda(x', x, t) = \lambda''(x', x, 2t - 1)$ for $1/2 \leq t \leq 1$ and $\lambda(x', x, t) = \lambda'(x', \lambda''(x', x, 0), 2t)$ for $0 \leq t \leq 1/2$. ∎

Finally, we recall that a space D is *countably dimensional* if $D = \bigcup_{i=1}^{\infty} D_i$ and D_i is zero-dimensional for every i. The following important assertion about countably dimensional subsets holds:

Proposition 2.7. *If a partial mapping $Z \hookleftarrow A \xrightarrow{\varphi} X$ is such that A or $\varphi(A)$ is countably dimensional then for every sequence of coverings $\omega_i \in \operatorname{cov} X$, $i \geq 1$, there exist countably many disjoint families σ_i of open sets in Z such that*

(2) $\sigma \rightleftharpoons \bigcup_{i=1}^{\infty} \sigma_i$ *covers A and $\varphi(\sigma_i) \succ \omega_i$ for all $i \geq 1$.*

The proof is perfectly analogous to that, for example, in [6, p. 10].

§ 3. The Approximative Criterion for Extendibility of Mappings

In the product $X \times J$ of a metric space X and the half-open interval $J = [0, 1)$, consider an open covering ω *adherent to the top* $X \times \{1\}$. This means that, for every point $a \in X \times \{1\}$ and every neighborhood $U = U(a)$ in $X \times [0, 1]$, there is a neighborhood $V = V(a)$ such that the star of V relative to ω lies in U. We note just away that one of the possibilities of arranging such adherent covering relates to a sequence of open coverings $\omega_i \in \operatorname{cov} X$, $i \geq 1$, satisfying the conditions:

(1) $\omega_n \circ \omega_n \succ \omega_{n-1}$ for all $n > 1$;

(2) for every point $x \in X$, the sequence of the sets $\{\omega_n(x)\}$ converges to this point.

We call such sequences *shallowing*. If we now fix an open covering $\Delta = \{\Delta_n\}$ of multiplicity 2 by intervals Δ_n for which $0 \in \Delta_1$, $\Delta_n \cap \Delta_{n+1} \neq \varnothing$ for all $n \geq 1$ (in this case obviously $\operatorname{diam} \Delta_n \to 0$ as $n \to \infty$) then the open covering $\omega = \{\omega_n \times \Delta_n \mid n = 1, 2, \dots\}$ of the product $X \times J$ is adherent to $X \times \{1\}$.

The following theorem allows us to reduce the problem of exact extension of partial mappings to the less burdensome problem of approximate extension.

Theorem 3.1 (the approximative criterion for extendibility of a partial mapping). *Let an open covering $\omega \in \operatorname{cov} X \times [0,1)$ adhere to the top $X \times \{1\}$. Then the partial mapping $Z \supseteq A \xrightarrow{f} X$ has a global (neighborhood) extension if and only if the partial mapping $Z \times J \supseteq A \times J \xrightarrow{f \times \mathrm{Id}} X \times J$ has an ω-extension to $Z \times J$ (the neighborhood $A \times J$).*

From Theorem 3.1 we derive the following corollary.

Theorem 3.2. *If the product $X \times J$ is an approximative absolute (neighborhood) extensor then $X \in \mathrm{A[N]E}$.*

Proof of Theorem 3.1. Only sufficiency is nontrivial. Therefore, it remains to construct an extension for an arbitrary partial mapping $Z \supseteq A \xrightarrow{f} X$. By hypothesis, for the partial mapping $Z \times J \supseteq A \times J \xrightarrow{g} X \times J$, where $g = f \times \mathrm{Id}_J$, there exists a mapping $\tilde{g} : Z \times J \to X \times J$ (a mapping $\tilde{g} : U \to X \times J$) such that $(g, \tilde{g}\restriction_{A \times J}) \prec \omega$.

Assertion 3.3. *The mapping $d : A \times [0,1] \to X \times [0,1]$, defined by the formula $d\restriction_{A \times J} = \tilde{g}\restriction_{A \times J}$, $d\restriction_{A \times \{1\}} = f \times \{1\}$, is continuous.*

Proof. It suffices to verify continuity of d at the point $a \times \{1\}$. To this end, fix an arbitrary neighborhood $V \times [t,1]$ of the point $b = (f(a), 1)$ and find another neighborhood $V_1 \times [r,1]$, $t < r < 1$, of this point such that $\operatorname{St}(V_1 \times [r,1]; \omega) \subseteq V \times [t,1]$. Since the mapping $f : A \to X$ is continuous, there is a neighborhood $O(a) \subseteq V_1$ of a such that $f(O(a)) \subseteq V_1$. Using the condition of ω-proximity of $g = f \times \mathrm{Id}_J$ and $\tilde{g}\restriction_{A \times J}$, we easily conclude that $d(O(a) \times [r,1]) \subseteq \operatorname{St}(V_1 \times [r,1]; \omega) \subseteq V \times [t,1]$. ■

The following is easy and we omit the proof.

Assertion 3.4. *Suppose that a mapping $\alpha : F \cup W \to T$ is defined on the union of a closed set F and an open set W of a space S and that the restrictions $\alpha\restriction_F$ and $\alpha\restriction_W$ of this mapping to F and W are continuous. Then there exists a closed set F, $F_1 \supseteq F$, of S such that $\alpha\restriction_{F_1}$ is continuous and $F \cap W \subseteq \operatorname{Int} F_1$.*

If we put $S \rightleftharpoons Z \times I$, $T \rightleftharpoons X \times I$, $F \rightleftharpoons A \times I \cup Z \times \{0\}$, $W \rightleftharpoons Z \times J$ ($F \rightleftharpoons A \times I$, $W \rightleftharpoons U$ respectively), and $\alpha = d \cup \tilde{g}$, then there exists a closed set H, $H \supseteq Z \times I$, such that

$$A \times I \cup Z \times J \supseteq H \supseteq A \times I \cup Z \times \{0\} \quad (A \times I \cup U \supseteq H \supseteq A \times I),$$

and also

$$A \times [0,1) \cup Z \times \{0\} \subseteq \operatorname{Int} H \quad (\text{resp. } A \times [0,1) \subseteq \operatorname{Int} H),$$

while the restriction of α to H is continuous. Now, we use the following assertion whose proof is easy and also omitted.

Assertion 3.5. *There are a sequence of neighborhoods $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_i \supseteq \operatorname{Cl}(V_{i+1})$, $\bigcap V_i = A$, and a monotone increasing numeric sequence $0 = r_0 < r_1 < r_2 < \dots$, $\lim r_i = 1$, for which $V_i \times [0, r_i] \subseteq H$.*

Let $\xi_i : Z \to [r_{i-2}, r_{i-1}]$, $i \geq 2$, be continuous functions such that $\xi_i\restriction_{\operatorname{Bd} V_i} \equiv r_{i-1}$ and $\xi_i\restriction_{\operatorname{Bd} V_{i-1}} \equiv r_{i-2}$. Then the function $\xi : Z \to [0,1]$, $\xi\restriction_{Z \setminus V_1} = 0$, $\xi\restriction_{V_{i-1} \setminus V_i} = \xi_i$ for $i \geq 2$, $\xi\restriction_A \equiv \mathrm{Id}$, is continuous and $(v, \xi(v)) \in H, v \in Z$ (resp. $(v, \xi(v)) \in H$, $v \in V_1$). The sought extension $\hat{f}$ of the partial mapping f is given by the formula $\hat{f}(v) = \alpha(v, \xi(v))$, $v \in Z$ ($v \in V_1$). ■

Closing the section, we formulate a few more easy facts about the interrelation of the classes ANE and A-ANE. These facts are not used in the article and therefore their proofs are omitted. Sometimes it is reasonable to transform Theorem 3.2 into the following assertion.

Theorem 3.6. *A necessary and sufficient condition for* $X \in \mathrm{A[N]E}$ *is as follows:*

(3) *There are an* A-A[N]E*-space* P *and mappings* $\alpha : X \times J \to P$ *and* $\beta : P \to X \times J$ *such that* $(\beta \circ \alpha, \mathrm{Id}_{X\times J}) \prec \omega$ (*here* $\omega \in \operatorname{cov} X \times [0, 1)$ *stands for an open covering adherent to the top* $X \times \{1\}$).

Which necessary conditions should we impose on an approximative absolute extensor X to guarantee the membership $X \in \mathrm{AE}$? The following theorem shows that it suffices to require that X is locally equiconnected.

Theorem 3.7. *If* $X \in \text{A-A[N]E} \cap \mathrm{LEC}$ *then* X *is an absolute* (*neighborhood*) *extensor.*

§ 4. Application of the Approximative Criterion for Extendibility of Mappings

Almost all criteria for absolute extensors and retracts can be deduced from Theorems 3.1 and 3.2 rather easily. Therefore, it is natural to call these theorems the main criteria for ANE-spaces.

With these theorems taken into account, the problem of exact extension of mappings for a wide class of spaces is solved by reducing it to that of approximative extension of mappings. An ω-extension $\varphi : Z \to X$ of a partial mapping $Z \supseteq A \xrightarrow{\varphi} X$ is in turn sought usually in the form of the composite of the canonical mapping $\theta : X \to \mathfrak{N}\langle\sigma\rangle$, generated by some covering $\sigma \in \operatorname{cov} Z$, and a continuous mapping $\mu : \mathfrak{N}\langle\sigma\rangle \to Z$. Which sufficient conditions on μ would guarantee ω-proximity of the composite $\mu \circ \theta$ to φ? To answer this question, we recall some facts that can be traced back to Michael [13].

Definition 4.1. Assume given a mapping $\varphi : A \to X$ between metric spaces and coverings $\sigma = \{V_a\} \in \operatorname{cov} A$ and $\gamma, \lambda \in \operatorname{cov} X$. We say that a mapping $\mu : \mathfrak{N}\langle\sigma\rangle \to X$ from the nerve $\mathfrak{N}\langle\sigma\rangle$ into X satisfies the (λ, γ)-*condition relative to* φ if

(1) μ is a γ-realization;

(2) $\mu(\langle V\rangle) \subseteq N(\varphi(a); \lambda)$ for all $V \in \sigma$ and $a \in V$.

Theorem 4.2. *The composite* $A \xrightarrow{\theta} \mathfrak{N}\langle\sigma\rangle \xrightarrow{\mu} X$ *of an arbitrary canonical mapping* θ *with a mapping* μ *satisfying the* (λ, γ)*-condition relative to a mapping* $\varphi : A \to X$ *is* $(\lambda \circ \gamma)$*-close to* φ.

Proof. Suppose that $a \in A$, $a \in \bigcap V_i$, and $\theta(a) = \sum \alpha_i\langle V_i\rangle$. Since μ is a γ-realization, we have $\{\mu\circ\theta(a), \mu\langle V_i\rangle \mid i\} \subseteq U \in \gamma$. Condition (2) implies that $\{\varphi(a), \mu\langle V_i\rangle \mid i\} \subseteq W \in \lambda$. This yields the sought proximity between the mappings: $\operatorname{dist}(\mu \circ \theta(a), \varphi(a)) \prec \lambda \circ \gamma$. ■

In the rest of this section we give a series of theorems which use Theorem 3.2 and imply both Theorem 1.3 and Haver's and Toruńczyk's theorems.

Theorem 4.3. *Suppose that one of the following two conditions holds for a countably dimensional metric space* X:

(3) $X \in \mathrm{LC}$;

(4) X *has an open base of weakly homotopy trivial sets; i.e.,* $\pi_i X = 0$ *for all* $i \geq 0$.

Then $X \in$ A-ANE.

Proof. We establish a stronger fact: If in the case (3) or (4) for a partial mapping $Z \supseteq A \xrightarrow{f} X$ one of the two spaces A and $f(A)$ is countably dimensional then for every covering $\omega \in \operatorname{cov} X$ there is a neighborhood ω-extension $\hat{f} : U \to X$ of f.

The case of local contractivity of X. Given an arbitrary natural number $i \geq 0$ and an arbitrary point $x \in X$, we can choose neighborhoods $V_i(x) \subseteq U_i(x) \subseteq V_{i-1}$ and contractions $F^{x,i} : V_i(x) \times I \to U_i(x)$ of $V_i(x)$ within $U_i(x)$ to x such that the covering $\{U_0(x) \mid x \in X\}$ is a refinement of ω and the covering $\{U_i(x) \mid x \in X\}$ is a starlike refinement of the covering $\{V_{i-1}(x) \mid x \in X\}$ for all $i \geq 1$. For definiteness, we consider $F^{x,i} \restriction_{V_i(x)\times\{0\}} = \mathrm{Id}$ and $F^{x,i} \restriction_{V_i(x)\times\{1\}} = x$.

Since A or $f(A)$ is countably dimensional, by Proposition 2.7 there are open families σ and σ_i, $i \geq 2$, in Z for which $\sigma = \bigcup_{i=1}^{\infty} \sigma_i$ covers A, $\operatorname{mult.} \sigma_i = 1$, and $f(\sigma_i \cap A) \succ \{V_i(x) \mid x \in X\}$. Without loss of generality, we may assume that the nerves of σ and $\sigma\restriction_A$ are isomorphic to each other; i.e.,

$$\bigcap_{i=1}^{k} W_i \neq \varnothing \Longleftrightarrow \bigcap_{i=1}^{k} (W_i \cap A) \neq \varnothing$$

for $W_i \in \sigma$. Our purpose is to extend f to the carrier $\bigcup \sigma$ of σ which is a neighborhood of A.

To this end, we consider the nerve $\mathfrak{N}\langle\sigma\rangle = \mathfrak{N}$ and the canonical mapping $\theta : \bigcup \sigma \to \mathfrak{N}$. The sought mapping f is constructed as the composite of θ and some mapping $g : \mathfrak{N} \to X$. Before defining g, we emphasize the important circumstance that the set of vertices of an arbitrary simplex of the nerve is furnished with some natural linear order. If $\Delta = \langle W_0, \dots, W_k\rangle$, $W_i \in \sigma_{n_i}$, is a k-dimensional simplex then $\bigcap_{i=0}^{k} W_i \neq \varnothing$ and all numbers n_i are distinct (this important observation follows from the fact that the multiplicity of the system σ_{n_i} equals one). Therefore, we can linearly order the vertices of the simplex Δ so that the indices n_i of the system σ_i constitute an increasing sequence $n_0 < n_1 < \dots < n_k$.

For $W \in \sigma_i$ we have $f(W \cap A) \subseteq S \in \{V_i(x)\}$, and since $\operatorname{St}(S; \{U_i(x)\}) \subseteq V_{i-1}(x_W)$ for some point x_W, this yields

(5) $\operatorname{St}(f(W \cap A); \{U_i(x)\}) \subseteq V_{i-1}(x_W)$.

Define $g_0 : \mathfrak{N}^{(0)} \to X$ by the formula $g_0(\langle W\rangle) = x_W$. Next, we successively extend g_0 to some mappings $g_i : \mathfrak{N}^i \mapsto X$, $i = 1, \dots, k-1$, defined on the ith skeleton of the nerve, each time requiring the validity of the condition

(6) for every simplex $\Delta^{k-1} = \langle W_0, \dots, W_{k-1}\rangle$, $k > 1$,

$$f(W_0 \cap A) \cup g_{k-1}(\Delta^{k-1}) \subseteq U_{n_0-1}(x_{W_0}).$$

Construct $g_k : \mathfrak{N}^{(k)} \to X, g_k\restriction_{\mathfrak{N}^{(k-1)}} = g_{k-1}$, assuming that $g_{k-1} : \mathfrak{N}^{(k-1)} \to X$ satisfying (6) are already available.

Lemma 4.4. *If $\Delta^k = \langle W_0, W_1, \dots, W_k\rangle$ is a k-dimensional simplex of $\mathfrak{N}$ then $g_{k-1}(\Delta^{k-1}) \subseteq V_{n_0-1}(x_{W_0})$, where $\Delta^{k-1} = \langle W_1, \dots, W_k\rangle$ is a face of Δ^k.*

Proof of the lemma. By (6) we have $f(W_1 \cap A) \cup g_{k-1}(\Delta^{k-1}) \subseteq U_{n_1-1}(x_{W_1})$. Since $\bigcap_{i=0}^{k} W_i \cap A \neq \varnothing$ and so $f(W_0 \cap A) \cap f(W_1 \cap A) \neq \varnothing$, it follows that

$$f(W_1 \cap A) \cup g_{k-1}(\Delta^{k-1}) \subseteq \operatorname{St}(f(W_0 \cap A); \{U_{n_1-1}(x)\}).$$

By (5) we have $\operatorname{St}(f(W_0 \cap A); U_{n_1-1}(x)) \subseteq V_{n_1-2}(x_{W_0})$. Since $n_0 \leq n_1 - 1$; therefore, $V_{n_1-2}(x_{W_0}) \subseteq V_{n_0-1}(x_{W_0}) \subseteq V_{n_0-1}(x_{W_0})$ and so $g_{k-1}(\Delta^{k-1}) \subset V_{n_0-1}(x_{W_0})$. ■

Now, the mapping $g_k\restriction_{\Delta^k}$ can be soundly defined by the formula

$$g_k(t \cdot \langle W_0\rangle + (1-t) \cdot v) = F^{x_{W_0}, n_0-1}(g_{k-1}(v), t), \quad \text{where } \in \langle W_1, \dots, W_k\rangle.$$

From the inclusion $\operatorname{Im} F^{x_{W_0}, n_0-1} \subseteq U_{n_0-1}(x_{W_0})$ we can easily deduce the following fact.

Lemma 4.5. $f(W_0 \cap A) \cup g_k(\Delta^k) \subseteq U_{n_0-1}(x_{W_0})$.

Lemma 4.6. $g_k\restriction_{\partial\Delta^k} = g_{k-1}$.

Proof of Lemma 4.6. Put $u = t \cdot \langle W_0\rangle + (1-t) \cdot v \in \partial\Delta^k$. If $t = 0$ then $u = v \in \langle W_1, \dots, W_k\rangle$ and $g_k(u) = g_{k-1}(v) = g_{k-1}(u)$. If $t > 0$ then $v \in \partial\langle W_1, \dots, W_k\rangle$. The formula for g_k readily yields the equalities $g_k(u) = F^{x_{W_0}, n_0-1}(g_{k-1}(v), t)$ and $g_{k-1}(u) = F^{x_{W_0}, n_0-1}(g_{k-2}(v), t)$. Since the equality $g_{k-1}\restriction_{\partial\langle W_1, \dots, W_k\rangle} = g_{k-2}$ has been already established, we have $g_{k-2}(v) = g_{k-1}(v)$, which completes the proof of the lemma. ■

We now demonstrate that the sought ω-extension $\hat{f} : \bigcup \sigma \to X$ is given by $\hat{f} = g \circ \theta$; this will finish the proof of (3). Indeed, if $a \in A$, $a \in \bigcap_{i=1}^{k} W_i$, and $\theta(a) = \sum_{i=1}^{k} \alpha_i \cdot \langle W_i \rangle$, then by condition (6) applied to the simplex $\langle W_0, \dots, W_k \rangle$ we have

$$f(W_0 \cap A) \cup g(\theta(a)) \subseteq U_{n_0-1}(x_{W_0}) \subseteq V_{n_0-2}(x_{W_0});$$

i.e., $f(a)$ and $\hat{f}(a)$ are ω-close (because $n_0 \geq 2$ while $\{V_{n_0-2}(x)\} \succ \omega$ for $n_0 - 1 \geq 1$).

The case in which X has an open base of weakly homotopy trivial sets. For all $i \geq 0$ and an arbitrary point $x \in X$ we can choose neighborhoods $V_i(x)$ such that

(a) all homotopy groups $\pi_m(V_i(x))$ are trivial;

(b) the covering $\{V_0(x) \mid x \in X\}$ is a refinement of ω;

(c) the covering $\{V_i(x) \mid x \in X\}$ is a starlike refinement of the covering $\{V_{i-1}(x) \mid x \in X\}$ for all $i \geq 1$.

Proceeding as in the first part of this theorem, we construct open (in Z) systems $\sigma = \bigcup_{i \geq 1} \sigma_i$ such that $\operatorname{mult.} \sigma_i = 1$, $f(\sigma_i \bigcap A) \succ \{V_i(x) \mid x \in X\}$, and $\bigcup \sigma \supset A$. Without loss of generality, we assume that $\mathfrak{N}\langle\sigma\rangle = \mathfrak{N}\langle\sigma \restriction_A\rangle$. As before we linearly order the vertices of the simplices $\Delta = \langle W_0, \dots, W_k \rangle$, $W_i \in \sigma_{n_i}$, in the nerve $\mathfrak{N}\langle\sigma\rangle \rightleftharpoons \mathfrak{N}$ of σ, requiring that the indices n_i of the systems σ_i constitute an increasing sequence $n_0 < n_1 < \dots < n_k$.

We construct the sought mapping f as the composite of the canonical mapping $\theta : \bigcup \sigma \to \mathfrak{N}$ and some mapping $g : \mathfrak{N} \to X$.

For $W \in \sigma_i$, we have $f(W \cap A) \subseteq S \in \{V_i(x)\}$, and since $\operatorname{St}(S; \{V_i(x)\}) \subseteq V_{i-1}(x_W)$ for some point x_W, this yields

(7) $\operatorname{St}(f(W \cap A); \{V_i(x)\}) \subseteq V_{i-1}(x_W)$.

Define $g_0 : \mathfrak{N}^{(0)} \to X$ by the formula $g_0(\langle W \rangle) = x_W$. Now, we successively extend g_0 to mappings $g_i : \mathfrak{N}^i \mapsto X$, $i = 1, \dots, k-1$, defined on the ith skeleton, each time requiring validity of the condition

(8) for every simplex $\Delta^{k-1} = \langle W_0, \dots, W_{k-1} \rangle$, $W_i \in \sigma_{n_i}, k > 1$,

$$f(W_0 \cap A) \cup g_{k-1}(\Delta^{k-1}) \subseteq V_{n_0-1}(x_{W_0}).$$

By analogy to Lemma 4.4, we prove the next

Lemma 4.7. *If $\Delta^k = \langle W_0, \dots, W_k \rangle$ is a k-dimensional simplex of $\mathfrak{N}$ then*

$$g_{k-1}(\Delta_0^{k-1}) \subset V_{n_0-1}(x_{W_0})$$

and so

$$g_{k-1}(\partial \Delta^k) \subseteq V_{n_0-1}(x_{W_0})$$

(here $\partial \Delta^k = \bigcup \Delta_i^{k-1}$, $\Delta_i^{k-1} = \langle W_0, \dots, \check{W}_i, \dots, W_k \rangle$).

Since $V_{n_0-1}(x)$ is $(k-1)$-connected, there is a mapping

$$g_k : \Delta^k \to V_{n_0-1}(x_{W_0}), \quad g_k \restriction_{\partial \Delta^k} = g_{k-1}.$$

So we have constructed a mapping $g_k : \mathfrak{N}^{(k)} \to X$ that extends g_{k-1}, is defined on the k-dimensional skeleton, and satisfies (8).

Thus, the mapping $g : \mathfrak{N} \to X$ is constructed; moreover, condition (8) is valid for every simplex Δ. Demonstrate that the sought ω-extension $\hat{f} : \bigcup \sigma \to X$ is given by $\hat{f} = g \circ \theta$. Indeed, if $a \in A$, $a \in \bigcap V_i$, and $\theta(a) = \sum a_i \cdot \langle V_i \rangle$ then by condition (8) applied to the simplex $\langle W_0, \dots, W_k \rangle$ we have

$$f(W_0 \cap A) \cup g(\theta(a)) \subseteq V_{n_0-1}(x_{W_0}).$$

In view of $\{V_{n_0-1}(x)\} \succ \omega$ for $n_0 - 1 \geq 0$, this implies the ω-proximity of f and $\hat{f}$. ■

Theorem 4.8. *Suppose that a metric space X has an open base $\mathfrak{B}$ all of whose finite intersections are homotopy trivial (and so the elements of the base are homotopy trivial themselves). Then $X \in$ A-ANE.*

Proof. Obviously, $X \in \mathrm{LC}^n$ for all n. The Mayer–Vietoris theorem [14] easily implies the following

Assertion 4.9. *If $W_i \in \mathfrak{B}$, $i = 1, \dots, n$, have a nonempty intersection then all homotopy groups of $\bigcup_{i=1}^n W_i$ are trivial.*

Suppose that $\omega = \{U_\beta\} \in \operatorname{cov} X$ and let $Z \supseteq A \xrightarrow{f} X$ be an arbitrary partial mapping. Without loss of generality, we may assume that the covering ω consists of elements of the base $\mathfrak{B}$.

Delate the covering $\sigma = f^{-1}(\omega) \in \operatorname{cov} A$ to a system $\tilde\sigma = \{V_\gamma\}$ of open sets in Z (this means that $\bigcap_{i=1}^m V_{\gamma_i} \neq \varnothing$ implies $V_{\gamma_i} \cap A \in \sigma$ and $\bigcap_{i=1}^m (V_{\gamma_i} \cap A) \neq \varnothing$). Assigning to each element $V_\gamma \in \tilde\sigma$ an element $U_\beta \in \omega$ such that $f(V_\gamma \cap A) \subseteq U_{\beta(\gamma)}$, we thereby define the mapping $g_0 : \mathfrak{N}^{(0)} \to X, g_0(\langle V_\gamma\rangle) \in U_{\beta(\gamma)}$ on the zero-dimensional skeleton of the nerve $\mathfrak{N}(\tilde\sigma)$. Now, we successively extend g_0 to mappings $g_i : \mathfrak{N}^i \mapsto X, i = 1, \dots, k-1$, defined on the ith skeleton of the nerve, each time requiring validity of the condition

(9) for every simplex $\Delta^{k-1} = \langle V_0, \dots, V_{k-1}\rangle$, $k > 1$,

$$\left(\bigcup_{i=0}^{k-1} f(V_i \cap A)\right) \cup g_{k-1}(\Delta^{k-1}) \subseteq \bigcup_{i=0}^{k-1} U_{\beta(\gamma_i)}.$$

Note that condition (9) follows easily from $g_{k-1}(\partial\Delta^k) \subseteq \bigcup_{i=0}^k U_{\beta(\gamma_i)}$ for every k-dimensional simplex $\Delta^k = \langle V_0, \dots, V_k\rangle$.

Since $\bigcap_{i=0}^k U_{\beta(\gamma_i)} \supseteq f((\cap V_i) \cap A) \neq \varnothing$, by Assertion 4.9 we have $\bigcup_{i=0}^k U_{\beta(\gamma_i)} \in C^\infty$. Therefore, there is an extension $g_k : \Delta^k \to \bigcup_{i=0}^k U_{\beta(\gamma_i)}$ of g_{k-1}.

From (9) it follows easily that the mapping g is an ω^2-realization. From Theorem 4.2 we obtain the ω^4-proximity of f to the composite $g \circ (\theta\restriction_A) : A \to X$ and so $g \circ \theta : \bigcup \tilde\sigma \to X$ is a sought ω^4-extension of f. By the arbitrariness of ω, this implies $X \in$ A-ANE. ■

§ 5. Fiberwise Trivial Mappings

Consider the projection $p \rightleftharpoons \mathrm{pr}_M : M \times N \to M$ of the product of metric spaces onto the first factor. Let the image $p(X)$ of a (not necessarily closed) subset $X \subset M \times N$ lie in $Y \subset M$. Denote the restriction of p to X by $\pi : X \to Y$. In general, we require neither surjectivity of π nor even density of $\pi(X)$ in Y.

We introduce some new notions. We say that an embedding $A \hookrightarrow B$ of subsets A and B of the product $M \times N$ is *fiberwise contractible within* p if there is a mapping $g : p(A) \to B$, $p \circ g = \mathrm{Id}_{p(A)}$, and a homotopy $H_t : A \to B$ such that

(1) $p \circ H_t = p$ for all $t \in I$ (the condition that the homotopy H_t is fiberwise);

(2) $H_0 = \mathrm{Id}_A$;

(3) $H_1 = g \circ p$ (the condition that H_1 factors through the projection p).

In this section we fix a collection Ω_X of neighborhoods of X such that

(4) the image $p(\mathscr{U})$ includes Y for all $\mathscr{U} \in \Omega_X$;

(5) for every neighborhood $\mathscr{U} \in \Omega_X$ and every point $y \in Y$, there is a neighborhood $O \rightleftharpoons O(y) \subset M$ such that $O \times \mathrm{pr}_N(O^\bullet \cap X) \subset \mathscr{U}$ (here and below, $C^\bullet$ stands for the product $C \times N = p^{-1}(C)$, where $C \subset M$).

Suppose that $\mathscr{U}, \mathscr{V} \in \Omega_X$ and $\mathscr{V} \subset \mathscr{U}$. We say that a mapping $\pi : X \to Y$ is *fiberwise $\mathscr{U}\mathscr{V}$-contractible within the projection* p if there exists a neighborhood $W \subset \pi\mathscr{V}$ of Y such that the embedding $W^\bullet \cap \mathscr{V} \hookrightarrow W^\bullet \cap \mathscr{U}$ is fiberwise contractible within p. We say that a mapping $\pi : X \to Y$ is *fiberwise trivial within* p if for every neighborhood $\mathscr{U} \in \Omega_X$ there exists a smaller neighborhood $\mathscr{V} \in \Omega_X$, $\mathscr{V} \subset \mathscr{U}$, such that the mapping $\pi : X \to Y$ is fiberwise $\mathscr{U}\mathscr{V}$-contractible within p. We say that π is *locally fiberwise trivial within the projection* p if for every neighborhood $\mathscr{U} \in \Omega_X$ there exist a neighborhood $\mathscr{V} \in \Omega_X$, $\mathscr{V} \subset \mathscr{U}$, and a family $\sigma = \{O(y) \mid y \in Y\}$ of open sets in M which cover Y and are such that $p(\mathscr{V}) \supset \bigcup\sigma$ and $O(y)^\bullet \cap \mathscr{V} \hookrightarrow O(y)^\bullet \cap \mathscr{U}$ is fiberwise contractible within p for all $y \in Y$.

Proposition 5.1. *Suppose that $\mathscr{V}_1 \supset \mathscr{V}_2 \supset \mathscr{V}_3 \supset \dots$ is a sequence of neighborhoods in Ω_X and the space Y lies in the union $W = \bigcup_{i=1}^{\infty} W_i$ of open subsets of M. If the conditions*

(6) $p(\mathscr{V}_{i+1}) \supset W_i$ *for all $i \geq 1$;*

(7) *the embedding $W_i^\bullet \cap \mathscr{V}_{i+1} \hookrightarrow W_i^\bullet \cap \mathscr{V}_i$ is fiberwise contractible within p for all $i \geq 1$*

are satisfied then $\mathscr{V} \rightleftharpoons \bigcup_{i=1}^{\infty} W_i^\bullet \cap \mathscr{V}_{i+1} \in \Omega_X$ and the projection π is fiberwise $\mathscr{V}_1\mathscr{V}$-contractible within p.

Proof. By [13] there is a countable open system $\{W_i'\}_{i=1}^{\infty}$ in M which covers Y and for which $\mathrm{Cl}_W W_i' \subset W_i$. Clearly, $\mathrm{Cl}_W F^n \subset E^n$, where $F^n = \bigcup_{i=1}^{n} W_i'$ and $E^n = \bigcup_{i=1}^{n} W_i$.

By assumption (7), there exist mappings $g_i : W_i \to \mathscr{V}_i$ and homotopies $G_i : (W_i^\bullet \cap \mathscr{V}_{i+1}) \times I \to W_i^\bullet \cap \mathscr{V}_i$ such that

(9) $p \circ (G_i)_t = p$ for all $t \in I$;

(10) $(G_i)_0 = \mathrm{Id}$;

(11) $(G_i)_1 = g_i \circ p$.

The plan of the further proof of the proposition consists in constructing mappings $h_n : E^n \to \mathscr{V}_1$ and homotopies $H_n : ((E^n)^\bullet \cap \mathscr{V}_{n+1}) \times I \to \mathscr{V}_1$, $n \geq 1$, such that

(12) $p \circ (H_n)_t = p$ for all $t \in I$;

(13) $(H_n)_0 = \mathrm{Id}$;

(14) $(H_n)_1 = h_n \circ p$;

(15) $H_{n+1}\restriction((F^n)^\bullet \cap \mathscr{V}_{n+2}) = H_n\restriction((F^n)^\bullet \cap \mathscr{V}_{n+2})$.

Thereby $h_{n+1}\restriction F^n$ will coincide with $h_n\restriction F^n$ and the formulas

(α) $h(y) = h_n(y)$, where y belongs to the neighborhood $F^\infty = \bigcup_{i=1}^{\infty} F^n = \bigcup_{i=1}^{\infty} W_i'$ of Y in M and the index n is such that $y \in F^n$,

(β) $H(x,t) = H_n(x,t)$, where $x \in \mathscr{V} \cap (F^\infty)^\bullet$ and the index n is such that $x \in (F^n)^\bullet \cap \mathscr{V}_{n+1}$,

will soundly define a continuous mapping $h : F^\infty \to \mathscr{V}_1$ and a continuous homotopy $H : \mathscr{V} \cap (F^\infty)^\bullet \to \mathscr{V}_1$. This will establish that the projection π is fiberwise $\mathscr{V}_1\mathscr{V}$-contractible.

Take h_1 and H_1 to be g_1 and G_1. Suppose that $h_1, \dots, h_n$ and $H_1, \dots, H_n$, satisfying (12)–(15), have been already constructed. In the metric space $C \rightleftharpoons E^n \cup W_{n+1} = E^{n+1}$, consider the closed sets $A \rightleftharpoons W_{n+1} \setminus E^n$ and $\mathrm{Cl}_C B$, where $B \rightleftharpoons (E^n \setminus W_{n+1}) \cup F^n$. Once $\mathrm{Cl}_W(F^n) \subset E^n$, it follows that A and $\mathrm{Cl}_C B$ are disjoint and hence there is an Urysohn function $\gamma : C \to [0,2]$ such that $\gamma^{-1}(0)$ is a neighborhood of B and $\gamma^{-1}(2)$ is a neighborhood of A. Then the functions $\alpha(c) = 2 - \max(1, \gamma(c))$ and $\beta(c) = \min(1, \gamma(c))$ carry C into the interval $[0,1]$ and possess the following properties:

(16) $\alpha^{-1}(1) \cup \beta^{-1}(1) = C$, i.e., $\alpha(x) = 1$ or $\beta(x) = 1$ for every point $x \in C$;

(17) $\beta^{-1}(0)$ coincides with $\gamma^{-1}(0)$ and is a neighborhood of B, whereas $\alpha^{-1}(0)$ coincides with $\gamma^{-1}(2)$ and is a neighborhood of A.

Define the mapping $H_{n+1} : ((E^{n+1})^\bullet \cap \mathscr{V}_{n+2}) \times I \to \mathscr{V}_1$ by the formula

$$H_{n+1}(x,t) = \begin{cases} H_n(x,t) & \text{if } x \in B^\bullet \cap \mathscr{V}_{n+2},\ t \in I; \\ G_{n+1}(x,t) & \text{if } x \in A^\bullet \cap \mathscr{V}_{n+2},\ t \in I; \\ H_n(G_{n+1}(x, \beta(p(x)) \cdot t), \alpha(p(x)) \cdot t) & \text{if } x \in (E^n \cup W_{n+1})^\bullet \cap \mathscr{V}_{n+2},\ t \in I. \end{cases}$$

Since the value of the function $\alpha(p(x))$ at the point $p(x) \in W_{n+1} \setminus E^n = A$ equals 0 while $(H_n)_0 = \mathrm{Id}$; therefore, H_{n+1} is a well defined continuous homotopy.

Clearly, $(H_{n+1})_0 = \mathrm{Id}$. The fact that the homotopies H_n and G_{n+1} are fiberwise implies that so is the homotopy $H_{n+1} : (H_{n+1})_t \circ p = p$ for all $t \in I$. Since $(H_n)_1$ and $(G_{n+1})_1$ carry all points of a single fiber to one point while (16) implies that $\beta(p(x)) = 1$ or $\alpha(p(x)) = 1$, it follows that $(H_{n+1})_1$ also carries all points of a single fiber to one point. If $p(x) \in F^n$ (i.e., $\gamma(p(x)) = 0$) then it is easy to see that $\beta(p(x)) = 0$ and $\alpha(p(x)) = 1$. Hence, $H_{n+1}(x,t) = H_n(x,t)$; i.e., (15) holds.

Consider the mapping $h_{n+1} : E_{n+1} \to \mathscr{V}_1$ defined by the formula

$$h_{n+1}(y) = \begin{cases} h_n(y) & \text{if } \ y \in \beta^{-1}(0); \\ H_n(G_{n+1}(g_{n+1}(y), \beta(y)), \alpha(y)) & \text{if } \ y \in W_{n+1}. \end{cases}$$

Clearly, this mapping is well defined at the points $y \in \beta^{-1}(0) \cap W_{n+1}$. It remains to establish the continuity of h_{n+1} and the coincidence of $(H_{n+1})_1$ with $h_{n+1} \circ p$, which can be done without effort. It is also easy to prove that the neighborhood $\mathscr{V}$ satisfies conditions (4) and (5), i.e., $\mathscr{V} \in \Omega_X$. ■

We now formulate a series of conditions on X that will guarantee validity of the properties of local fiberwise triviality and fiberwise triviality for the projections of a graph (Propositions 5.2–5.4).

Proposition 5.2. *Suppose that $Z \hookleftarrow A \xrightarrow{\varphi} X \in \mathrm{LC}$ is a partial mapping and $G_\varphi \rightleftharpoons \{(a, \varphi(a)) \mid a \in A\} \subset Z \times X$ is the graph of φ. Then the projection $\pi : G_\varphi \to A$ of G_φ to A is locally fiberwise trivial within the projection $p : Z \times X \to Z$.*

Proof. It is easy to observe that the family Ω_{G_φ} coincides with the collection of all neighborhoods of the graph G_φ in $Z \times X$. (This observation will be utilized in Propositions 5.3–5.5 too.)

Let $\mathscr{U} \supset G_\varphi$ be a closed neighborhood in $Z \times X$. To prove the proposition, we have to find a neighborhood $\mathscr{V} \in \Omega_{G_\varphi}$, $G_\varphi \subset \mathscr{V} \subset \mathscr{U}$, in $Z \times X$ such that

(a) For every point $a_0 \in A$, there is a neighborhood $O = O(a_0) \subset Z$ for which the embedding $O^\bullet \cap \mathscr{V} \hookrightarrow O^\bullet \cap \mathscr{U}$ is fiberwise contractible, where $O^\bullet \rightleftharpoons O \times X$.

To start, we consider the multivalued mapping $\Phi : A \rightsquigarrow \mathbb{R}^+$, $\Phi(a) = \{r > 0 \mid \mathrm{N}(a; r) \times \mathrm{N}(\varphi(a); r) \subset \mathscr{U}\} \subset \mathbb{R}^+$. It is easy to see that Φ is a lower semicontinuous convex-valued mapping; hence, by Dowker's theorem [13, 5.5.20] there exists a continuous selection $r : A \to (0, \infty)$. We have another multivalued mapping $\Psi : A \rightsquigarrow \mathbb{R}^+$, $\Psi(a) = \{r(a) \geq t > 0 \mid \mathrm{N}(\varphi(a); t)$ contracts in $\mathrm{Cl}(\mathrm{N}(\varphi(a); r(a)))$ to a point$\} \subset \mathbb{R}^+$ which is lower semicontinuous and convex-valued and which has a continuous selection $t : A \to (0, \infty)$ by Dowker's theorem; moreover, $t(a) \leq r(a)$.

We use continuity of φ and diminish the neighborhood $\mathrm{N}(a; r(a))$ to a neighborhood $W(a)$ such that

(b) $\mathrm{N}(\varphi(a'); t(a')/2) \subset \mathrm{N}(\varphi(a); t(a))$ for every point $a' \in W(a) \cap A$.

By paracompactness of A, there exists a family $\sigma' = \{W'(a) \subset W(a) \mid a \in A\}$ of neighborhoods of points in A which is a starlike refinement of $\sigma \rightleftharpoons \{W(a) \mid a \in A\}$, $\sigma' \circ \sigma' \succ \sigma$. Assign to each point $a \in A$ a point $z_a \in A$ such that

$$\mathrm{St}_{\sigma'}(a) \rightleftharpoons \bigcup_{a \in W'(b)} W'(b) \subset W(z_a).$$

Proposition 2.1 yields

(c) $\bigcap_{i=1}^n W'(a_i) \neq \varnothing$ implies $\bigcup_{i=1}^n W'(a_i) \subset \bigcap_{i=1}^n W(z_{a_i})$.

Finally, define the sought neighborhood $\mathscr{V} \in \Omega_{G_\varphi}$ by

$$\mathscr{V} \rightleftharpoons \bigcup\nolimits_{a \in A} W'(a) \times \mathrm{N}(\varphi(a); t(a)/2).$$

Clearly, $G_\varphi \subset \mathscr{V} \subset \mathscr{U}$. To verify (a), consider an arbitrary point $a_0 \in A$ and its neighborhood $O = W'(a_0)$ and demonstrate that the embedding $O^\bullet \cap \mathscr{V} \hookrightarrow O^\bullet \cap \mathscr{U}$ admits fiberwise contraction.

Take $z \in O$. The explicit formulas for $\mathscr{V}$ and O imply that

(d) $(z \times X) \cap \mathscr{V} = \{z\} \times \bigcup_{\lambda \in \Lambda_z} \mathrm{N}(\varphi(a_\lambda); t(a_\lambda)/2)$;

(e) $(z \times X) \cap \mathscr{U} \supset \{z\} \times \bigcup_{\lambda \in \Lambda_z} \mathrm{N}(\varphi(z_{a_\lambda}); r(z_{a_\lambda}))$, where $\Lambda_z = \{\lambda \mid W'(a_\lambda) \ni \{z\}\}$.

Since $W'(a_0) \ni \{z\}$ for all $z \in O$, the index λ_0, corresponding to $W'(a_0)$, belongs to $\bigcap_{z \in O} \Lambda_z$. If we show that $\bigcup_{z \in O} \bigcup_{\lambda \in \Lambda_z} \mathrm{N}(\varphi(a_\lambda); t(a_\lambda)/2)$ contracts to a point in the set $\bigcap_{z \in O} \bigcup_{\lambda \in \Lambda_z} \mathrm{N}(\varphi(z_{a_\lambda}); r(z_{a_\lambda}))$, then $(z \times X) \cap \mathscr{V}$ will contract to a point in $(z \times X) \cap \mathscr{U}$ (moreover, the contraction would be independent of z) and so the embedding $O^\bullet \cap \mathscr{V} \hookrightarrow O^\bullet \cap \mathscr{U}$ will be fiberwise contractible.

We turn the reader's attention to the fact that, by (c), $\bigcup_{z \in O} \bigcup_{\lambda \in \Lambda_z} a_\lambda \subset W(z_{a_{\lambda_0}})$, whereas (b) implies that

$$\bigcup\nolimits_{z \in O} \bigcup\nolimits_{\lambda \in \Lambda_z} \mathrm{N}(\varphi(a_\lambda); t(a_\lambda)/2) \subset \mathrm{N}(\varphi(z_{a_{\lambda_0}}); t(z_{a_{\lambda_0}})).$$

Since $t : A \to (0, \infty)$ is a selection of Ψ, it follows that $\mathrm{N}(\varphi(z_{a_{\lambda_0}}); t(z_{a_{\lambda_0}}))$ contracts in $\mathrm{N}(\varphi(z_{a_{\lambda_0}}); r(z_{a_{\lambda_0}}))$ to a point, which establishes the sought property (a). ■

Proposition 5.3. *Suppose that an embedding $X' \hookrightarrow X$ is* LEC, $X' \in$ ANE, *and $Z \hookleftarrow A \xrightarrow{\varphi} X'$ is a partial mapping. Then the projection $\pi : G_\varphi \to A$ of the graph of φ to A is fiberwise trivial within the projection $p : Z \times X \to Z$.*

Proof. Let a closed neighborhood $\mathscr{O} \subset X' \times X$ and a mapping $\lambda : \mathscr{O} \times I \to X$ be taken from the definition of LEC-embedding. Fix a closed neighborhood $\mathscr{U} \supset G_\varphi$ in $Z \times X$. Since $X' \in$ ANE, the mapping φ admits an extension $\hat{\varphi} : W \to X'$ to some closed neighborhood $W \supset A$ such that $G_{\hat{\varphi}} \subset \mathscr{U}$.

Consider the multivalued mapping $\Phi : W \rightsquigarrow \mathbb{R}^+$, $\Phi(w) = \{r > 0 \mid \mathrm{N}(w; r) \times \mathrm{N}_X(\hat{\varphi}(w); r) \subset \mathscr{U}$ and $\mathrm{N}_{X'}(\hat{\varphi}(w); r) \times \mathrm{N}_X(\hat{\varphi}(w); r) \subset \mathscr{O}\}$. Obviously, Φ is a lower semicontinuous convex-valued mapping; hence, by Dowker's theorem there exists a continuous selection $r : W \to (0, \infty)$.

We have one more multivalued mapping $\Psi : W \rightsquigarrow \mathbb{R}^+$, $\Psi(w) = \{r(w) \geq t > 0 \mid \lambda(\mathrm{N}_{X'}(\hat{\varphi}(w); \varepsilon) \times \mathrm{N}_X(\hat{\varphi}(w); t) \times I) \subset \mathrm{N}_X(\hat{\varphi}(w); r(w))$ for some $\varepsilon\}$ which is also lower semicontinuous and convex-valued and so by Dowker's theorem has a continuous selection $t : W \to (0, \infty)$; moreover, $t(w) \leq r(w)$.

We define the sought neighborhood $\mathscr{V} \in \Omega_{G_\varphi}$ by

$$\mathscr{V} \rightleftharpoons \bigcup\nolimits_{w \in \operatorname{Int} W} w \times \mathrm{N}_X(\hat{\varphi}(w); t(w)).$$

Obviously, $G_\varphi \subset \mathscr{V} \subset \mathscr{U}$. The fiberwise contraction of $\mathscr{V}$ in $\mathscr{U}$ is arranged as follows: if $w \in \operatorname{Int} W$ then $F_t(w, x) \rightleftharpoons w \times \lambda(\hat{\varphi}(w), x, t) \in w \times \mathrm{N}_X(\hat{\varphi}(w), r(w)) \subset \mathscr{U} \cap (w \times X)$, where $x \in \mathrm{N}(\hat{\varphi}(w), t(w))$, $0 \leq t \leq 1$. Clearly, F_t is a homotopy connecting $F_1 = \mathrm{Id}_{\mathscr{V}}$ with the mapping F_0 which factors through the projection p, i.e., $F_0 = \hat{\varphi} \circ p$. ■

The following assertion is a strengthening of Proposition 5.3 and has a similar proof.

Proposition 5.4. *Suppose that an embedding $X' \hookrightarrow X$ is equiconnected and the space X' is represented as a countable union of subspaces F_i, $i \geq 1$; moreover, each partial mapping $Z \hookleftarrow A \xrightarrow{\varphi} F_i$ has a neighborhood extension $\hat{\varphi} : U \to X'$. Then for every $i \geq 1$ the projection $\pi : G_\varphi \to A$ of the graph of an arbitrary partial mapping $Z \hookleftarrow A \xrightarrow{\varphi} F_i$ to A is fiberwise trivial within the projection $p : Z \times X \to Z$.*

Closing the section, we state a strengthening of Proposition 1.7 which allows us to interrelate fiberwise trivial mappings and A-ANE-spaces.

Proposition 5.5. *Let X be a closed subset of a normed space Z. If the projection $\pi_\varphi : G_\varphi \to X$ of the graph $G_\varphi \subset Z \times X$ of a partial mapping $Z \hookleftarrow X \xrightarrow{\varphi \rightleftharpoons \mathrm{Id}_X} X$ is fiberwise trivial within the projection $p \rightleftharpoons \mathrm{pr}_Z : Z \times X \to Z$, then $X \in$ A-ANE.*

Proof. We demonstrate that for an arbitrary covering $\omega = \{W_\gamma \mid \gamma \in \Gamma\} \in \operatorname{cov} X$ there exists a mapping $\widetilde{\varphi} : W \to X$, where W is a neighborhood of A in Z, such that $\operatorname{dist}(\varphi, \widetilde{\varphi}\restriction_X) \prec \omega$. Afterwards $X \in$ A-ANE by the following lemma which appears easily from the definition of A-ANE and the fact that $Z \in$ AE.

Lemma 5.6. *If a partial mapping $Z \hookleftarrow X \xrightarrow{\varphi \rightleftharpoons \mathrm{Id}_X} X$ has a neighborhood ω-extension for every covering $\omega \in \operatorname{cov} X$ then $X \in$ A-ANE.*

Consider a covering $\omega' = \{W'_\beta\} \in \operatorname{cov} X$ which is a starlike refinement of ω: $\omega' \circ \omega' \succ \omega$. By Proposition 2.1, there exists a mapping $\beta \mapsto \gamma = \gamma(\beta)$ of the index sets which satisfies 2.1(1). Also, consider an open system $\sigma = \{S_\lambda\}$ in Z covering X and such that $\{\varphi(S_\lambda \cap X)\} \succ \omega'$. Suppose that $\varphi(S_\lambda \cap X) \subset W'_{\beta = \beta(\lambda)} \subset W_{\gamma = \gamma(\beta)}$.

Since the mapping $\pi_\varphi : G_\varphi \to X$ is fiberwise trivial within $p : Z \times X \to Z$, for the neighborhood $\mathscr{U} \rightleftharpoons \bigcup_\lambda S_\lambda \times W'_{\beta(\lambda)} \in \Omega_{G_\varphi}$ of the graph G_φ there are neighborhoods W, $Z \supset W \supset X$, and $\mathscr{V}$, $G_\varphi \subset \mathscr{V} \subset \mathscr{U}$, such that $p(\mathscr{V}) \supset W$ and $W^\bullet \cap \mathscr{V}$ contracts fiberwise in $W^\bullet \cap \mathscr{U}$. Hence, there exists a mapping $\widetilde{\varphi} : W \to X$ such that $(x, \widetilde{\varphi}(x)) \in \mathscr{U}$ for all $x \in X$. We demonstrate that $\widetilde{\varphi}$ is a sought mapping.

Given $x_0 \in X$, put $\Lambda' \rightleftharpoons \{\lambda \mid x_0 \in S_\lambda\}$. It is easy to see that $x_0^\bullet \cap \mathscr{U} = \bigcup_{\lambda\in\Lambda'} x_0 \times W'_{\beta(\lambda)} = x_0 \times \left(\bigcup_{\lambda\in\Lambda'} W'_{\beta(\lambda)}\right)$. Since $x_0 = \varphi(x_0) \in \bigcap_{\lambda\in\Lambda'} \varphi(S_\lambda \cap A) \subset \bigcap_{\lambda\in\Lambda'} W'_{\beta(\lambda)}$, property 2.1(1) implies that $x_0^\bullet \cap \mathscr{U} \subset x_0 \times \bigcap_{\lambda\in\Lambda'} W_{\gamma(\beta(\lambda))}$. In view of $(x_0, \widetilde{\varphi}(x_0)) \in \mathscr{U}$, we have $\widetilde{\varphi}(x_0) \in \bigcap_{\lambda\in\Lambda'} W_{\gamma(\beta(\lambda))}$. Thus, $\varphi(x_0)$ and $\widetilde{\varphi}(x_0)$ belong to the element $W_{\gamma(\beta(\lambda_0))}$ of ω, where λ_0 is the element of Λ' for which $\varphi(x_0) \in W'_{\beta(\lambda_0)}$. ■

§ 6. Proof of Theorem 1.6

Theorem 6.1. *Suppose that a space Y is represented as a countable union $\bigcup_{i=1}^\infty Y_{2i-1} \cup D$ of closed subspaces Y_{2i-1} and a countably dimensional subspace D. If*

(a) *the restriction $\pi_{2i-1} : X_{2i-1} \to Y_{2i-1}$ of the projection $\pi : X \to Y$ to $X_{2i-1} = \pi^{-1}(Y_{2i-1})$ is fiberwise trivial within the projection $p : M \times N \to M$ for every $i \ge 1$;*

(b) *the projection π is locally fiberwise trivial within p in case $D \ne \varnothing$,*

then the projection π is fiberwise trivial within p.

Proof. Let Ω_X (Ω_{X_i}) be a collection of neighborhoods of X (X_i) in $M \times N$ satisfying (4) and (5) of § 5. Fix $\mathscr{V}_1 \in \Omega_X$. Since the projection π_1 is fiberwise trivial within p, there exists $\mathscr{V}_2' \in \Omega_{X_1}$ such that π_1 is a fiberwise $\mathscr{V}_1\mathscr{V}_2'$-contractible projection. Since X_1 is closed in Y, it is easy to construct $\mathscr{V}_2 \in \Omega_X$ such that π_1 is fiberwise $\mathscr{V}_1\mathscr{V}_2$-contractible within $p : M \times N \to M$. Since the projection π is locally fiberwise trivial within p, there is an open family $\sigma_1 = \{O_1(y) \mid y \in Y\}$ in M covering Y and there is a neighborhood $\mathscr{V}_3 \subset \mathscr{V}_2$, $\mathscr{V}_3 \in \Omega_X$, such that $p(\mathscr{V}_3) \supset \bigcup \sigma_1$ and the embedding $(O_1(y))^\bullet \cap \mathscr{V}_3 \hookrightarrow (O_1(y))^\bullet \cap \mathscr{V}_2$ is fiberwise contractible within p for all $y \in Y$.

Similarly, for $i \ge 4$ and $j \ge 2$ we construct neighborhoods $\mathscr{V}_i \in \Omega_X$, $\mathscr{V}_i \subset \mathscr{V}_{i-1}$, and families $\sigma_j = \{O_j(y) \mid y \in Y\}$ of open sets in M which cover Y and are such that

(1) $\sigma_j \succ \sigma_{j-1}$;

(2) π_j is a fiberwise $\mathscr{V}_{2j-1}\mathscr{V}_{2j}$-contractible projection within p;

(3) $p(\mathscr{V}_{2j+1}) \supset \bigcup \sigma_j$ and the embedding $(O_j(y))^\bullet \cap \mathscr{V}_{2j+1} \hookrightarrow (O_j(y))^\bullet \cap \mathscr{V}_{2j}$ is fiberwise contractible within the projection p for all $y \in Y$.

Since D is countably dimensional, by Proposition 2.7

$$D \subset \bigcup_{j=1}^\infty \bigcup_{\lambda\in\Lambda_j} D_{2j}(\lambda),$$

where $\{D_{2j}(\lambda) \mid \lambda \in \Lambda_j\}$ is a family of open sets in M which has multiplicity 1 and is a refinement of σ_j. Put

$$W_{2j} = \bigcup_{\lambda\in\Lambda_j} D_{2j}(\lambda) \quad \text{for } j \ge 1.$$

Also, consider a neighborhood W_{2j-1} for which $p(\mathscr{V}_{2j}) \supset W_{2j-1} \supset Y_{2j-1}$ and the neighborhood $(W_{2j-1})^\bullet \cap \mathscr{V}_{2j} \hookrightarrow (W_{2j-1})^\bullet \cap \mathscr{V}_{2j-1}$ is fiberwise contractible within p. Clearly, $p(\mathscr{V}_{j+1}) \supset W_j$ for all $j \ge 1$ and the embedding $W_j^\bullet \cap \mathscr{V}_{j+1} \hookrightarrow W_j^\bullet \cap \mathscr{V}_j$ is fiberwise contractible within p. Thereby conditions (6) and (7) of Proposition 5.1 are satisfied. Hence, $\mathscr{V} \rightleftharpoons \bigcup_{j=1}^\infty W_j^\bullet \cap \mathscr{V}_{j+1} \in \Omega_X$ and π is fiberwise $\mathscr{V}_1\mathscr{V}$-contractible within p. This completes the proof of the theorem. ■

Proof of Theorem 1.6. Since the class of spaces X satisfying the hypothesis of Theorem 1.6 is invariant under the product by $[0, 1)$, in view of Theorem 1.2 it suffices to prove that $X \in$ A-ANE.

Let $Z \hookleftarrow X \xrightarrow{\varphi \rightleftharpoons \mathrm{Id}_X} X$, $G_\varphi \subset Z \times X$, $\pi_\varphi : G_\varphi \to X$, and $p : Z \times X \to Z$ be taken from Proposition 5.5. We demonstrate that π_φ is a fiberwise trivial projection within p. Then Proposition 5.5 would imply $X \in$ A-ANE. (The existence of a closed embedding of X in a normed vector space Z follows from [10].)

Since $X \in$ LC, by Proposition 5.2 π_φ is a locally fiberwise trivial projection within p. Moreover, Proposition 5.4 implies that

$$\pi_\varphi\restriction_{\pi_\varphi^{-1}(X_i)} : \pi_\varphi^{-1}(X_i) \to X_i, \quad i \ge 1,$$

is a fiberwise trivial projection within p. Since $X = D \cup \bigcup_{i=1}^\infty X_i$, all conditions of Theorem 6.1 are satisfied and so π_φ is a fiberwise trivial projection within p. ■

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