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A NEW CONSTRUCTION OF SEMI-FREE ACTIONS ON MENGER MANIFOLDS

SERGEI M. AGEEV AND DUŠAN REPOVŠ

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ABSTRACT. A new construction of semi-free actions on Menger manifolds is presented. As an application we prove a theorem about simultaneous coexistence of countably many semi-free actions of compact metric zero-dimensional groups with the prescribed fixed-point sets: Let G be a compact metric zero-dimensional group, represented as the direct product of subgroups G_i, M a μ^n-manifold and $\nu(M)$ (resp., $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then, given closed subsets $X_i, i \geq 1$, of M, there exists a G-action on M such that (1) $\nu(M)$ and $\Sigma(M)$ are invariant subsets of M; and (2) each X_i is the fixed point set of any element $g \in G_i \setminus \{e\}$.

0. Introduction

The following remarkable theorem was proved by Iwamoto [11]:

Theorem (0.1). *Let G be a compact metric zero-dimensional group, M a μ^n-manifold and $\nu(M)$ (resp. $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then for every closed subset X of M, there exists a semi-free G-action on M such that X is the fixed-point set of every element $g \in G \setminus \{e\}$, and $\nu(M)$ and $\Sigma(M)$ are invariant subsets of M.*

This theorem is a significant generalization of the Dranishnikov free action theorem [8] and Sakai's results [13], [14]. In the present paper we give a new construction of semi-free actions, and prove a simultaneous coexistence of countably many semi-free actions of compact metric zero-dimensional groups with the prescribed fixed-point sets.

Theorem (0.2). *Let G be a compact metric zero-dimensional group, represented as the direct product $\prod G_i$ of subgroups G_i, M a μ^n-manifold and $\nu(M)$ (resp., $\Sigma(M)$) its pseudo-interior (resp., pseudo-boundary). Then, given closed subsets $X_i, i \geq 1$, of M, there exists a G-action on M such that:*

(1) *$\nu(M)$ and $\Sigma(M)$ are invariant subsets of M; and*
(2) *each X_i is the fixed-point set of every element $g \in G_i \setminus \{e\} \subset \prod G_i$.*

Recall that an *action* of G on a space X is a homomorphism $T\colon G \to \operatorname{Aut} X$ of the group G into the group $\operatorname{Aut} X$ of all autohomeomorphisms of X, such that the

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map $G \times X \to X$, given by $(g, x) \mapsto T(g)(x) = g \cdot x$, is continuous. A space X with a fixed action of G is called a *G-space*.

A subset $A \subset X$ (resp., a point $a \in X$) is said to be *invariant* (resp., *fixed*) if $G \cdot A = \{g \cdot a \mid g \in G, a \in A\} = A$ (resp., $G \cdot a = a$). A G-space X is said to be *free* (resp., *semi-free*) if $g \cdot x \neq x$, for every $x \in X$ and $g \neq e$ (resp., $g \cdot x \neq x$, for every nonfixed point x and $g \neq e$).

The proof of Theorem (0.2) is presented in detail only for the case of the Menger compactum μ^n. By Pontryagin's theorem [12], each G_i can be considered as a closed subgroup of the product $\prod_{k=1}^{\infty} H_{ik}$ of nontrivial finite groups with the following property: Every nontrivial element $g_i = (g_{ik}) \in G_i$ has infinitely many nontrivial coordinates.

Next, fix the Cantor compactum $\mathcal{C} = \mu^0$ with the pseudo-interior $\nu = \nu(\mu^0)$ and the pseudo-boundary $\Sigma = \Sigma(\mu^0)$. The key to the construction of the desired semi-free action lies in the canonical surjection

$$r\colon I^n \times H \times \mathcal{C} \to \mu^n, \text{ where } H = \prod_i H_i \text{ and } H_i = \prod_k H_{ik},$$

and a discontinuous action Ψ of H on $I^n \times H \times \mathcal{C}$ such that:

(3) the restriction of Ψ on $G_k \subset H_k$ is a discontinuous semi-free action with respect to $r^{-1}(X_k)$;
(4) $\Psi(h, (x, u, v)) = (x, u', v)$, where $h \in H, x \in I^n, u \in H, v \in \mathcal{C}$ (i.e. Ψ changes only the middle coordinate);
(5) $r(I^n \times H \times \nu)$ is the pseudo-interior $\nu(\mu^n)$ and $r(I^n \times H \times \Sigma)$ is the pseudo-boundary $\Sigma(\mu^n)$;
(6) $r^{-1}(m)$ is a rectangle subset of $I^n \times H \times \mathcal{C}$, for every $m \in \mu^n$; and
(7) $\Phi(h, r(x, u, v)) = r(x, u', v)$, where $(x, u', v) = \Psi(h, (x, u, v))$, is a continuous action of H on μ^n, and the restriction of Φ on G_k is a continuous semi-free action with respect to X_k.

The proof of Theorem (0.2) in the general case (for an arbitrary μ^n-manifold) is analogous to the special case of μ^n and will be discussed in the last chapter.

1. Preliminaries

All spaces are assumed to be separable metric and all maps to be continuous. Recall that a space X is said to be $(k-1)$-*connected* (C^{k-1}) if the homotopy group $\pi_i(X)$ is trivial, for every $i < k$. A space X is said to be *locally* $(k-1)$-*connected* (LC^{k-1}) if for every $x \in X$ and every neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ with the property that every map $\alpha\colon S^i = \partial B^{i+1} \to V, i < k$, extends to $\tilde{\alpha}\colon B^{i+1} \to U$.

By the Kuratowski-Dugundji theorem [5], $X \in \mathrm{LC}^{k-1} \cap \mathrm{C}^{k-1}$ (resp., $X \in \mathrm{LC}^{k-1}$) if and only if $X \in \mathrm{AE}(k)$, i.e. X is an absolute extensor in dimension k (resp., $X \in \mathrm{ANE}(k)$, i.e. X is an absolute neighborhood extensor in dimension k). A family $\{X_\alpha\}$ of sets $X_\alpha \subset X$ is said to be *equi*-LC^{k-1} if for every $x \in \bigcup X_\alpha$ and every neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ with the property that every partial map $Z \hookleftarrow A \xrightarrow{f} V \cap X_\alpha, \dim Z \leq k$, extends to $g\colon Z \to U \cap X_\alpha$, $g|_A = f$. The following criteria are convenient for verification of connectivity properties of spaces (cf. [1] and [4], respectively):

Proposition (1.1). *Let $\{Z_\alpha\}$ be a closed cover of the compactum Z. Then the following assertions hold:*

(a) $Z \in \mathrm{AE}(n)$ *if and only if for every neighborhood $U(z)$ of $z \in Z$, there exists a neighborhood $V(z)$ (if $U(z) = Z$ then we require $V(z) = Z$), such that for every map $\varphi\colon S^k \to V(z), k < n$, and for every $\nu > 0$, there exists a map $\psi\colon B^{k+1} \to Z$, with $(\varphi, \psi\,|_{S^k}) < \nu$.*

(b) $\{Z_\alpha\} \in$ equi-LC^{n-1}, *if and only if for every neighborhood $U(z)$ of $z \in Z$, there exists a neighborhood $V(z)$, such that for every map $\varphi\colon S^k \to V(z) \cap Z_\alpha, k < n$, and for every $\nu > 0$, there exists a map $\psi\colon B^{k+1} \to U(z) \cap Z_\alpha$ with $(\varphi, \psi\,|_{S^k}) < \nu$.*

Proposition (1.2). *Let $\mathcal{P} = \{p_\alpha\}$ be a closed cover of an $\mathrm{ANE}(r)$-compactum Z and suppose that $p_{\alpha_1} \cap p_{\alpha_2} \cap \cdots \cap p_{\alpha_t} \neq \emptyset$ implies $p_{\alpha_1} \cap p_{\alpha_2} \cap \cdots \cap p_{\alpha_t} \in \mathrm{AE}(r+1-t)$, for every $t \leq r$. Then $Z \in \mathrm{AE}(r)$ if and only if the nerve $\mathcal{N}\langle\mathcal{P}\rangle$ of $\mathcal{P}$ is $(r-1)$-connected.*

We shall need the following three results of Bestvina [3].

Theorem (1.3). *A locally compact space X is μ^k-manifold if and only if $X \in \mathrm{ANE}(k)$, $\dim X = k$ and X has the disjoint k-disks property, namely $X \in \mathrm{DD}^k\,\mathrm{P}$. If, in addition, X is a compact $\mathrm{AE}(k)$, then X is homeomorphic to the Menger universal compactum μ^k.*

Theorem (1.4). *For every μ^k-manifold M^k, there exists a* PL *manifold R of dimension $\geq 2k+1$ with a triangulation L and a proper map $f\colon R^{(k)} \to M$ which induces isomorphisms of homotopy groups of* $\dim < k$ *and homotopy groups of ends of* $\dim < k$ *(here $R^{(k)}$ is the k-skeleton with respect to L).*

Theorem (1.5). *Let $f\colon M_1 \to M_2$ be a proper map between μ^k-manifolds which induces isomorphisms of homotopy groups of* $\dim < k$ *and homotopy groups of ends of* $\dim < k$. *Then f is properly $(k-1)$-homotopic to a homeomorphism.*

Let M be a μ^k-manifold. By $\mathcal{Z}_M$, we shall denote the collection of all Z-sets in M. A $\mathcal{Z}_M$-absorber A of M is called the *pseudo-boundary* $\Sigma(M)$ of M and the complement $M \setminus A$ is called the *pseudo-interior* $\nu(M)$ of M. The topological types of the pseudo-boundaries and the pseudo-interiors of M are unique [2]. The following criterion was proved in [7].

Proposition (1.6). *Let $\{A_i\}_{i=1}^\infty$ be an increasing sequence of Z-sets in a μ^k-manifold M with the following properties:*

(1) *for every $\epsilon > 0$, there exists $m > 0$ such that A_m is ϵ-dense in M;*
(2) *each A_i is a Z-set in A_{i+1};*
(3) $\{A_i\}_{i=1}^\infty$ *is* equi-LC^{k-1} *and*
(4) *A_i is a μ^k-manifold.*

Then $\bigcup_{i=1}^\infty A_i$ is a pseudo-boundary of M.

The pseudo-boundary and the pseudo-interior of the zero-dimensional Menger manifold $\mathcal{C} = \mu^0$ can be described in the following manner:

Definition (1.7). (5) A subset $R \subset \prod_{i=1}^s T_i, 1 \leq s \leq \infty$, is called a *rectangle*, if $R = \prod_{i=1}^s T_i'$, where $T_i' \subset T_i$ for every i.

(6) A rectangle subset $R_1 = \prod_{i=1}^{s} T_i'$ is said to have *an infinite codimension* in a rectangle subset $R_2 = \prod_{i=1}^{s} T_i''$, whenever $R_1 \subset R_2$ and the set $\{i \in \mathbb{N} \mid T_i'' \setminus T_i' \neq \emptyset\}$ is infinite.

Proposition (1.8). *Let $\mathcal{C} = \prod_{i=1}^{\infty} T_i$ be the Cantor compactum. Then there exists a sequence of rectangle subsets $R_1 \subset R_2 \subset \dots$ such that*

(7) *$R_i \cong \mathcal{C}$ for every i;*
(8) *every R_i has infinite codimension in R_{i+1}; and*
(9) *$R = \bigcup_{i=1}^{\infty} R_i$ is a dense subset in $\mathcal{C}$.*

According to Proposition (1.6), R is the pseudo-boundary of $\mathcal{C}$.

2. The canonical decomposition of $I^n \times \prod T_i$

Let us consider a partition $\Delta_m, m \geq 1$, of the unit segment I, determined by the subset $\delta_m = \{\frac{2i-1}{2^m} | 0 < i \leq 2^{m-1}\}$. The product $\underbrace{\Delta_m \times \dots \times \Delta_m}_{n} = (\Delta_m)^n$ of the partitions Δ_m is a partition of the cube $I^n = \underbrace{I \times \dots \times I}_{n}$ into $(2^{m-1}+1)^n$ cubes of diameter $\leq \sqrt{n} \cdot 2^{1-m}$. The union F_m of all boundaries of these cubes satisfies the following properties:

(1) $F_m = \{x = (x_1, \dots, x_n) \in I^n | x_i \in \delta_m, \text{ for some } \ i\}$; and
(2) for every $x \in I^n$, the set $\{m | x \in F_m\}$ has less than $n+1$ elements.

Fix a sequence $\{T_i, i \geq 1\}$ of finite sets T_i with $|T_i| > 1$, and construct (in a canonical way) an upper semi-continuous decomposition $\mathcal{T}$ of $Q = I^n \times \prod_{i=1}^{\infty} T_i$. The quotient space $P = Q/\mathcal{T}$, generated by $\mathcal{T}$, will have properties (3)-(5) below:

(3) P is compact and $\dim P = n$;
(4) P satisfies the disjoint n-disks property (DD^nP); and
(5) $P \in \mathrm{C}^{n-1} \cap \mathrm{LC}^{n-1}$.

By Theorem (1.3), P and the Menger compactum μ^n are homeomorphic. Let $x \in I^n$ and $t = (t_i)_{i=1}^m \in \prod_{i=1}^{m} T_i, 1 \leq m \leq \infty$. (Convention: $m+1 = \infty$ in case $m = \infty$.) Let us denote $\mathcal{T}_i(x,t) = t_i$, if $x \notin F_i$; $\mathcal{T}_i(x,t) = T_i$, if $x \in F_i$, and $\mathcal{T}(x,t) = \prod_{i=1}^{m} \mathcal{T}_i(x,t)$.

Proposition (2.1). *The family $\mathcal{T}_m = \{x \times \mathcal{T}(x,t) | x \in I^n, t \in \prod_{i=1}^{m} T_i\}, 1 \leq m \leq \infty$, yields an upper semi-continuous decomposition of $Q_m = I^n \times \prod_{i=1}^{m} T_i$:*

(6) *for every (x_0, t_0) and $\epsilon > 0$, there exists $\delta > 0$ such that* $\operatorname{dist}((x,t),(x_0,t_0)) <$ *δ implies $\alpha_H(x \times \mathcal{T}(x,t), x_0 \times \mathcal{T}(x_0,t_0)) < \epsilon$ (here $\alpha_H(A,B) = \inf\{\gamma > 0 | A \subset \mathrm{N}(B;\gamma)\}$ is the* Hausdorff deflection*; cf.* [10, *p.* 98, (7.7.1)]*).*

Proof. By definition, it is evident that

(7) $x \times \mathcal{T}(x,t) \cap x' \times \mathcal{T}(x',t') \neq \emptyset \iff x = x'$ and $t_i = t_i'$ for all $i < m+1$ such that $x \notin F_i \iff x \times \mathcal{T}(x,t) \equiv x' \times \mathcal{T}(x',t')$ (we use a convention that $m+1 = \infty$ in case $m = \infty$).

Fix $x_0 \in I^n, t_0 \in \prod_{i=1}^{m} T_i$ and $\epsilon > 0$. Let us choose $\delta > 0$ and a finite number $p \leq m$ such that

(8) $\operatorname{dist}(x,x') < \delta$ and the coincidence of the first p coordinates of t and t' implies $\operatorname{dist}((x,t),(x',t')) < \epsilon/2$;

(9) $\operatorname{dist}((x,t),(x',t')) < \delta$ implies the coincidence of the first p coordinates of t and t' and, naturally, $\operatorname{dist}(x,x') < \delta$; and

(10) $\delta < \min\{\operatorname{dist}(x_0,F_i) | x_0 \notin F_i, i \leq p\}$.

Let $\operatorname{dist}((x,t),(x_0,t_0)) < \delta$. It easily follows from (9) and (10) that $\operatorname{dist}(x,x_0) < \delta$, the first p coordinates of t and t_0 coincide and $\mathcal{T}_i(x,t) \subset \mathcal{T}_i(x_0,t)$, for every $i \leq p$. Then $\alpha_H(x \times \mathcal{T}(x,t), x_0 \times \mathcal{T}(x_0,t)) < \epsilon/2$ by (8). The coincidence of the first p coordinates of t and t_0 implies $\alpha_H(x_0 \times \mathcal{T}(x_0,t), x_0 \times \mathcal{T}(x_0,t_0)) < \epsilon/2$. Hence it follows that $\alpha_H(x \times \mathcal{T}(x,t), x_0 \times \mathcal{T}(x_0,t_0)) \leq \alpha_H(x \times \mathcal{T}(x,t), x_0 \times \mathcal{T}(x_0,t)) + \alpha_H(x_0 \times \mathcal{T}(x_0,t), x_0 \times \mathcal{T}(x_0,t_0)) < \epsilon$. □

Corollary (2.2). *For every $1 \leq m \leq \infty$, the following assertions hold:*

(a) *the quotient space $P_m = Q_m/\mathcal{T}_m$ is a compactum; and*

(b) *if $m < \infty$, then the map $p_m \colon P_m \to P_{m-1}$, defined by setting*

$$p_m(r_m(x,(t_1,\ldots,t_m))) = r_m(x,(t_1,\ldots,t_{m-1})),$$

is well-defined and continuous and the following diagram is commutative:

$$\begin{tikzcd} Q_m \arrow[r, "r_m"] \arrow[d, "q_m"'] & P_m \arrow[d, "p_m"] \\ Q_{m-1} \arrow[r, "r_{m-1}"] & P_{m-1} \end{tikzcd}$$

where q_m is the projection on $Q_{m-1} = I^n \times \prod_{i=1}^{m-1} T_i$.

For $m = \infty$ we have the map $r \colon Q = I^n \times \prod_{i=1}^{\infty} T_i \to P = P_\infty$. Clearly, P is the limit of the inverse system $\{P_1 \xleftarrow{p_2} P_2 \xleftarrow{p_3} P_3 \xleftarrow{p_4} \ldots\}$. Since r_m has finitely many fibers for $m < \infty$, it follows that $\dim P_m = \dim Q_m = n$. Since the dimension of the limit of the inverse system of n-dimensional spaces is less than or equal to n, we have $\dim P \leq n$.

Proposition (2.3). $\dim P = n$ *and* $P \in DD^nP$.

Proof. Every $a \in T_{m+1}$ naturally generates the embedding $s_{m,a} \colon P_m \to P_{m+1}$ of P_m into P_{m+1} defined by $s_{m,a}(r_m(x,t)) = r_{m+1}(x,t')$, where $t' = (t,a)$. It is evident that $s_{m,a}$ is a section of p_{m+1}. Hence P contains a copy of P_m with $\dim P_m = n$. Therefore, $\dim P = n$.

One can easily conclude from (2) that the images of the following compositions are disjoint:

$$\operatorname{Im}(s_{m+n,a_{m+n+1}} \circ \cdots \circ s_{m,a_{m+1}}) \cap \operatorname{Im}(s_{m+n,a'_{m+n+1}} \circ \cdots \circ s_{m,a'_{m+1}}) = \emptyset,$$

for every $a_{m+n+1} \neq a'_{m+n+1}, \ldots, a_{m+1} \neq a'_{m+1}$. Consequently, P satisfies DD^nP. □

Remark (2.4). If R' is a rectangle subset of infinite codimension in a rectangle subset R'' of $\prod T_i$, then $P' = r(I^n \times R')$ is a Z-set in $P'' = r(I^n \times R'')$ and $P'' \in \mathrm{DD}^n\,\mathrm{P}$.

In the next section we shall prove that $P'' \in \mathrm{AE}(n)$; so it will follow by the Bestvina theorem (1.3) that P'' is homeomorphic to the Menger compactum μ^n.

3. Connectivity properties of subsets of P

Let the factor T_p, $p < m+1$, be represented as the disjoint union $T'_p \coprod T''_p$ (we assume $m+1 = m$ for $m = \infty$). By replacing T_p with T'_p, we obtain the decomposition $\mathcal{T}'_m$ of $Q'_m = I^n \times T'_p \times \prod\{T_i | i < m+1, i \neq p\}, 1 \leq m \leq \infty$, and the quotient map $r'_m \colon Q'_m \to P'_m = Q'_m / \mathcal{T}'_m$. The decomposition $\mathcal{T}''_m$ of the space Q''_m and the map $r''_m \colon Q''_m \to P''_m$ are defined analogously. Note that $Q'_m \coprod Q''_m = Q_m$. It is clear that $r_m(Q'_m) = \hat{P}'_m \subset P_m$ and $r_m(Q''_m) = \hat{P}''_m \subset P_m$ are naturally homeomorphic to P'_m and P''_m, respectively.

Proposition (3.1). $\hat{P}'_m \cap \hat{P}''_m = r_m(F_p \times \prod\limits_{i=1}^{m} T_i)$.

Definition (3.2). An *index* of $x \in I$ (briefly $\operatorname{ind} x$) is defined to be $i \in \mathbb{N}$, if $x \in \delta_i$, and ∞, if $x \notin \delta_j$ for every j. An index of $b = (b_1, \ldots, b_n) \in I^n$ is defined to be $\operatorname{ind}(b) = \{\operatorname{ind}(b_i) | i \leq n\} \subset \mathbb{N} \cup \{\infty\}$.

Let J be a closed segment from the partition $\Delta_m, m < \infty$. If $J \subset \operatorname{Int} I$, then there exists a unique point $b' \in J$ with $\operatorname{ind}(b') < m$ (b' is the midpoint of J). If $J \cap \{0,1\} = b'$, then J does not contain points with index less than m. We define $\operatorname{Ind}(J) = \operatorname{ind}(b')$ in the first case and $\operatorname{Ind}(J) = \infty$ in the second one. In both cases point b' is said to be the *center* of J. Let $J = [\frac{2k+1}{2^m}, \frac{2k+3}{2^m}] \in \Delta_m$. Then $\operatorname{Ind}(J) < m$ and $b' = \frac{2k+2}{2^m}$ is the center of J.

The partition $(\Delta_m)^n$ of I^n yields new partitions $\mathcal{Q}'_m$ of $Q_m = I^n \times \prod\limits_{i=1}^{m} T_i$ and $\mathcal{Q}_m$ of $Q = I^n \times \prod\limits_{i=1}^{\infty} T_i$ defined as follows:

$$\mathcal{Q}'_m = \{\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b,t) | \mathcal{J} = \prod_{i=1}^{n} J_i \in (\Delta_m)^n, b = (b_i) \in I^n,$$

$$t \in \prod_{i=1}^{m} T_i,\ \text{each } b_i \text{ is the center of } \ J_i\}$$

and $\mathcal{Q}_m = \{\mathcal{E}' \times \prod\limits_{i=m+1}^{\infty} T_i | \mathcal{E}' \in \mathcal{Q}'_m\}$. The partitions $\mathcal{Q}_m$ of Q and $\mathcal{Q}'_m$ of Q_m consist of regular closed subsets.

Theorem (3.3). *Let $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b,t)$ be an arbitrary element of the partition $\mathcal{Q}'_m, m < \infty$, and $\mathcal{J}_{p_1 \ldots p_w} = \mathcal{J} \cap F_{p_1 \ldots p_w} \neq \emptyset$, where $1 \leq p_1 < \cdots < p_w \leq s, s \geq m, 0 \leq w \leq n$, and*

$$F_{p_1 \ldots p_w} = \begin{cases} \bigcap\limits_{k=1}^{w} F_{p_k} & \text{if } w \geq 1; \\ I^n & \text{if } w = 0. \end{cases}$$

Then for every rectangle subset $R = \prod_{i=1}^{s} T_i'$ *with the property* $(*)$:

$$\text{“}|T_i'| > 1 \textit{ implies } i \geq m \textit{ or } i \in \operatorname{ind}(b)\text{”},$$

the compactum $X = r_s(\mathcal{J}_{p_1\ldots p_w} \times R)$ *is a nonempty* $\mathrm{AE}(n-w)$.

Corollary (3.4). *Let* $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b,t) \in \mathcal{Q}_m', m < \infty$, *and* $m \leq p_1 < \cdots < p_w \leq s, 0 \leq w \leq n$. *Then for rectangle subsets* $R_1 \subset \prod_{i=1}^{m} T_i$ *and* $R_2 \subset \prod_{i=m+1}^{s} T_i$, *the compactum* $r_s(\mathcal{J}_{p_1\ldots p_w} \times (\mathcal{T}(b,t) \cap R_1) \times R_2)$ *is a nonempty* $\mathrm{AE}(n-w)$.

The following preliminary facts precede the proof of Theorem (3.3). Let us consider a finite set A represented as the disjoint union $A_1 \coprod \cdots \coprod A_w$. We shall study the connectivity properties of the subpolyhedron $K = K^v_{A_1\ldots A_w}, v \leq w$, of the $(|A|-1)$-dimensional simplex Δ: $K = \bigcup\{\langle a_1 \ldots a_v\rangle |\ \langle a_1 \ldots a_v\rangle$ is a $(v-1)$-dimensional simplex spanned by the vertices $a_k \in A_{i_k}, 1 \leq k \leq v$, where $i_k \neq i_{k'}$ if $k \neq k'\}$.

Lemma (3.5). $K^v_{A_1\ldots A_w} \in C^{v-2}$ *for each* $v \leq w$.

Proof. We now proceed by induction on $\theta = e \cdot w + f + v$, where $e = \max|A_i|, f = |\{i\ :\ |A_i| = e\}|$. It is evident that $\theta' = e' \cdot w' + f' + v' < \theta$ if $e' < e$ and $w' \leq w, v' \leq v$ or $e' \leq e, f' < f$ and $w' \leq w, v' \leq v$.

The basis of the induction corresponds to $\theta = 1 \cdot v + v + v = 3v$. In this case K is the $(v-1)$-skeleton of Δ, which is $(v-2)$-connected. The same is valid for $e = 1$.

Assume the validity of the lemma for all $\theta' < \theta$. Suppose, without loss of generality, that $|A_1| = e > 1$ and $A_1 = A_1' \coprod A_1'', |A_1'| \leq |A_1''| < e$. By the inductive hypothesis ($\theta' < \theta$ and $\theta'' < \theta$), $K^v_{A_1' A_2\ldots A_w}$ and $K^v_{A_1'' A_2\ldots A_w}$ are $(v-2)$-connected and their intersection

$$K^v_{A_1' A_2\ldots A_w} \cap K^v_{A_1'' A_2\ldots A_w} = K^v_{A_2\ldots A_w} \cup K^{v-1}_{A_2\ldots A_w} = \begin{cases} K^v_{A_2\ldots A_w} & \text{if } w \geq v; \\ K^{v-1}_{A_2\ldots A_w} & \text{if } w = v \geq 2. \end{cases}$$

By the Van Kampen and the Mayer-Vietoris argument [15], $K^v_{A_1 A_2\ldots A_w} \in \mathrm{C}^{v-2}$. $\square$

Lemma (3.6). *Let* $\mathcal{J} = \prod_{i=1}^{n} J_i \in (\Delta_m)^n, b = (b_i) \in I^n$, *each* b_i *is the center of* J_i, *and* $\mathcal{J}_{p_1\ldots p_w} \neq \emptyset$ *for* $1 \leq p_1 < \cdots < p_w, 0 \leq w \leq n$. *Then* $\mathcal{J}_{p_1\ldots p_w} \in \mathrm{AE}(n-w)$.

Proof. Let $\pi_i \colon I^n \to I$ be the projection onto the ith factor. Then

$$F_{p_1} = \bigcup_{k=1}^{n} \coprod_{q \in \delta_{p_1}} \Pi_{k,q}, \text{where } \Pi_{k,q} = \pi_k^{-1}(q) = \{(x_1,\ldots,x_n) \in I^n | x_k = q\}.$$

Hence $\mathcal{J}_{p_1\ldots p_w} = \bigcup_{k=1}^{n} \coprod\{\Pi'_{k,q} | q \in A_k\}$, where $\Pi'_{k,q} = \mathcal{J} \cap \Pi_{k,q} \cap F_{p_2\ldots p_w}$, $A_k = \{q \in \delta_{p_1} | \Pi'_{k,q} \neq \emptyset\}$. It is easy to see that $\mathcal{J}_{p_1\ldots p_w} \neq \emptyset$ implies $p_k \geq m$ or $p_k \in \operatorname{ind}(b)$, for every $k \leq w$.

We shall omit the proof of the following fact which easily follows from the note mentioned above:

Lemma (3.7). $\bigcap_{i=1}^{r} \Pi'_{k_i q_i} \neq \emptyset$ *for every* $k_1 < k_2 < \cdots < k_r, 0 \leq r \leq v = n+1-w, q_i \in A_{k_i}$.

We now proceed by induction on $\sigma = n + w$. The basis of induction is obvious. Assume the validity of the lemma for all $\sigma' < \sigma$. Then $\Pi'_{k,q} \in \mathrm{AE}((n-1)-(w-1))$ and moreover, for every $k_1 < k_2 < \cdots < k_r, 0 \le r \le v = n+1-w, q_i \in A_{k_i}$, we have $\bigcap_{i=1}^{r} \Pi'_{k_i q_i} = (\mathcal{J} \cap \bigcap_{i=1}^{r} \Pi_{k_i q_i}) \cap F_{p_2} \cap \cdots \cap F_{p_w} \in \mathrm{AE}((n-r)-(w-1)) = \mathrm{AE}((n-w+1)-r)$ (since $\bigcap_{i=1}^{r} \Pi_{k_i q_i}$ is an $(n-r)$-dimensional cube, parallel the $(n-r)$-dimensional plane $\{(x_1, \ldots, x_n) | x_{k_i} = 0, i \le r\}$ and $\mathcal{J} \cap \bigcap_{i=1}^{r} \Pi_{k_i q_i}$ is a nonempty element of $(\Delta_m)^{n-r}$).

Hence the intersections of the $\Pi_{k_i q_i}$'s are correctly connected (in the sense of Bestvina [3]) and their nerve coincides with $K^v_{A_1 \ldots A_n} = K^{n+1-w}_{A_1 \ldots A_n}$, which is $\mathrm{AE}(v-1) = \mathrm{AE}(n-w)$ by Lemma (3.5).

To complete the proof we apply Proposition (1.2) to the cover $\{\Pi'_{kq}\}$ of $F_{p_1 \ldots p_w}$. $\square$

Proof of Theorem (3.3). We prove by induction on $\theta = s \cdot e + f - w$, where $e = \max\{|T'_i| : 1 \le i \le s\}, f = |\{i : |T'_i| = e, 1 \le i \le s,\}|$ (e, f, s and w are variables). Let us note that $\theta' = s' \cdot e' + f' - w' < \theta$ provided $e' = 1, s' = m, f' = m$ and $w' = n$ or $e' = e, s' = s, f' < f$ and $w' = w$ or $e' < e, s' = s$ and $w' = w$ or $e' = e, s' = s, f' = f$ and $w' = w$.

If $e = 1$, then each $|T_i| = 1, i > m$, and therefore $X_{p_1 \ldots p_w} \cong \mathcal{J}_{p_1 \ldots p_w}$ which is $\mathrm{AE}(n-w)$ by Lemma (3.6). Hence the basis of the induction is verified.

Assume the validity of Theorem (3.3) for all $\theta' < \theta$ and let $e > 1$. Pick $i_0 \le s$ with $|T'_{i_0}| = e$ and represent $T'_{i_0} = \widehat{T}_{i_0} \coprod \check{T}_{i_0}$, $|\widehat{T}_{i_0}| \le |\check{T}_{i_0}| < e$. Then $X_{p_1 \ldots p_w}$ is the union of

$$Y' = X'_{p_1 \ldots p_w} = r_s(\mathcal{J}_{p_1 \ldots p_w} \times \widehat{R}) \subset \widehat{P}'_s \quad \text{and}$$

$$Y'' = X''_{p_1 \ldots p_w} = r_s(\mathcal{J}_{p_1 \ldots p_w} \times \check{R}) \subset \widehat{P}''_s,$$

where the rectangle subsets $\widehat{R}$ and $\check{R}$ coincide with the rectangle subsets $R \cap (\prod_{i \ne i_0} T_i \times \widehat{T}_{i_0})$ and $R \cap (\prod_{i \ne i_0} T_i \times \check{T}_{i_0})$, which satisfy the property $(*)$ (for notation see the beginning of Section 3). By the inductive hypothesis ($\theta' \le \theta'' < \theta$), we have $Y', Y'' \in \mathrm{AE}(n-w)$. Let us prove that $Y' \cap Y'' \in \mathrm{AE}(n-w-1)$. It will then follow by the Van Kampen and the Mayer-Vietoris argument [15], that $X_{p_1 \ldots p_w} = Y' \cup Y'' \in \mathrm{AE}(n-w)$.

The proof splits naturally into two parts: $i_0 \in \{p_1 \ldots p_w\}$ and $i_0 \notin \{p_1 \ldots p_w\}$.

Part (1). $i_0 \notin \{p_1 \ldots p_w\}$. In this case, $Y' \cap Y'' \cong r_s(\mathcal{J}_{p_1 \ldots p_w i_0} \times R)$. Since $w' = w + 1$ and $e' = e, s' = s, f' = f$, the inductive variable θ' for $Y' \cap Y''$ is less than θ. By the inductive assumption $Y' \cap Y'' \in \mathrm{AE}(n-w-1)$.

Part (2). $i_0 \in \{p_1 \ldots p_w\}$. This implies that $Y' = r_s(\mathcal{J}_{p_1 \ldots p_w} \times \widehat{R}) \cong r_s(\mathcal{J}_{p_1 \ldots p_w} \times R) = X_{p_1 \ldots p_w}$. Hence $X_{p_1 \ldots p_w} \in \mathrm{AE}(n-w)$. $\square$

Theorem (3.8). *Let $\mathcal{R}$ be the family of all rectangle subsets $R \subset \prod_{i=1}^{\infty} T_i$. Then*

(a) $P_R = r(I^n \times R)$ *is an* $\mathrm{AE}(n)$, *for every* $R \in \mathcal{R}$; *and*
(b) $\{P_R = r(I^n \times R) | R \in \mathcal{R}\}$ *is an* equi-LC^{n-1}*-family.*

Proof. (a) Fix a point $(x, t) \in I^n \times R$ and $r(x, t) = z$. For every $\epsilon > 0$, it is possible to choose a number $m \ge 1$ and an element $\mathcal{E}' = \mathcal{J} \times \mathcal{T}(b, t) \in \mathcal{Q}'_m$ such that

(1) the neighborhood $U = \mathrm{N}(z,\epsilon) \cap P_R$ of z in P_R contains a closed neighborhood $V = r(\mathcal{J} \times (\mathcal{T}(b,t) \cap R_1) \times R_2 \times R_3)$ of z in P_R, where R is represented as the product $R_1 \times R_2 \times R_3, R_1 = \prod_{i=1}^{m} T_i', R_2 = \prod_{i=m+1}^{s} T_i'$ and $R_3 = \prod_{i=s+1}^{\infty} T_i'$ (s is a sufficiently large number).

By Corollary (3.4), $V_s = r_s(\mathcal{J} \times (\mathcal{T}(b,t) \cap R_1) \times R_2) \subset P_s$ is $\mathrm{AE}(n)$ for every s. Pick an arbitrary constant $\nu > 0$ and s such that Id_P and the projection p_s^∞ are ν-close. Since P_s is naturally embedded into P (cf. Proposition (2.3)), we can consider V_s as a subset of V. Now let $\varphi\colon S^k \to V, k < n$, be an arbitrary map. The composition $p_s^\infty \circ \varphi\colon S^k \to V_s$, which is ν-close to φ, is extended on $\psi\colon B^{k+1} \to V_s \hookrightarrow V$. By Proposition (1.1a), $P_R \in \mathrm{AE}(n)$.

(b) The proof is analogous to (a), because the choice of the constant $\nu > 0$ does not depend on the choice of the family $\{P_R\}$. □

Corollary (3.9). $P = r(Q) \in \mathrm{AE}(n)$ *(and hence, by Section 2, is also homeomorphic to* μ^n*).*

4. Construction of the action of the zero-dimensional compact group G on P

By Pontryagin's theorem [12], every compact metric zero-dimensional group G_i can be considered as a closed subgroup of the product $\prod_{k=1}^{\infty} H_{ik}$ of nontrivial finite groups H_{ik} with the following property:

(1) every nontrivial element $g_i = (g_{ik}) \in G_i$ has infinitely many nontrivial coordinates.

Let us identify some factors of $Q = I^n \times \prod_{i=1}^{\infty} T_i$ with finite groups: $H_{kl} \equiv T_{m_{kl}},\ 1 \le k,l < \infty$, such that $\mathbb{N}$ can be represented as a disjoint union of $\{m_{kl} | k,l\}$ and some infinite subset. Therefore, $\prod_{i=1}^{\infty} T_i$ is represented as the product of $\mathcal{D} = \prod_{k,l \in \mathbb{N}} T_{m_{kl}}$ and the Cantor compactum $\mathcal{C}$. According to Proposition (1.8), there exists a sequence $R_1 \subset R_2 \subset \dots$ of rectangle subsets on $\mathcal{C}$, satisfying (1.8)(7)-(9). Then $\mathcal{D} \times R_1 \subset \mathcal{D} \times R_2 \subset \dots$ is the sequence of the rectangle subsets of $\mathcal{D} \times \mathcal{C}$ also satisfying (1.8)(7)-(9).

By Theorem (3.8), $\{r(I^n \times \mathcal{D} \times R_i)\}$ is an equi-LC^{n-1}-family of the Menger compactum P. By Remark (2.4), each $\{r(I^n \times \mathcal{D} \times R_i)\}$ is a Z-set in $r(I^n \times \mathcal{D} \times R_{i+1})$. Since $\bigcup_{i=1}^{\infty} r(I^n \times \mathcal{D} \times R_i)$ is a dense subset of P, it follows by Proposition (1.6) that the pseudo-boundary $\Sigma(P)$ coincides with $\bigcup_{i=1}^{\infty} r(I^n \times \mathcal{D} \times R_i)$.

To construct a semi-free action Φ of $G \subset H$ on P as in Theorem (0.2), we first define a discontinuous action Ψ of H on Q. Let $\mathrm{N}(B, \mathcal{Q}_m)$ be the star of a subset $B \subset Q$ with respect to the partition $\mathcal{Q}_m = \{\mathcal{E}' \times \prod_{i=m+1}^{\infty} T_i | \mathcal{E}' \in \mathcal{Q}_m'\}$ of $Q = I^n \times \prod_{i=m+1}^{\infty} T_i$. Note that $\mathrm{N}(B, \mathcal{Q}_m)$ is a closed neighborhood of B and

(2) $\bigcap_{m=1}^{\infty} \mathrm{N}(B, \mathcal{Q}_m) = r^{-1} r(B)$; and

(3) if $(x,u,v) \in \mathrm{Bd}(\mathrm{N}(B,\mathcal{Q}_m))$, then $\mathrm{ind}(x) = m$.

Let $B_k = r^{-1}(X_k) \subset I^n \times \mathcal{D} \times \mathcal{C}$. Now the action Ψ of H on Q is defined by setting for $h = (h_{kl}) \in H$ and $z = (x,u,v) \in I^n \times \mathcal{D} \times \mathcal{C}$,

(4) $\Psi(h,z) = (x,u',v)$, where $u' = (u'_{m_{kl}})$ and

$$u'_{m_{kl}} = \begin{cases} u_{m_{kl}} & \text{if } (x,u,v) \in \mathrm{Int}(\mathrm{N}(B_k,\mathcal{Q}_{m_{kl}})),\\ h_{kl} \cdot u_{m_{kl}} & \text{otherwise.} \end{cases}$$

It is obvious that

(5) $r(I^n \times \mathcal{D} \times A)$ is an invariant subset for every $A \subset \mathcal{C}$.

Let $(x,u,v) \notin B_k$ and $g = (g_{kl}) \in G_k \backslash \{e\}$. Since $\mathcal{M}_k = \{m_{kl} | (x,u,v) \in \mathrm{N}(B_k,\mathcal{Q}_{m_{kl}})\}$ is finite, whereas $\mathcal{G} = \{l | g_{kl} \neq e\}$ is infinite, we have

(6) B_k is the fixed-point set for every $g = (g_{kl}) \in G_k \backslash \{e\}$.

Next the action Φ of H on P is defined by $\Phi(h, r(x,u,v)) = r(\Psi(h,(x,u,v)))$. As the preimage $r^{-1}(z)$ is a rectangle subset $x \times \mathcal{T}(x,t)$ of Q, the action Φ is well-defined. It follows from (5)-(6) and the formula for Φ that $\Sigma(P)$ and $\nu(P)$ are invariant subsets and X_k is the fixed-point set for every $g \in G_k \backslash \{e\}$.

To complete the proof of Theorem (0.2), we need the following fact:

Proposition (4.1). *The action $\Phi : G \times P \to P$ is continuous.*

Proof. It suffices to verify that for every $(x_0,t_0) \in Q, \epsilon > 0$ and $g_0 \in G$, there exist a neighborhood $O(g_0)$ of g_0 and $\delta > 0$, such that $\mathrm{dist}((x,t),(x_0,t_0)) < \delta$ implies $\alpha_H(g \cdot (x \times \mathcal{T}(x,t)), g_0 \cdot (x_0 \times \mathcal{T}(x_0,t_0))) < \epsilon$, for every $g \in O(g_0)$.

We shall use the upper semi-continuity of the decomposition $\mathcal{T}$ (Proposition (2.1)). Let us choose $\delta > 0$ and $p \in \mathbb{N}$, satisfying (8)-(10) in Section 2, and a neighborhood $O(g_0)$ of g_0 such that the first p coordinates of every element $g \in O(g_0)$ coincide. Then it follows that $g \cdot (x \times \mathcal{T}(x,t)) = x \times \mathcal{T}(x_0, g \cdot t)$ and $\alpha_H(x \times \mathcal{T}(x, g \cdot t), x_0 \times \mathcal{T}(x_0, g_0 \cdot t_0)) < \epsilon$, analogously to the proof of Proposition (2.1). □

5. Epilogue

Let M be a μ^k-manifold. By Theorem (1.4), there exist a PL manifold R of dimension $\geq 2k+1$ with a triangulation L, and a proper map $f\colon R^{(k)} \to M$ which induces isomorphisms of homotopy groups of dim $< k$ and of homotopy groups of ends of dim $< k$, where $R^{(k)} = |L^{(k)}|, L^{(k)}$ is the k-skeleton of L. If $R^{(k)}$ would admit a cubical triangulation, then by applying Sections 2-4 word-by-word, we could construct the map $r\colon R^{(k)} \times \prod T_i \to (R^{(k)} \times \prod T_i)/\mathcal{T} = N^k$ into N^k, which by Theorem (1.3) is a μ^k-manifold. The natural projection $\pi'\colon R^{(k)} \times \prod T_i \to R^{(k)}$ generates the proper retraction $\pi\colon N^k \to R^{(k)}$, which would induce isomorphisms of homotopy groups of dim $< k$ and of homotopy groups of ends of dim $< k$. The composition $f \circ \pi$ would then be a proper map between μ^k-manifolds which would induce isomorphisms of homotopy groups of dim $< k$ and of homotopy groups of ends of dim $< k$. By Theorem (1.5), $f \circ \pi$ would be properly $(k-1)$-homotopic to a homeomorphism and therefore $(R^{(k)} \times \prod T_i)/\mathcal{T}$ would be homeomorphic to M^k. As in Section 4, we could construct the desired action of $\prod G_i$ on $N = (R^{(k)} \times \prod T_i)/\mathcal{T}$.

Problem (5.1). *Is every simplicial complex homeomorphic to a complex which admits a cubical triangulation?*

Without referring to the solvability of this problem, we give an outline of the proof of Theorem (0.2) in the case of μ^k-manifold M. To this end, we consider the handlebody decomposition $H_m = \{\mathrm{St}(v; \beta^{m+2}L) \mid v \in (\beta^{m+1}L)^{(0)}\}$ of $R^{(k)}$ according to $\beta^m L$, where $\beta^m L$ is the mth barycentric subdivision of the triangulation L and $(\beta^{m+1}L)^{(0)}$ are the vertices of the $(m+1)$-st barycentric subdivision.

By F_m we denote the union of all the boundaries of all elements of the partition H_m. Clearly, the analogue of (2) in Section 2 holds:

(1) for every $x \in R^{(k)}$, the set $\{m | x \in F_m\}$ has less than $k+1$ elements.

All assertions from Sections 2-4 go through in our case, whereas their proofs do not differ substantially and the reader can supply the details.

Only the fact that the proper retraction $\pi\colon (R^{(k)} \times \prod T_i)/\mathcal{T} \to R^{(k)}$ induces isomorphisms of homotopy groups of dim $< k$ and of homotopy groups of ends of dim $< k$, requires a proof. Due to the limited space we must leave the details to the readers. However, we wish to point out that they do not significantly differ from the techniques used in our proofs above.

As in Section 4 we can construct the desired action of $\prod G_i$ on $(R^{(k)} \times \prod T_i)/\mathcal{T}$, where $\prod T_i$ is chosen as $\prod H_{kl} \times \mathcal{C}$.

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References

1. S. M. Ageev, *An equivariant generalization of Michael's selection theorem*, Mat. Zametki **57:4** (1995), 499–508 (in Russian); English transl. in Math. Notes **57:3-4** (1995), 345-350. MR **96h:**54017
2. C. Bessaga and A. Pelczyński, *Selected Topics in Infinite-dimensional Topology*, PWN, Warsaw, 1975. MR **57:**17657
3. M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc. **71(380)** (1988). MR **89g:**54083
4. K. Borsuk, *On the imbedding of systems of compacta in simplicial complexes*, Fund. Math. **35** (1948), 217–234. MR **10:**391b
5. K. Borsuk, *Theory of Retracts*, PWN, Warsaw, 1967. MR **35:**7306
6. G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972. MR **54:**1265
7. A. Chigogidze, K. Kawamura and E. D. Tymchatyn, *Nöbeling spaces and pseudo-interiors of Menger compacta*, Topol. Appl. **68** (1996), 33-65. MR **97c:**54032
8. A. N. Dranishnikov, *Free actions of* 0*-dimensional compact groups*, Izv. Akad. Nauk SSSR Ser. Mat. **52:1** (1988), 212-228 (in Russian); English transl. in Math. USSR Izv. **32:1** (1989), 217–232. MR **90e:**57065
9. A. N. Dranishnikov, *Absolute extensors in dimension n and dimension-raising n-soft maps*, Uspehi Mat. Nauk **39:5** (1984), 55-95 (in Russian); English transl. in Russian Math. Surv. **39:5** (1984), 63–111. MR **86c:**54017
10. V. V. Fedorchuk and V. V. Filippov, *General Topology. Basic Constructions*, Moscow Univ. Press, Moscow, 1988 (in Russian).
11. Y. Iwamoto, *Fixed-point sets of transformation groups of Menger manifolds, their pseudo-boundaries and their pseudo-interiors*, Topol. Appl. **68** (1996), 267–283. MR **97a:**54035
12. L. S. Pontryagin, *Topological Groups*, Gordon and Breach, New York, 1966.

13. K. Sakai, *Free actions of zero-dimensional compact groups on Menger manifolds*, Proc. Amer. Math. Soc. **122** (1994), 647–648. MR **95c:**57057
14. K. Sakai, *Semi-free actions of zero-dimensional compact groups on Menger compacta*, Proc. Amer. Math. Soc. **125** (1997), 2809–2813. MR **97j:**54043
15. E. H. Spanier, *Algebraic Topology*, Mc Graw-Hill, New York, 1966. MR **35:**1007

Department of Mathematics, Brest State University, 224011 Brest, Belarus
E-mail address: ageev@highmath.brsu.brest.by

Institute for Mathematics, Physics and Mechanics, University of Ljubljana, 1001 Ljubljana, Slovenia
E-mail address: dusan.repovs@fmf.uni-lj.si