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The complement $Q_E(n)$ of the point Eucl of Euclidean space in the Banach–Mazur compactum $Q(n)$ is a Q-manifold

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The problem of determining the topological type of the Banach–Mazur compactum $Q(n)$ arose in the Polish school of the geometric theory of Banach space and became widely known in topological circles after West's publication [1] (p. 544). Significant progress has been achieved recently in the study of the topology of $Q(n)$. In 1996 Fabel proved [2] that $Q(2)$ is an absolute extensor. It was proved [3], [4] in the same year that all the compacta $Q(n)$ are absolute extensors ($Q(n) \in \mathrm{AE}$). Finally, in [5] (concisely in [6]) a negative solution was obtained to the problem of homeomorphism of $Q(n)$ to the Hilbert cube Q (see also [7]):

Theorem 1. *$Q(2)$ is not homeomorphic to Q.*

The key idea of the proof of Theorem 1 lies in establishing the homotopic non-triviality of $Q(2) \setminus \{\mathrm{Eucl}\}$, where $\{\mathrm{Eucl}\} \in Q(n)$ is the Euclidean point to which corresponds the isometry class of standard n-dimensional Euclidean space. This in turn is a consequence of the non-triviality of the 4-dimensional cohomology group $\mathrm{H}^4\big(Q(2) \setminus \{\mathrm{Eucl}\}, \mathbb{Q}\big)$ with rational coefficients. The connection between Banach–Mazur compacta and Smith's theory of periodic homeomorphisms was first established in [5] (p. 7) and [6]. The study of $Q(2)$ was continued in [8], where it was proved that $Q(2)$ is the one-point compactification of a Q-manifold, which in turn implies its non-homogeneity by Theorem 1. The natural problem of the structure of $Q(n)$, $n > 2$, was reduced to a plausible conjecture in convex geometry. It turns out that the following is true in general:

Theorem 2. *$Q_E(n) = Q(n) \setminus \{\mathrm{Eucl}\}$ is a Q-manifold.*

The Banach–Mazur compactum $Q(n)$ is, by definition, the space of isometry classes of n-dimensional Banach spaces (see [8] and [7] for a detailed acquaintance with the topology of a Banach–Mazur compactum). This compactum admits a representation as a quotient space of the space $C(n)$ of compact convex centrally symmetric (with respect to the origin) bodies in $\mathbb{R}$. The space $C(n)$ with the Hausdorff metric can be endowed with the action of the general linear group $\mathrm{GL}(n) \times C(n) \to C(n)$, $T \cdot V = T(V)$, where $T\colon \mathbb{R}^n \to \mathbb{R}^n \in \mathrm{GL}(n)$ and $V \in C(n)$. The orbit space $C(n)/\mathrm{GL}(n)$ is naturally homeomorphic to the Banach–Mazur compactum.

John's well-known theorem asserts that for any convex body $V \in C(n)$ there exists a unique ellipsoid $E_V \in C(n)$ (called the Löwner ellipsoid) that contains V and has minimal Euclidean volume. The minimality of $\mathrm{vol}\, E_V$ implies the $\mathrm{GL}(n)$-invariance of E_V, that is, $E_{T\cdot V} = T \cdot E_V$ for any $T \in \mathrm{GL}(n)$. The continuous dependence of E_V on V in the Hausdorff metric is proved in [4]. Therefore, the orbit space $Q(n) = C(n)/\mathrm{GL}(n)$ is homeomorphic to $L(n)/O(n)$, where

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$L(n)$ consists of those $V \in C(n)$ whose Löwner ellipsoid coincides with the unit ball B^n [4]. Moreover, $Q_E(n) = L_E(n)/O(n)$, where $L_E(n) = L(n) \setminus \{B^n\}$. Consequently, Theorem 2 is reduced to the following.

Theorem 3. *$L_E(n)/O(n)$ is a Q-manifold.*

Since $Q(n) \in \mathrm{AE}$ by [4], it follows that $L(n)/O(n) \in \mathrm{AE}$, and therefore $L_E(n)/O(n) \in \mathrm{ANE}$. Further, the proof of Theorem 3 (and consequently of Theorem 2) reduces by Toruńczyk's characterizing criterion [10] to the following fact.

Theorem 4. *For any $\delta > 0$ there exist $O(n)$-maps $f_i\colon L_E(n) \to L_E(n)$, $i \in \{1,2\}$, such that* $\operatorname{dist}(f_i, \mathrm{Id}_{L_E(n)}) < \delta$ *and* $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \varnothing$.

The desired maps are in turn obtained from the following two homotopies in which the disjointness of the images is achieved due to the different character of the sets of extremal points. A point a of a convex set $V \subset \mathbb{R}^n$ is said to be *extremal* if $V \setminus \{a\}$ is convex. The set $\operatorname{Extr}(V)$ of extremal points of V lies on the relative boundary $\operatorname{rbd} V$ and V coincides with the convex hull $\operatorname{Conv}(\operatorname{Extr}(V))$ of its set of extremal points. If $\operatorname{Extr}(V) = \operatorname{rbd}(V)$, then V is said to be *elliptically convex*, otherwise V is *not elliptically convex*.

Theorem 5. *There exists an $O(n)$-homotopy $H\colon L(n) \times [0,1] \to L(n)$ such that*:

(a) $H_0 = \mathrm{Id}$;
(b) *if $V \in L(n)$ and $t \in [0,1]$, then $H_t(V) = B^n$ if and only if $V = B^n$*;
(c) *$H_t(V)$ is elliptically convex for any $V \in L(n)$ and $t > 0$.*

Theorem 6. *There exists an $O(n)$-homotopy $F\colon L(n) \times [0,1] \to L(n)$ such that*:

(d) $F_0 = \mathrm{Id}$;
(e) *$F_t(V)$ is not elliptically convex for any $V \in L_E(n)$ and $t > 0$.*

The proof of Theorem 5 is obtained constructively. Let $\Psi\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be defined by the formula $\Psi(x,t) = \Psi_t(x) = (1+t)x/(1+t\cdot\|x\|) \in \mathbb{R}^n$. It is clear that for any $t \in (0,1]$ the following hold:

(1) Ψ_t is a continuous $O(n)$-embedding;
(2) $V \subseteq \Psi_t(V) \subseteq B^n$, and therefore $\Psi_t(V) \in L^n$ for every $V \in L(n)$;
(3) if $\Psi_t(V) = B^n$ for $V \in L(n)$, then $V = B^n$;
(4) $\Psi_t(V)$ is an elliptically convex body for every $V \in L(n)$ ([11], p. 95).

The proof of Theorem 6 is obtained with the help of an equivariant partition of unity subordinate to an equivariant covering of the complement Z of the set $A = L(n) \times \{0\} \cup \{B^n\} \times [0,1]$ in the space $X = L(n) \times [0,1]$. Here to an element v of the covering we must assign a set that is not elliptically convex, and then an unambiguously defined continuous map is obtained by Minkowski averaging with respect to the partition of unity and by the retraction of $C(n)$ onto $L(n)$ given by the Löwner ellipsoid [4]. Then the Minkowski sum, and therefore its affine image as well, turn out to be not elliptically convex compact bodies.

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