Doklady Mathematics, Vol. 58, No. 2, 1998, pp. 190–193. Translated from Doklady Akademii Nauk, Vol. 362, No. 2, 1998, pp. 151–154.

MATHEMATICS

Extensors of Noncompact Lie Groups

S. M. Ageev*, S. A. Bogatyi**, D. Repovs***, and P. Fabel****

Presented by Academician A.T. Fomenko November 28, 1996

Received December 3, 1996

INTRODUCTION

One old and well-known problem of the geometry of Banach spaces is studying the topological properties of the set $Q(n)$ of classes of isomeric n-dimensional Banach spaces with the Banach–Mazur metric

$$d(E, F) = \ln\inf\{\|T\|\cdot$$
$$\cdot\|T^{-1}\|\,|\, T: E \to F \text{ is an isomorphism}\}.$$

It is known [1] that the space $Q(n)$ is compact.

The main problem is to establish whether or not $Q(n)$ is isomorphic to the Hilbert cube I^∞ (the West problem [2, p. 544]). By the Torunszyk criterion [3], a compact space X is homeomorphic to I^∞ if and only if X is an absolute extensor and has the disjoint disk property DD^mP in any dimension m; therefore, the basic conjecture that the specified spaces are homeomorphic breaks into two parts:

(A) $Q(n) \in$ AE;

(B) $Q(n)$ has the m-dimensional disjoint disk property DD^mP for any m.

The one-to-one correspondence between norms in $\mathbb{R}^n$ and the set $C(n)$ of convex compact symmetric bodies given by the Minkowski functional makes it possible to reduce studying $Q(n)$ to studying the natural action of the group $GL(n)$ of all linear nondegenerate transformations of $\mathbb{R}^n$ on the space $C(n)$ with the topology generated by the Hausdorff metric. Namely, the quotient space $C(n)/GL(n)$ of $C(n)$ with respect to the equivalence relation induced by the natural action of $GL(n)$ (i.e., the space of orbits) is homeomorphic to the Banach–Mazur compact space $Q(n)$. This allows us to apply results of the theory of topological groups of transformations. The application of the equivariant theory of extensors to solve the stated problem is based on the following two theorems.

(i) *If the convex structure on a space X is consistent with the action of a compact Lie group G and there is at least one point fixed with respect to this action, then X is an equivariant absolute extensor* [4].

(ii) *If a compact group G acts on a metric space X that is an equivariant absolute extensor, then the orbit space X/G is an absolute extensor* [5].

The convex structure on $C(n)$ determined by the Minkowskii linear combination of convex bodies is consistent with the action of the group $GL(n)$. Thus, (i) and (ii) imply that $C(n)/H \in$ AE for any compact subgroup H. Unfortunately, $GL(n)$ is not a compact group, and the principal difficulty, at least in problem (A), is the passage from compact to locally compact groups and study of the extensor properties of their actions.

The second approach to solving the problem is to reduce the question about the quotient space of the action of a noncompact group to the question about the quotient space of the action of a compact group. This second approach allowed P. Fabel to prove conjecture (A) for $n = 2$ (without the use of the equivariant theory of extensors but with the application of some spaces of conformal mappings). On May 23, 1996, at the Session in memory of Borsuk and Kuratowski in Warszaw, Fabel reported about the Banach–Mazur compact space $Q(2)$; after that, S. Ageev applied the theory of $GL(n)$-extensors to show that the compact space $Q(n)$ is locally contractible for all n (the contractibility of the compact spaces $Q(n)$ was known, although unpublished). However, because the space $Q(n)$ is strongly infinite-dimensional, the proved inclusion $Q(n) \in LC \cap C$ does not imply that $Q(n) \in$ AE. In addition, Ageev and S. Bogatyi noticed that the reduction of the question about the action of a noncompact group $GL(n)$ to the question about the action of the compact group $O(n)$ that was suggested by Fabel makes it possible to prove [by applying Theorems (i) and (ii) stated above] Conjecture (A) in the general case [6]. In June, 1996, Ageev and D. Repovs deeper developed the theory of extensors of locally compact groups, which made it possible, in particular, to strengthen the inclusion $Q(n) \in$ LC to $Q(n) \in$ AE, i.e., to give the second proof of Conjecture (A).

This work can be considered an exposition of the principles of the theory of extensors of locally compact groups, which is nevertheless advanced to a degree that

* Brest State University, Brest, Belarus
** Moscow State University, Vorob'evy gory, Moscow, 119899 Russia
*** Ljubljana University, Ljubliana, Slovenia
**** Mississippi State University, the United States of America

makes it possible to prove (A) without the use of the theory of convex bodies (for example, the Loevner ellipsoid). Moreover, not only $Q(n) = C(n)/\mathrm{GL}(n)$, but also the space $C(n)/H$ of orbits for any closed subgroup $H \subset \mathrm{GL}(n)$ is an absolute extensor. The equivariant local contractibility of the $\mathrm{GL}(n)$-space $C(n)$, which is established in this work, makes it possible to make a step forward in proving Conjecture (B) and, therefore, the whole conjecture that $Q(n) \approx I^\infty$, namely, to prove that the open dense subset $Q_0(n) \subset Q(n)$ of classes of isometric nonsymmetric Banach spaces is a I^∞-variety.

1. PRELIMINARIES

Let G be a locally compact Lie group. An action of G on a space X is a homomorphism T of G into the group of all homeomorphisms of X such that the mapping $(g, x) \mapsto T(g)(x) = g \cdot x = gx$ from $G \times X$ into X is continuous. The space X is then called a G-space.

If $x \in X$, then the stabilizer of x is $G_x = \{g \in G \mid gx = x\}$ and the orbit of the point x is $G(x) = \{gx \mid g \in G\}$. The space of all orbits is denoted by X/G, and the natural mapping $\pi: X \to X/G$ defined by $\pi(x) = G(x)$ is called the orbit projection. The space of orbits X/G is endowed with the quotient topology generated by the mapping π.

The action of a noncompact group G is poorly consistent with the orbital structure of X: the orbit of a point may be dense in X, the space of orbits X/G non-Hausdorff, and two orbits with the same stabilizer nonhomeomorphic. Palais [7] defined a class of G-spaces on which a locally compact group acts and called it proper; this class does not have the mentioned shortcomings.

Definition 1. (a) For subsets $A, B \subset X$, $((A, B))$ is the subset $\{g \mid gA \cap B \neq \emptyset\}$ in G. We call the set A thin with respect to B if $((A, B))$ is precompact (i.e., contained in a compact subspace). Since $((A, B)) = ((B, A))^{-1}$, B is then thin with respect to A.

(b) A set A is called small if any point $x \in X$ has a neighborhood $O(x)$ thin with respect to A.

(c) A G-space X is called proper if it has a base of small neighborhoods.

We remind the reader that a continuous mapping $f: X \to Y$ of two G-spaces is called a G-mapping if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. The stabilizers are then in the inclusion relation $G_x \subset G_{f(x)}$.

Definition 2. A cut at a point x is a G-mapping $\varphi: U \to G(x)$ of some G-neighborhood U ($G \cdot U = U$) of the orbit $G(x)$ such that $\varphi(x) = x$. The preimage $\varphi^{-1}(x)$ of x is also called a cut or a G_x-kernel.

Palais Theorem. *A proper completely regular G-space X has a cut at any its point.*

In what follows, all our reasoning is concerned with separable metric proper G-spaces; the class of such spaces we denote by $\mathfrak{S}$.

Proposition 1 [7]. *Let $X, Y \in \mathfrak{S}$ and H is a compact subgroup of G. Then*

(a) *$X \times Y \in \mathfrak{S}$ and $G/H \in \mathfrak{S}$;*

(b) *the orbit $G(x)$ is closed in X, the stabilizer G_x is compact, and the natural mapping of G/G_x into $G(x)$ defined by $gG_x \mapsto gx$ is a homeomorphism;*

(c) *the space of orbits X/G is a metric separable space and* $\dim X/G \leq \dim X$;

(d) *the space X can be metrized by a metric d such that it is invariant [i.e., $d(gx, gx') = d(x, x')$] and the topology of the space of orbits X/G is generated by the metric $\tilde{d}([x], [y]) = \inf\{d(x', y') \mid x' \in G(x), y' \in G(y)\}$;*

(e) *if d is a complete invariant metric on X, then $(X/G, \tilde{d})$ is a complete space.*

A matrix group is a Lie group G that is subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(n)$ for some n.

Theorem 1. *A group G is matrix if and only if, for any its compact subgroup H, there exists a finite-dimensional linear G-space V where the stabilizer G_v of some point v coincides with H.*

Definition 3. A G-space X is called an equivariant absolute neighborhood extensor (symbolically, $X \in$ G-ANE) if any partial G-mapping $Z \hookleftarrow A \xrightarrow{\varphi} X$ defined on a closed equivariant subset A of the G-space Z from the class $\mathfrak{S}$ can be extended to a G-mapping $\hat{\varphi}: U \to X$ defined on some G-neighborhood U such that $A \subset U$. If $U = Z$, then $X \in$ G-AE; i.e., X is an equivariant absolute extensor.

The most important and nontrivial example of G-A[N]E-spaces for locally compact groups G are convex G-spaces.

2. THE PROOF OF CONJECTURE (A) WITH THE USE OF THE LOEVNER ELLIPSOID

Theorem 2. *The Banach–Mazur compact space $Q(n)$ is an absolute extensor.*

Proof. For $B \in C(n)$ and a subgroup $H \subseteq \mathrm{GL}(n)$, we put $[B]_H = \{T(B): T \in H\}$. By the John theorem [1], there exists a (unique) filled ellipsoid $E(B)$ (which is called the Loevner ellipsoid) that contains the body B and has the smallest volume. Let S_B be a linear mapping such that $S_B(E(B))$ is the unit ball. The required topological embedding $\sigma: C(n)/\mathrm{GL}(n) \to C(n)/\mathrm{O}(n)$ is defined by the formula $\sigma([B]_{\mathrm{GL}(n)}) = [S_B(B)]_{\mathrm{O}(n)}$. The inverse mapping (retraction) $r: C(n)/\mathrm{O}(n) \to C(n)/\mathrm{GL}(n)$ is defined by $r([B]_{\mathrm{O}(n)}) = [B]_{\mathrm{GL}(n)}$. To the $\mathrm{O}(n)$-space $C(n)$, Theorem (i) from the introduction applies. Since B^n is a fixed point of the action of $\mathrm{O}(n)$, we have $C(n) \in \mathrm{O}(n)$-AE. By Theorem (ii) from the introduction, we have $C(n)/\mathrm{O}(n) \in$ AE.

3. EQUIVARIANT EXTENSORS FOR LOCALLY COMPACT GROUPS

Theorem 3. *If G is a locally compact Lie group and X a convex G-space, then X is an equivariant absolute neighborhood extensor in the class of proper G-spaces; i.e., $X \in$ G-ANE.*

The proof of Theorem 3 is based on replacing the exact G-extension by an approximate one.

Definition 4. A closed subset F of a G-space Z cuts this space if F is small in Z and intersects every orbit; i.e., $F \cap G(z) \neq \emptyset$ for all $z \in Z$.

Proposition 2. *If a G-space Z is proper and space of orbits Z/G is metrizable, then there exists a cutting set F in Z.*

In the class of metric G-spaces, the orbit projection $\pi: X \to X/G$ of a proper G-space is generally not closed.

Proposition 3. *If the conditions of Proposition 2 are fulfilled, then*

(a) *the image $\pi(F)$ of a small closed set $F \subset Z$ is closed in the space of orbits Z/G;*

(b) *the restriction of π to a cutting set $F \subset Z$ is a closed mapping.*

Definition 5. A G-space X is called an approximative G-A[N]E space ($X \in$ G-AA[N]E) if, for any G-space Z from the class $\mathfrak{S}$, any closed small set F in Z, and any cover $\omega \in \mathrm{cov}(X)$), the following is true: any partial G-mapping $Z \hookleftarrow A \xrightarrow{\varphi} X$ has an approximative G-continuation $\tilde{\varphi}: Z \to X$ ($\tilde{\varphi}: U \to X$, where U is a G-neighborhood of A) such that the restrictions of φ and $\tilde{\varphi}$ to $A \cap F$ are ω-close.

Theorem 4. *Any convex G-space X is an approximative equivariant neighborhood extensor; i.e., $X \in$ G-AANE.*

Theorem 5. *If the product of a G-space X and the half-open interval $J = [0, 1)$ is a G-AANE, then the factor X is a G-ANE-space.*

Theorem 3 is obtained from the observation that the product of the convex G-space X and the half-open interval J is a convex G-space and, by virtue of Theorem 4, $X \times J \in$ G-AANE. Theorem 5 implies that $X \in$ G-ANE.

4. SPACES OF ORBITS OF EQUIVARIANT ABSOLUTE EXTENSORS

Theorem 6. *If G is a matrix group and X a proper G-ANE-space, then the space of orbits X/G is an ANE.*

First, we apply Theorems 1 and 3 to obtain the following preliminary result.

Proposition 4. *Let H be a compact subgroup in a matrix group G. Then G/H is a G-ANE-space.*

Let X be a metric G-space of diameter 1. It is easy to verify that the metric cone $\mathrm{Con}X = X \times [0, 1]/_m X \times \{0\}$ is a G-space.

Proposition 5. *If a metric G-space X is a G-ANE, then the metric cone $\mathrm{Con}X$ is a G-AE.*

Proposition 6. *Let Z be a proper G-space. Then, for any point $z \in Z$ and any $\varepsilon > 0$, there exists a G-mapping $\varphi: Z \to \mathrm{Con}G(z)$ such that $\varphi(z) = z$ and*

$$\mathrm{diam}\,\varphi^{-1}((V \cdot z) \times (0, 1]) < \varepsilon \begin{Bmatrix} \text{for some neighborhood } V \\ \text{of the stabilizer } G_z \in G \end{Bmatrix}.$$

Proposition 7. *Let G be a matrix group and X belong to $\mathfrak{S}$. Then there exist countably many finite-dimensional G-ANE-spaces R_k, $k \geq 1$, of class $\mathfrak{S}$ such that the topological G-embedding $i: X \hookrightarrow \prod_{k=1}^{\infty} \mathrm{Con}\,R_k$ holds.*

Proposition 8. *Let a G-space H be the limit of an inverse spectrum $\{H_1 \xleftarrow{q_1} H_2 \xleftarrow{q_2} H_3 \leftarrow \ldots\}$ of G-spaces H_i and G-mappings q_i. If the sets of fixed points H_i^G are no more than singletons for all i and the stabilizer G_h of any point $h \in H_i \setminus H_i^G$ is compact, then, for the space of orbits, the homeomorphism*

$$H/G \cong \varprojlim\{H_1/G \xleftarrow{\tilde{q}_1} H_2/G \xleftarrow{\tilde{q}_2} H_3/G \leftarrow \ldots\}$$

holds.

Proof of Theorem 6. Let us apply the condition of the theorem and fix a topological G-embedding

$$i: X \hookrightarrow \prod_{k=1}^{\infty} \mathrm{Con}\,R_k = D \text{ (see Proposition 7)}$$

and a closed topological embedding $j: X/G \hookrightarrow L$ of the space of orbits X/G into a linear normed space L. Clearly, $i \times (j \circ \pi) = e: X \hookrightarrow L \times D$ is a closed topological G-embedding. Since the image $e(X)$ contains no points with noncompact stabilizers, $e(X)$ does not intersect the closed set $L \times \{*\}$, where $\{*\}$ is the product of the vertices of the cones that are factors in D. Hence, $e(X)$ is contained in the proper open G-space $U' = L \times (D \setminus \{*\})$.

Since $L \times D \in$ G-AE by Propositions 4 and 5, we have $U' \in$ G-ANE. Because $X \in$ G-ANE, there exists a G-retraction $r: U \to X$ of some G-neighborhood U such that $e(X) \subset U \subset U'$. Hence, $\tilde{r}: U/G \to X/G$ is a retraction, and the inclusion $X/G \in$ ANE reduces to the other inclusion $U/G \in$ ANE.

Now, if we prove that $D/G \in$ AE, then we will have $(L \times D)/G = L \times (D/G) \in$ AE and, hence, that $U/G \in$ ANE as an open subset in the space of orbits. Thus, to complete the proof of Theorem 6, it remains to show that $D/G \in$ AE.

Put $D_m = \prod_{k=1}^{m} \operatorname{Con} R_k$, and let $q_m: D_{m+1} \to D_m$ be a projection. Since $R_k = G(x_k)$ is metrizable by a complete invariant metric, $\operatorname{Con} R_k$ and D_m are also metrizable by complete invariant metrics. Therefore, the space of orbits D_m/G is metrizable by a complete metric. The inclusion $D_m \in$ G-AE and Proposition 10 imply that $D_m/G \in \mathrm{LC} \cap \mathrm{C}$. By virtue of the finite dimensionality of this space, we obtain $D_m \in$ AE.

Since $\operatorname{Con} R_{m+1} \in$ AE, the projection q_m is a fiberwise G-contractible mapping; i.e., there exist fiberwise G-mappings $s: D_m \to D_{m+1}$ with $q_m \circ s = \mathrm{Id}$ and $H: D_{m+1} \times [0, 1] \to D_{m+1}$ with $q_m \circ H = q_m$ such that $H_0 = \mathrm{Id}$ and $\mathrm{Im}(H_1) = \mathrm{Im}(s)$. Passing to spaces of orbits, we obtain fiberwise contractible mappings $\tilde{q}_m: D_{m+1}/G \to D_m/G$; thus, $\tilde{q}_m$ is a CE-mapping (the fiber $\tilde{q}_m$ is contractible!). All conditions of the Curtis theorem [8] are fulfilled; hence, $\varprojlim\{D_i/G, q_i\}$ is an AE. By virtue of Propositions 8, this inverse limit coincides with the space of orbits D/G.

5. APPLICATIONS

The space $C(n)$ of convex bodies is a convex GL(n)-space. By virtue of Theorem 3, $C(n) \in$ GL(n)-ANE. However, $C(n)$ is additionally a proper space, which implies that $C(n)$ has the property of equivariant local contractibility and, by virtue of the theorem, that $Q(n) \in$ AE.

Proposition 9. *$C(n)$ is a proper* GL(n)*-space.*

Proposition 10. *If X is a proper* G-ANE*-space, then, for any G-neighborhood U of the orbit $G(x)$, there exists a G-neighborhood V and a G-mapping H: $V \times [0, 1] \to U$ such that $H_0 = \mathrm{Id}$; $\mathrm{Im}(H_1) \subset G(x)$, and $H_t \restriction G(x) = \mathrm{Id}$ for all $t \in I$.*

We call a Banach space nonsymmetric if the symmetry group of its unit ball centered at 0 is {Id, –Id}. Let $Q_0(n)$ denote the open dense subset of $Q(n)$ formed by the classes of isomeric nonsymmetric n-dimensional Banach spaces.

Theorem 7. *$Q_0(n)$ is a variety modeled by the Hilbert cube.*

ACKNOWLEDGMENTS

S.A.B. acknowledges the financial support of the Russian Foundation for Basic Research, project no. 95-01-01273.

REFERENCES

1. John, F., *Studies and Essays, Presented to R. Courant on His 60-th Birthday*, New York, 1948, pp. 187–204.
2. *Open Problems in Topology*, van Mill, J. and Reed, G.M., Eds., New York: Elsevier, 1990, Problem 899.
3. Torunczyk, H., *Fund. Math.*, 1980, vol. 106, pp. 31–40.
4. Murayama, M., *Osaka J. Math.*, 1983, vol. 20, no. 3, pp. 479–512.
5. Ageev, S.M., *Vestn. Mosk. Univ., Ser. Mat., Mekh.*, 1994, no. 1, pp. 11–16.
6. Ageev, S., Bogatyi, S., and Fabel, P., *VII Belarusskaya matematicheskaya konferentsiya* (VII Belarus Mathematical Conf.), Minsk, 1996, pp. 151–152.
7. Palais, R., *Ann. Math.*, 1961, vol. 73, pp. 295–323.
8. Curtis, D.W., *Fund. Math.*, 1971, vol. 72, no. 1, pp. 101–113.