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Banach–Mazur Compacta are Aleksandrov Compactifications of Q-manifolds

S. M. Ageev, S. A. Bogatyi, and D. Repovš

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Abstract—It is proved that, for all $n > 2$, the Banach–Mazur compactum $Q(n)$ is the compactification of a Q-manifold by a Euclidean point. For $n = 2$, this was known earlier.

Key words: *Banach–Mazur compactum, Q-manifold, elliptically convex set, action of a compact Lie group.*

1. INTRODUCTION

The problem of determining the topological type of the Banach–Mazur compactum $Q(n)$ goes back to the Polish school in the geometric theory of Banach spaces. A. Pełczyński noticed that elementary geometric arguments prove the contractibility of $Q(n)$. On the other hand, the Banach–Mazur compactum is closely related to the hyperspace of all convex bodies in $\mathbb{R}^n$. Taking into account these facts and Wojdysławski's problem of whether hyperspaces of a certain type are homeomorphic to the Hilbert cube Q, Pełczyński stated two conjectures, which have become widely known among topologists, especially after West's work [1]. These are:

(1) the space $Q(n)$ (for $n \geq 2$) is an absolute retract;

and the stronger conjecture

(2) the space $Q(n)$ (for $n \geq 2$) is homeomorphic to the Hilbert cube Q.

In 1996, Fabel proved that the compact space $Q(2)$ is an absolute retract (i.e., $Q(2) \in \mathrm{AE}$); the same year, it was shown [2] that $Q(n)$ is an absolute extensor for all $n \geq 2$. Finally, in 1997, a negative answer to the question about the existence of an isomorphism between $Q(n)$ and the Hilbert cube Q was obtained [3] (a short version was published in [4]; see also [5]).

Theorem 1.1. *$Q(2)$ and Q are not homeomorphic.*

The key point in the proof of Theorem 1.1 is the homotopic nontriviality of $Q(2)\setminus\{\mathrm{Eucl}\}$, where $\{\mathrm{Eucl}\} \in Q(n)$ is the Euclidean point corresponding to the isometry class of standard Euclidean n-space. In its turn, this nontriviality follows from the nontriviality of the 4-dimensional cohomology group $\mathrm{H}^4(Q(2) \setminus \{\mathrm{Eucl}\}, \mathbb{Q})$ with rational coefficients. Ideas from [6] were used to describe the structure of the Eilenberg–MacLane complexes in the Banach–Mazur compactum $Q(2)$ in [3], which made it possible to apply fairly advanced techniques of algebraic topology, such as calculations in cohomology rings and Smith's theory of periodic homeomorphisms [3, p. 7]. Note that the more recent paper [7] on Problem 2 uses such calculations[1] at the key point of the proof, on p. 224.

[1]The paper [8] by the same author with the same title contains bad mistakes. Lemma 6 asserts the existence of an $O(2)$-equivariant map which not is $O(2)$-equivariant. Because of this lemma, the groups $SO(2)$ and $O(2)$ are identified in the proof of the main result (Theorem 4), and the question of the final factorization by the group $\mathbb{Z}_2 = O(2)/SO(2)$, for which the Smith theory is employed in [4], does not even arise.

Apparently, revealing a deeper relationship between the Banach–Mazur compacta and algebraic topology would make it possible to go further into studying their topology, in particular, prove that the $Q(n)$ are not homeomorphic to the Hilbert cube for all $n > 2$.

In [9], the study of $Q(2)$ was continued; it was proved that $Q(2)$ is the one-point compactification of a Q-manifold, which (together with Theorem 1.1) implied its inhomogeneity. The naturally arising problem concerning $Q(n)$, where $n > 2$, was reduced to a certain assertion from convex geometry, which was likely to be true. In this paper, we return to this problem and prove the following theorem by applying the ideas of [9] to the notion of elliptic convexity.

Theorem 1.2. *$Q_{\mathcal{E}}(n) \rightleftharpoons Q(n) \setminus \{\mathrm{Eucl}\}$ is a Q-manifold.*

2. PRELIMINARIES

Let G be a compact Lie group. By an action of G on a space X we mean a homomorphism $T\colon G \to \operatorname{Aut} X$ of G to the group $\operatorname{Aut} X$ of all autohomeomorphisms of X such that the map $G \times X \to X$ defined by $(g, x) \mapsto T(g)(x) = g \cdot x$ is continuous. A space X with a fixed action of G is called a *G-space*.

The *isotropic subgroup* of a point x, or the *stabilizer* of x, is defined as $G_x = \{g \in G \mid g \cdot x = x\}$; the *orbit* of x is $G(x) = \{g \cdot x \mid g \in G\}$. The space of all orbits is denoted by X/G, and the natural map $\pi\colon X \to X/G$ defined by $\pi(x) = G(x)$ is called the *orbit projection*. The *orbit space* X/G is endowed with the quotient topology induced by π (see [10]).

Below we give one of the several equivalent definitions of the Banach–Mazur compactum $Q(n)$ (a detailed description of the topology of the Banach–Mazur compactum is contained in, e.g., [9, 5]). By $C(n)$ we denote the family of all compact convex centrally symmetric (with center of symmetry 0) bodies in $\mathbb{R}^n$. Measuring the distances between subsets of $\mathbb{R}^n$ by the Hausdorff metric ρ_H and defining linear combinations $\sum_{i=0}^{n} \lambda_i A_i$ by the Minkowski operation, we make $(C(n), \rho_H)$ into a locally compact convex space. Moreover, $C(n)$ can be endowed with the action of the general linear group $\mathrm{GL}(n) \times C(n) \to C(n)$ defined by

$$T \cdot V = T(V), \qquad \text{where} \quad T\colon \mathbb{R}^n \to \mathbb{R}^n \in \mathrm{GL}(n) \quad \text{and} \quad V \in C(n),$$

which is consistent with the convex structure of $C(n)$. It is well known that the orbit space $C(n)/\mathrm{GL}(n)$ is naturally homeomorphic to the Banach–Mazur compactum.

Recall that the Banach–Mazur compactum $Q(n) = C(n)/\mathrm{GL}(n)$ is also homeomorphic to the orbit space of an action of the orthogonal group $O(n)$ of $\mathbb{R}^n$. As is known (see [11]), for any convex body $V \in C(n)$, there exists a unique ellipsoid $E_V \in C(n)$ (called the *Levner ellipsoid*) which contains V and has minimal Euclidean volume. The minimality of $\operatorname{vol} E_V$ implies the $\mathrm{GL}(n)$-invariance of E_V; i.e., $E_{T \cdot V} = T \cdot E_V$ for any $T \in \mathrm{GL}(n)$. The continuity of the dependence of E_V on V with respect to the Hausdorff metric was proved in [2]. Therefore, the map $\mathcal{L}\colon C(n) \to \mathfrak{E}$ defined by $\mathcal{L}(V) = E_V$ is a $\mathrm{GL}(n)$-retraction of $C(n)$ onto the "elliptic" orbit $\mathfrak{E} = \mathrm{GL}(n) \cdot B^n$, where B^n is the unit ball ($\mathcal{L}$ is called the *Levner retraction*). Let $L(n) = \mathcal{L}^{-1}(B^n)$ be a cut which is a compact $O(n)$-space. In other words, $L(n)$ consists of those $V \in C(n)$ whose Levner ellipsoids coincide with B^n. It is easy to see that the orbit space $Q(n) = C(n)/\mathrm{GL}(n)$ is naturally homeomorphic to $L(n)/O(n)$. Therefore, $Q_{\mathcal{E}} \rightleftharpoons Q(n) \setminus \{\mathrm{Eucl}\}$ coincides with $L_{\mathcal{E}}(n)/O(n)$, where $L_{\mathcal{E}} \rightleftharpoons L(n) \setminus \{B^n\}$. Thus, Theorem 1.2 reduces to the following assertion.

Theorem 2.1. *$L_{\mathcal{E}}(n)/O(n)$ is a Q-manifold.*

A space X is an *absolute neighborhood extensor* ($X \in \mathrm{ANE}$) if any map $\varphi\colon A \to X$ defined on a closed subset A of a metric space Z (it is called a *partial map*) can be extended to some neighborhood $U \subset Z$ of A, i.e., there exists a $\tilde{\varphi}\colon U \to X$ such that $\tilde{\varphi}|_A = \varphi$. If we can always take $U = Z$, then X is an *absolute extensor* ($X \in \mathrm{AE}$). For metric spaces X, the notions of an absolute (neighborhood) extensor and an absolute (neighborhood) retract coincide. According to the Toruńszyk characterization theorem [12, 13], any locally compact space $X \in \mathrm{ANE}$ is locally homeomorphic to the Hilbert cube Q (i.e., it is a *Q-manifold*) if and only if X admits maps $f_i \colon X \to X$ for $i \in \{1, 2\}$ which are arbitrarily close to Id_X and $\mathrm{Im}\, f_1 \cap \mathrm{Im}\, f_2 = \varnothing$.

According to [2], $Q(n) \in \mathrm{AE}$; therefore, $L(n)/O(n) \in \mathrm{AE}$, and $Q_{\mathcal{E}} \cong L_{\mathcal{E}}(n)/O(n) \in \mathrm{ANE}$. Thus, using the Toruńszyk characterization, we can easily reduce the proof of Theorem 2.1 (and, hence, of Theorem 1.2) to proving the following assertion (see [9]).

Theorem 2.2. *There exist homotopies*

$$f_t \colon L_{\mathcal{E}}(n)/O(n) \to L_{\mathcal{E}}(n)/O(n) \quad \text{and} \quad g_t \colon L_{\mathcal{E}}(n)/O(n) \to L_{\mathcal{E}}(n)/O(n), \qquad 0 \le t \le 1,$$

such that $f_0 = g_0 = \mathrm{Id}$ *and* $\mathrm{Im}\, f_t \cap \mathrm{Im}\, g_s = \varnothing$ *for all* $0 < s \le t \le 1$.

It is well known [14] that there exists an $O(n)$-retraction $R \colon C(n) \to L(n)$ which maps $C_{\mathcal{E}}(n)$ to $L_{\mathcal{E}}(n)$. However, we shall need a more precise statement, which follows from geometric considerations.

Proposition 2.3 (see [9]). *There exists a continuous $O(n)$-retraction $\mathfrak{R} \colon C(n) \to L(n)$ such that $\mathfrak{R}(V)$ and V are affinely equivalent for any $V \in C(n)$.*

Proof. Let T be an element of $\mathrm{GL}(n)$ such that $T^{-1} \cdot B^n = \mathcal{L}(V)$. According to [15], the operator T can be represented as $T_2 \circ T_1$, where $T_2 \in O(n)$ and T_1 is self-adjoint. We set $\mathfrak{R}(V) = T_1(V)$ and leave the verification of all the required properties to the reader. □

Let (X, d) be a metric space of diameter 1. In [9], the erroneous formula

$$\rho((x,t),(x',t')) = \sqrt{t^2 + (t')^2 - 2tt' \cos\gamma}, \qquad \text{where} \quad \cos\gamma = \frac{(2 - d^2(x,x'))}{2},$$

for the metric on the cone $\mathrm{Con}\, X$ was given (although, this did not affect the correctness of the other results). The correct, slightly different, formula is largely known (see, e.g., [16, p. 91]); this is

$$\rho((x,t),(x',t')) = \sqrt{t^2 + (t')^2 - 2tt' \cos\gamma}, \qquad \text{where} \quad \gamma = d(x,x').$$

The authors thank S. Antonyan, who kindly pointed out this inaccuracy (not affecting the contents of [9]) in his thesis; additional information about this metric can be found in [7].

Next, we introduce a partial order on the compact Lie groups. For compact Lie groups K and H, we set $K < H$ if K is isomorphic to a proper subgroup of H. Clearly,

(α) if $K < H$, then either $\dim K < \dim H$ or $\dim K = \dim H$ and $\mathcal{C}_K < \mathcal{C}_H$, where $\mathcal{C}_H$ is the number of path-connected components of H.

If K is a closed subgroup of a compact Lie group H and $\dim K = \dim H$, then K is an open subgroup. This implies the following stronger result.

(β) Let K be a closed subgroup of a compact Lie group H. If $\dim K = \dim H$ and $\mathcal{C}_K = \mathcal{C}_H$, then $K = H$.

The verification of the following property of the order introduced above is fairly simple, and we leave it to the reader.

(γ) There exists no countable sequence of compact Lie groups $\{H_i\}$ such that

$$H_1 > H_2 > H_3 > \cdots > H_n > \dots.$$

Clearly, the pair $\mathrm{ind}\, H = (\dim H, \mathcal{C}_H)$ belongs to $\mathbb{N} \times \mathbb{N}$. Let us endow $\mathbb{N} \times \mathbb{N}$ with the lexicographic order. According to (α), the map $H \mapsto \mathrm{ind}\, H$ is order preserving.

From (γ) the following principle can be derived; it allows us to use induction on compact Lie groups.

Proposition 2.4. *Let $\mathcal{P}(H)$ be a property depending on the compact Lie group H. Suppose that*

(δ) *$\mathcal{P}(H)$ holds for the trivial group $H = \{e\}$ and*

(ε) *$\mathcal{P}(H)$ holds if $\mathcal{P}(K)$ holds for all $K < H$.*

Then $\mathcal{P}(H)$ holds for all groups H.

As an example, in [17], the following property satisfying (δ) and (ε) was considered: *$\mathcal{P}(H)$ holds if, for any metric H-space $X \in H$-*ANE*, the orbit space X/H is an* ANE.

3. PROOF OF THEOREM 2.2

Recall that a point a of a convex set $V \subset \mathbb{R}^n$ is said to be *extreme* if $V \setminus \{a\}$ is convex. It is well known that the set $\operatorname{Extr}(V)$ of all extreme points of V is contained in the relative boundary $\operatorname{rbd} V$, and V coincides with the convex hull $\operatorname{Conv}(\operatorname{Extr}(V))$. If $\operatorname{Extr}(V) = \operatorname{rbd}(V)$, then V is said to be *elliptically convex*; otherwise, V is *not elliptically convex.* Let us show that the following two assertions, which readily imply Theorem 2.2, are valid.

Theorem 3.1. *There exists an $O(n)$-homotopy $H\colon L(n) \times [0,1] \to L(n)$ such that*

(a) *$H_0 = \mathrm{Id}$ and $H_t^{-1}(B^n)$ is contained in the set of elliptically convex bodies for all $t \in [0,1]$;*

(b) *$H_t(V)$ is elliptically convex for any $V \in L(n)$ and $t > 0$.*

Theorem 3.2. *There exists an $O(n)$-homotopy $F\colon L(n) \times [0,1] \to L(n)$ such that*

(c) *$F_0 = \mathrm{Id}$ and*

(d) *$F_t(V)$ is not elliptically convex for any $V \in L_{\mathcal{E}}(n)$ and $t > 0$.*

To prove Theorem 2.2, it is sufficient to consider the two $O(n)$-homotopies

$$F\colon L(n) \times [0,1] \to L(n) \qquad \text{and} \qquad H \circ F\colon L(n) \times [0,1] \to L(n)$$

and pass to the orbit space.

Proof Theorem 3.1. Let $V \in L(n)$. Note that the convex body V is elliptically convex if and only if each supporting hyperplane of V intersects V in precisely one point [15]. Thanks to this criterion, we can relate the notion of elliptic convexity to Minkowski linear combinations.

Lemma 3.3. *Suppose that $V, V_i \in C(n)$ and $V = \sum_{i=1}^{p} \lambda_i \cdot V_i$, where all the λ_i are positive. Then V is elliptically convex if and only if V_i is elliptically convex for each i.*

Proof. Let Π and Π_i be parallel supporting hyperplanes of V and V_i, respectively. We set $A = V \cap \Pi$ and $A_i = V_i \cap \Pi_i$. Obviously, $\sum_{i=1}^{p} \lambda_i \cdot A_i \subset A$. It is easy to show that if $x_i \in V_i$ and $\sum_{i=1}^{p} \lambda_i \cdot x_i \in A$, then $x_i \in A_i$. Therefore, $A = \sum_{i=1}^{p} \lambda_i \cdot A_i$, and hence A consists of one point if and only if each A_i consists of one point. The proof is completed by applying the above criterion for elliptic convexity. □

We continue the proof of Theorem 3.1. Consider the homotopy $\psi_t\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, where $0 \le t \le 1$, defined by $\psi_t(x) \rightleftharpoons x/(1 + t \cdot \|x\|) \in \mathbb{R}^n$. Clearly, this is a continuous homotopy and, for any $t \in [0,1]$,

(1) ψ_t is an $O(n)$-homeomorphism,

(2) ψ_t maps any interval to a curve intersecting each ellipse in finitely many points, and

(3) $\psi_t(V)$ is an elliptically convex body (see [8, p. 95]) for any $V \in C(n)$.

It is easy to see from (1) that the continuous map

$$\Psi\colon L(n) \times [0,1] \to C(n), \qquad (V,t) \in L(n) \times [0,1] \mapsto \psi_t(V) \in C(n),$$

preserves the action of the group $O(n)$. The $O(n)$-homotopy $H \rightleftharpoons \mathfrak{R} \circ \Psi\colon L(n) \times [0,1] \to L(n)$, where $\mathfrak{R}$ is the retraction mentioned in Proposition 2.3, satisfies the requirements of Theorem 3.1; this follows from (2)–(3). We leave the details to the reader. □

Proof of Theorem 3.2. First, note that no additional extreme points arise in passing to the convex hull, i.e.,

(4) $\operatorname{Extr}(\operatorname{Conv} A) \subset A$ for any $A \subset \mathbb{R}^n$.

Therefore, if K is finite, then $\operatorname{Extr}\operatorname{Conv} K$ is also finite, and hence $\operatorname{Conv} K$ is not elliptically convex. The main idea of the proof of Theorem 3.2 is to find sufficiently many nonelliptically convex bodies in $C(n)$.

Lemma 3.4. *Suppose that $V \in L_{\mathcal{E}}(n)$ and $H = O(n)_V$ is the stabilizer of V. Then, for any finite set $L \subset \operatorname{Bd} V$ with $0 \in \operatorname{Int}(\operatorname{Conv} L)$, $W \rightleftharpoons \operatorname{Conv}(H \cdot L) \in C(n)$ is not elliptically convex. Moreover, the stabilizer $O(n)_W$ contains H.*

Proof. Since $\{\pm \operatorname{Id}\} \subset H$ and $0 \in \operatorname{Int}(\operatorname{Conv} L) \subset \operatorname{Conv}(H \cdot L)$, we have $\operatorname{Conv}(H \cdot L) \in C(n)$. Let us make the following elementary observation.

(5) If $A, B \in C(n)$ and $\operatorname{Bd} A \subset \operatorname{Bd} B$, then $A = B$ (and, therefore, $\operatorname{Bd} A = \operatorname{Bd} B$).

Suppose that, contrary to the assertion of the lemma, W is elliptically convex, i.e.,

$$\operatorname{Extr}(W) = \operatorname{Bd} W.$$

Then

$$\operatorname{Bd} W = \operatorname{Extr}(\operatorname{Conv}(H \cdot L)) \subset H \cdot L \subset \operatorname{Bd} V.$$

By virtue of (5), we have $V = W$. Since L is finite and $O(n)$ acts orthogonally on $\mathbb{R}^n$, $H \cdot L$ is contained in the disjoint union $\bigsqcup r_i \cdot S^{n-1}$ of finitely many concentric spheres. Since $\operatorname{Bd} W \subset H \cdot L$ and $\operatorname{Bd} W$ is connected, $\operatorname{Bd} W \subset r_{i_0} \cdot S^{n-1}$ for some i_0. It follows from (5) that $V = W = r_{i_0} \cdot B^n$, which contradicts $V \in L_{\mathcal{E}}(n)$.

Finally, $O(n)_W$ contains H because the action of $O(n)$ on $\mathbb{R}^n$ is orthogonal. □

The next step is finding a nonelliptically convex body with additional properties in an arbitrarily small neighborhood $V \in L_{\mathcal{E}}(n)$.

Proposition 3.5. *Suppose that $V \in L_{\mathcal{E}}(n)$ and $H = O(n)_V$ is the stabilizer of V. Then, for any $\varepsilon > 0$, there exists a finite set $L \subset \operatorname{Bd} V$ such that $0 \in \operatorname{Int}(\operatorname{Conv} L)$ and*

(i) *$W = \operatorname{Conv}(H \cdot L)$ is not elliptically convex,*
(ii) *V and W have the same $O(n)$-stabilizers, and*
(iii) *$\rho_H(V, W) < \varepsilon$.*

Proof. Applying Corollary 5.5 [19, Chap. 2] to the $O(n)$-space $C(n)$, we conclude that there exists a $\theta > 0$ such that

(6) if $U \in C(n)$ and $\rho_H(V, U) < \theta$, then the subgroup conjugate to the stabilizer $O(n)_U$ is contained in H, or, equivalently, $O(n)_U \subset H'$, where H' is some subgroup conjugate to H.

Lemma 3.6. *If $U \in C(n)$, $\rho_H(V, U) < \theta$, and $O(n)_U \supseteq H$, then $O(n)_U = H$.*

Proof. Since $H \subset O(n)_W$, (6) implies $H \subset H'$. The Lie groups H and H' are isomorphic; hence $\dim H = \dim H'$ and $\mathcal{C}(H) = \mathcal{C}(H')$. Property ($\beta$) implies $H = H'$. □

Clearly, there exists a finite set $L \subset \operatorname{Bd} V$ such that $0 \in \operatorname{Int}(\operatorname{Conv} L)$ and $\rho_H(V, \operatorname{Conv} L) < \theta$. Let $W \rightleftharpoons \operatorname{Conv}(H \cdot L)$. Then

$$V \supseteq \operatorname{Conv}(H \cdot L) = W \supseteq \operatorname{Conv} L$$

and, therefore, $\rho_H(V, W) < \theta$. Since

$$O(n)_W = O(n)_{\operatorname{Conv}(H \cdot L)} \supset O(n)_{H \cdot L} \supseteq H,$$

Lemma 3.6 implies $O(n)_W = H$. □

Now, we apply Proposition 3.5 to construct an ample family of equivariant retractions to nonelliptically convex orbits.

Lemma 3.7. *There exist an open $O(n)$-cover $\omega = \{\mathcal{U}_\gamma\}$ of the $O(n)$-space $Z \rightleftharpoons L_{\mathcal{E}}(n) \times (0,1]$ and a family $\Omega = \{r_\gamma\colon \mathcal{U}_\gamma \to P_\gamma\}$ of $O(n)$-maps such that*

(e) *for any γ, $P_\gamma = (O(n)\cdot Q_\gamma) \times \{t_\gamma\}$, where $Q_\gamma \in C(n)$ is not elliptically convex and $t_\gamma \in (0,1]$;*
(f) *the cover $\{\mathcal{U}_\gamma\}$ is $O(n)$-adjacent to $A \rightleftharpoons L(n)\times[0,1] \setminus Z = \{B^n\}\times[0,1] \cup L(n)\times\{0\}$*[2]*;*
(g) *for any $O(n)$-orbit $\mathcal{O}(a) \subset A$, where $a \in A$, and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\operatorname{dist}(r_\gamma, \mathrm{Id}_{\mathcal{U}_\gamma}) < \varepsilon$ provided that $\mathcal{U}_\gamma$ is contained in the δ-neighborhood (with respect to the Hausdorff metric) $N(\mathcal{O}(a);\delta)$ of the orbit $\mathcal{O}(a)$ (or, briefly, $\operatorname{dist}(r_{\gamma_i}, \mathrm{Id}) \to 0$ provided that $\mathcal{U}_{\gamma_i} \to \mathcal{O}(a) \subset A$).*

Proof. Suppose that $Q \in L_{\mathcal{E}}(n)$, $t \in (0,1]$, and $R = \{g\cdot Q, t) \mid g \in O(n)\} \subset Z$ is the orbit of Q. By the Palais cut Theorem [20], there exists an $O(n)$-retraction $r'_R\colon \mathcal{V}_R \to R$ ($r'_R \restriction_R = \mathrm{Id}$), where $\mathcal{V}_R \subset Z$ is an invariant neighborhood of R. We can assume that

(7) the cover $\{\mathcal{V}_R\}$ is $O(n)$-adjacent to A and
(8) $\operatorname{dist}(r'_{R_i}, \mathrm{Id}) \to 0$ provided that $\mathcal{V}_{R_i} \to \mathcal{O}(a) \subset A$.

By Proposition 3.5, for any orbit $R = (O(n)\cdot Q)\times\{t\}$, we can find an orbit $R' = (O(n)\cdot Q')\times\{t\}$ such that

(9) the body Q' is not elliptically convex and
(10) there exists an $O(n)$-homeomorphism $s_R\colon R \to R'$ such that $\operatorname{dist}(s_{R_i}, \mathrm{Id}) \to 0$ provided that $\mathcal{V}_{R_i} \to \mathcal{O}(a) \subset A$.

The cover $\{\mathcal{V}_R\}$ and the family of compositions $r_R = s_R \circ r'_R\colon \mathcal{V}_R \to R'$ are the required ω and Ω. $\square$

We proceed to complete the proof of Theorem 3.2. Let $\{\lambda_\gamma\colon Z \to [0,1]\}$ be an equivariant partition of unity subordinate to the cover $\{\mathcal{U}_\gamma\}$, and let $\mathfrak{R}$ be the retraction from Proposition 2.3. We define the required $O(n)$-map F as

$$F(V,t) = \begin{cases} \mathfrak{R}\circ\left(\sum_\gamma \lambda_\gamma(V,t)\cdot r_\gamma(V,t)\right) & \text{if } (V,t)\in Z,\\ F(V,t) = V & \text{if } (V,t) \in A. \end{cases}$$

By Lemma 3.3, the finite sum $\sum_\gamma \lambda_\gamma(V,t)\cdot r_\gamma(V,t)$, where $(V,t)\in Z$, is not an elliptically convex body. Since $\mathfrak{R}(W)$ and W are affinely equivalent (by Proposition 2.3), $F(V,t)$ for $(V,t)\in Z$ is not elliptically convex either. The continuity of F at the points $(V,t)\in A$ follows from (8). $\square$

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[2]We say that a family $\{B_\gamma\}$ of G-subsets of X contained in $X \setminus A$ is *G-adjacent to the G-set A* if, for any $x\in A$ and any neighborhood $\mathcal{O}(x)\subset X$, there exists a neighborhood $\mathcal{O}_1(x)\subset X$ such that $B_\gamma \subset G\cdot\mathcal{O}(x)$ provided that $B_\gamma \cap G\cdot\mathcal{O}_1(x) \neq \varnothing$.

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(S. M. Ageev) A. S. Pushkin Brest State University, Belarus
E-mail: ageev@highmath.brsu.brest.by
(S. A. Bogatyi) M. V. Lomonosov Moscow State University
E-mail: bogatyi@mech.math.msu.su
(D. Repovš) Ljubljana University, Slovenia
E-mail: dusan.repovs@fmf.uni-lj.si