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Nodal solutions for double phase Kirchhoff problems with vanishing potentials

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Abstract. We consider the following (p,q)-Laplacian Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)\Delta_p u-\left(c+d\int_{\mathbb{R}^3}|\nabla u|^q\,dx\right)\Delta_q u$$
$$+V(x)\left(|u|^{p-2}u+|u|^{q-2}u\right)=K(x)f(u)\quad \text{in } \mathbb{R}^3,$$

where $a,b,c,d>0$ are constants, $\frac{3}{2}<p<q<3$, $V:\mathbb{R}^3\to\mathbb{R}$ and $K:\mathbb{R}^3\to\mathbb{R}$ are positive continuous functions allowed for vanishing behavior at infinity, and f is a continuous function with quasicritical growth. Using a minimization argument and a quantitative deformation lemma we establish the existence of nodal solutions.

Keywords: (p,q)-Kirchhoff, nodal solutions, vanishing potentials, Nehari manifold

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1. Introduction

This paper deals with the existence of least energy nodal solutions for the following class of quasilinear problems

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)\Delta_p u-\left(c+d\int_{\mathbb{R}^3}|\nabla u|^q\,dx\right)\Delta_q u$$
$$+V(x)\left(|u|^{p-2}u+|u|^{q-2}u\right)=K(x)f(u)\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a,b,c,d>0$ are constants, $\frac{3}{2}<p<q<3$, $V:\mathbb{R}^3\to\mathbb{R}$ and $K:\mathbb{R}^3\to\mathbb{R}$ are positive functions, and f is a continuous function with quasicritical growth.

In recent years, a considerable interest has been devoted to the study of this general class of problems due to the fact that they arise in applications in physics and related sciences.

When $a=c=1$ and $b=d=0$, equation (1.1) becomes a (p,q)-Laplacian problem of the type

$$-\Delta_p u-\Delta_q u+V(x)\left(|u|^{p-2}u+|u|^{q-2}u\right)=K(x)f(u)\quad \text{in } \mathbb{R}^3. \tag{1.2}$$

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As underlined in [27], this equation is related to the more general reaction-diffusion system

$$u_t = \operatorname{div}\big(D(u)\nabla u\big) + c(x,u) \quad \text{and} \quad D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

which appears in plasma physics, biophysics and chemical reaction design.

In these applications, u represents a concentration, $\operatorname{div}(D(u)\nabla u)$ is the diffusion with the diffusion coefficient $D(u)$, and the reaction term $c(x,u)$ relates to source and loss processes. Usually, the reaction term $c(x,u)$ is a polynomial of u with variable coefficient (see [27]). This kind of problem has been widely investigated by many authors, see for instance [27,31,32,39,42–44,46] and the references therein. In particular, in [17], using a minimization argument and a quantitative deformation lemma, the authors proved the existence of nodal solutions for the following class of (p,q) problems

$$-\operatorname{div}\big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\big) + V(x)b\big(|u|^p\big)|u|^{p-2}u = K(x)f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 3$, $2 \leqslant p < N$, $a, b, f \in \mathcal{C}^1(\mathbb{R})$, and V, K are continuous and positive functions (see also [16]).

We stress that in the nonlocal framework, only few recent works deal with the fractional (p,q)-Laplacian. In [25] the authors established the existence, nonexistence and multiplicity for a nonlocal (p,q)-subcritical problem. Ambrosio [7] obtained an existence result for a critical fractional (p,q)-problem via mountain pass theorem. In [21] the authors investigated the existence of infinitely many nontrivial solutions for a class of fractional (p,q)-equations involving concave-critical nonlinearities in bounded domains. Hölder regularity result for nonlocal double phase equations has been established in [29]. Applying suitable variational and topological arguments, in [12] the authors obtained a multiplicity and concentration result for a class of fractional problems with unbalanced growth. We also mention [1,11,36] for other interesting results.

We underline that there is a huge bibliography concerning the nonlinear Schrödinger equation (that is when $p = q = 2$ in (1.2))

$$-\Delta u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

and we would like to point out that an important class of problems associated with (1.3) is the so called *zero mass* case, which occurs when the potential V vanishes at infinity. Using several variational methods, many authors attacked this equation; see for instance [2,3,5,6,18,19,22].

When $a = c$, $b = d (\neq 0)$ and $p = q = 2$, problem (1.1) becomes the following Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

This problem is related to the stationary analogue of the Kirchhoff equation [41]

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

for all $x \in (0,L)$ and $t \geqslant 0$. This equation is an extension of the classical D'Alembert wave equation taking into account the changes in the length of the strings produced by transverse vibrations. In (1.4),

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$u(x,t)$ is the lateral displacement of the vibrating string at the coordinate x and the time t, L is the length of the string, h is the cross-section area, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial axial tension.

The early studies dedicated to the Kirchhoff equation (1.4) were done by Bernstein [20] and Pohozaev [50]. However, the Kirchhoff equation (1.4) began to attract the attention of more researchers only after the work by Lions [45], in which the author introduced a functional analysis approach to study a general Kirchhoff equation in arbitrary dimension with external force term. For more details on classical Kirchhoff problems we refer to [13,15,48,49]. In [34] the authors established the existence of a least energy nodal solution to the following class of nonlocal Schrödinger–Kirchhoff problems

$$M\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} V(x)u^2\,dx\right)\left(-\Delta u + V(x)u\right) = K(x)f(u) \quad \text{in } \mathbb{R}^3.$$

Moreover, when the problem presents symmetry, they proved the existence of infinitely many nontrivial solutions. We also mention [30,33] where the existence of nodal solutions for problems like (1.4) has been obtained.

In the nonlocal framework, Fiscella and Valdinoci [35] proposed the following stationary Kirchhoff model driven by the fractional Laplacian

$$\begin{cases} -M(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2_s^* = \frac{2N}{N-2s}$, $N > 2s$, $s \in (0,1)$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function which behaves like $M(t) = a + bt$, with $b \geqslant 0$, and f is a continuous function. Based on a truncation argument and the mountain pass theorem, the authors established the existence of a non-negative solution to (1.5) for any $\lambda > \lambda^* > 0$, where λ^* is an appropriate threshold. We also mention [8–10,14,47,51] in which the authors dealt with existence and multiplicity of solutions for (1.5), while concerning the existence and multiplicity of sign-changing solutions for fractional Kirchhoff problems only few results appear in the literature [24,26,40].

Finally, if $a = c$, $b = d(\neq 0)$ and $p = q \neq 2$, we have the following p-Laplacian Kirchhoff-type equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^p\,dx\right)\Delta_p u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Very recently, in [38] using a minimization argument and the Nehari manifold method, the authors investigated the existence of least energy nodal (or sign-changing) solutions to (1.6). We also mention [23,28,37,55] for results regarding Schrö dinger–Kirchhoff equations involving the p-Laplacian.

Motivated by the interest shared by the mathematical community toward (p,q)-Laplacian problems, the goal of the present paper is to study the existence of nodal solutions to (1.1). In order to state precisely our main result, we first introduce the main assumptions on the potentials V and K and on the nonlinearity f.

We assume that $V, K : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions and we say that $(V,K) \in \mathcal{K}$ if the following conditions are satisfied (see [3]):

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(VK_1) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$;

(VK_2) If $(\mathcal{A}_n) \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure $|\mathcal{A}_n| \leqslant R$, for all $n \in \mathbb{N}$ and for some $R > 0$, then

$$\lim_{r\to\infty} \int_{\mathcal{A}_n \cap \mathcal{B}_\varrho^c(0)} K(x)\, dx = 0,$$

uniformly in $n \in \mathbb{N}$, where $\mathcal{B}_\varrho^c(0) := \mathbb{R}^3 \setminus \mathcal{B}_\varrho(0)$.

Furthermore, one of the following conditions is satisfied:

(VK_3) $\frac{K}{V} \in L^\infty(\mathbb{R}^3)$;

or

(VK_4) There exists $m \in (q, q^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{q^*-m}{q^*-p}}} \to 0 \quad \text{as } |x| \to \infty.$$

Let us point out that the hypotheses on the functions V and K characterize problem (1.1) as a *zero mass* problem.

Regarding the nonlinearity f, we assume that $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and f fulfills the following conditions:

(f_1) $\lim_{|t|\to 0} \frac{f(t)}{|t|^{2p-1}} = 0$ if (VK_3) holds,

$(\tilde{f}_1)$ $\lim_{|t|\to 0} \frac{f(t)}{|t|^{m-1}} = 0$ if (VK_4) holds, with $m \in (q, q^*)$ defined in (VK_4),

(f_2) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^{q^*-1}} = 0$,

(f_3) $\lim_{t\to\infty} \frac{F(t)}{t^{2q}} = \infty$, where $F(t) := \int_0^t f(\tau)\, d\tau$,

(f_4) The map $t \mapsto \frac{f(t)}{|t|^{2q-1}}$ is strictly increasing for all $|t| > 0$.

We note that from assumption (f_4) it follows that $t \mapsto \frac{1}{2q} f(t)t - F(t)$ is increasing for $t \geqslant 0$ and also that $t \mapsto \frac{1}{2q} f(t)t - F(t)$ is decreasing for $t \leqslant 0$ (see Remark 2.1 below).

Our main result can be stated as follows:

Theorem 1.1. *Assume that $(V, K) \in \mathcal{K}$ and f satisfies conditions (f_1) (or $(\tilde{f}_1)$) and (f_2)–(f_4). Then problem (1.1) admits a least energy sign-changing weak solution. If in addition, f is an odd function, then (1.1) has infinitely many nontrivial solutions.*

A *weak solution* of problem (1.1) is a function $u \in \mathbb{E}$ such that

$$\begin{aligned} &a \int_{\mathbb{R}^3} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, dx + b\left(\int_{\mathbb{R}^3} |\nabla u|^p\, dx\right) \int_{\mathbb{R}^3} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, dx \\ &\quad + c \int_{\mathbb{R}^3} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi\, dx + d\left(\int_{\mathbb{R}^3} |\nabla u|^q\, dx\right) \int_{\mathbb{R}^3} |\nabla u|^{q-2} \nabla u \cdot \nabla \varphi\, dx \\ &\quad + \int_{\mathbb{R}^3} V(x)\big(|u|^{p-2} u\varphi + |u|^{q-2} u\varphi\big)\, dx - \int_{\mathbb{R}^3} K(x) f(u)\varphi\, dx = 0 \end{aligned} \tag{1.7}$$

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for all $\varphi \in \mathbb{E}$, where

$$\mathbb{E} = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^3) \cap \mathcal{D}^{1,q}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)\big(|u|^p + |u|^q\big)\, dx < \infty \right\}.$$

By a sign-changing weak solution to problem (1.1) we mean a function $u \in \mathbb{E}$ that satisfies (1.7) with $u^+ = \max\{u, 0\} \neq 0$ and $u^- = \min\{u, 0\} \neq 0$.

The proof of Theorem 1.1 is achieved by using suitable variational techniques inspired by [4,16,17,34]. In order to study (1.1) we consider the following functional $\mathcal{I} : \mathbb{E} \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{I}(u) = {} & \frac{a}{p}\int_{\mathbb{R}^3} |\nabla u|^p\, dx + \frac{b}{2p}\left(\int_{\mathbb{R}^3} |\nabla u|^p\, dx\right)^2 + \frac{c}{q}\int_{\mathbb{R}^3} |\nabla u|^q\, dx + \frac{d}{2q}\left(\int_{\mathbb{R}^3} |\nabla u|^q\, dx\right)^2 \\ & + \int_{\mathbb{R}^3} V(x)\left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right) dx - \int_{\mathbb{R}^3} K(x)F(u)\, dx.\end{aligned}$$

It is easy to check that $\mathcal{I} \in \mathcal{C}^1(\mathbb{E}, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathcal{I}'(u), \varphi\rangle = {} & a\int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx + b\left(\int_{\mathbb{R}^3} |\nabla u|^p\, dx\right)\int_{\mathbb{R}^3} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx \\ & + c\int_{\mathbb{R}^3} |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi\, dx + d\left(\int_{\mathbb{R}^3} |\nabla u|^q\, dx\right)\int_{\mathbb{R}^3} |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi\, dx \\ & + \int_{\mathbb{R}^3} V(x)\big(|u|^{p-2}u\varphi + |u|^{q-2}u\varphi\big)\, dx - \int_{\mathbb{R}^3} K(x)f(u)\varphi\, dx.\end{aligned}$$

Then, we define the nodal set

$$\mathcal{M} = \big\{ w \in \mathcal{N} : w^{\pm} \neq 0, \langle \mathcal{I}'(w), w^+\rangle = \langle \mathcal{I}'(w), w^-\rangle = 0 \big\},$$

where

$$\mathcal{N} = \big\{ u \in \mathbb{E} \setminus \{0\} : \langle \mathcal{I}'(u), u\rangle = 0 \big\}.$$

In order to get least energy nodal (or sign-changing) solutions to (1.1), we minimize the functional $\mathcal{I}$ on the nodal set $\mathcal{M}$. Then we prove that the minimum is achieved and, by using a variant of the quantitative deformation lemma, we show that it is a critical point of $\mathcal{I}$. Finally, when the nonlinearity f is odd, we obtain the existence of infinitely many nontrivial weak solutions not necessarily nodals. We point out that our paper extends the results obtained in [34,38].

Problem (1.1) is called nonlocal due to the presence of the Kirchhoff term $(\int_{\mathbb{R}^3} |\nabla u|^p\, dx)\Delta_p u$, this causes some mathematical difficulties which makes the study of such a class of problems particularly interesting. We underline that here we are considering the sum of two Kirchhoff terms: $(\int_{\mathbb{R}^3} |\nabla u|^p\, dx)\Delta_p u$ and $(\int_{\mathbb{R}^3} |\nabla u|^q\, dx)\Delta_q u$, with $p < q$. Moreover, due to the fact that the nonlinearity f is only continuous, one cannot apply standard $\mathcal{C}^1$-Nehari manifold arguments due to the lack of differentiability of the associated Nehari manifold $\mathcal{N}$. We were able to overcome this difficulty by borrowing some abstract

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critical point results obtained in [53]. Furthermore, to produce nodal solutions, instead of using the Miranda Theorem to get critical points of $g_u(\xi,\lambda) = \mathcal{I}(\xi u^+ + \lambda u^-)$ we use an iterative process to build a sequence which converges to a critical point of $g_u(\xi,\lambda)$.

The paper is organized as follows. In Section 2 we introduce the variational structure. In Section 3 we give some preliminary results which overcome the lack of differentiability of the Nehari manifold. Section 4 is devoted to some technical lemmas used in the proof of the main result. In Section 5 we prove Theorem 1.1.

1.1. Notations

We denote by $\mathcal{B}_R(x)$ the ball of radius R with center x and we set $\mathcal{B}_R^c(x) = \mathbb{R}^3 \setminus \mathcal{B}_R(x)$. Let $1 \leqslant r \leqslant \infty$ and $A \subset \mathbb{R}^3$. We denote by $|u|_{L^r(A)}$ the $L^r(A)$-norm of the function $u : \mathbb{R}^3 \to \mathbb{R}$ belonging to $L^r(A)$. When $A = \mathbb{R}^3$, we shall simply write $|u|_r$.

2. Variational framework

Let us introduce the space

$$\mathbb{E} = \left\{ u \in \mathcal{D}^{1,p}(\mathbb{R}^3) \cap \mathcal{D}^{1,q}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)\big(|u|^p + |u|^q\big)\,dx < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \big(a|\nabla u|^p + V(x)|u|^p\big)\,dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^3} \big(c|\nabla u|^q + V(x)|u|^q\big)\,dx \right)^{\frac{1}{q}}.$$

Let us define the Lebesgue space

$$L_K^r(\mathbb{R}^3) = \left\{ u : \mathbb{R}^3 \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^3} K(x)|u|^r\,dx < \infty \right\}$$

equipped with the norm

$$\|u\|_{L_K^r(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} K(x)|u|^r\,dx \right)^{\frac{1}{r}}.$$

We recall the following continuous and compactness results whose proofs can be found in [17]:

Lemma 2.1. *Assume that* $(V, K) \in \mathcal{K}$.

(i) *If* (VK_3) *holds, then* $\mathbb{E}$ *is continuously embedded in* $L_K^r(\mathbb{R}^3)$ *for every* $r \in [q, q^*]$.
(ii) *If* (VK_4) *holds, then* $\mathbb{E}$ *is continuously embedded in* $L_K^m(\mathbb{R}^3)$.

Lemma 2.2. *Assume that* $(V, K) \in \mathcal{K}$.

(i) *If* (VK_3) *holds, then* $\mathbb{E}$ *is compactly embedded in* $L_K^r(\mathbb{R}^3)$ *for every* $r \in (q, q^*)$.

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(ii) *If* (VK_4) *holds, then* $\mathbb{E}$ *is compactly embedded in* $L^m_K(\mathbb{R}^3)$.

The last lemma of this section is a compactness result related to the nonlinearity (see [17]).

Lemma 2.3. *Assume that* $(V, K) \in \mathcal{K}$ *and* f *satisfies* (f_1)–(f_2) *or* $(\tilde{f}_1)$–(f_2). *If* (u_n) *is a sequence such that* $u_n \rightharpoonup u$ *in* $\mathbb{E}$, *then*

$$\int_{\mathbb{R}^3} K(x)F(u_n)\,dx \to \int_{\mathbb{R}^3} K(x)F(u)\,dx$$

and

$$\int_{\mathbb{R}^3} K(x)f(u_n)u_n\,dx \to \int_{\mathbb{R}^3} K(x)f(u)u\,dx.$$

We conclude this section by giving the following useful remarks.

Remark 2.1. Let us point out that from assumption (f_4) it follows that $t \mapsto \frac{1}{2q}f(t)t - F(t)$ is increasing for $t \geqslant 0$. Indeed, let $0 < t_2 < t_1$, then using (f_4) twice we get

$$\begin{aligned}
\frac{1}{2q}f(t_1)t_1 - F(t_1) &= \frac{1}{2q}f(t_1)t_1 - F(t_2) - \int_{t_2}^{t_1} f(\tau)\,d\tau \\
&= \frac{1}{2q}f(t_1)t_1 - F(t_2) - \int_{t_2}^{t_1} \frac{f(\tau)}{\tau^{2q-1}}\tau^{2q-1}\,d\tau \\
&> \frac{1}{2q}f(t_1)t_1 - F(t_2) - \frac{f(t_1)}{t_1^{2q-1}}\int_{t_2}^{t_1} \tau^{2q-1}\,d\tau \\
&= \frac{1}{2q}f(t_1)t_1 - F(t_2) - \frac{f(t_1)}{t_1^{2q-1}}\frac{t_1^{2q}-t_2^{2q}}{2q} \\
&= \frac{1}{2q}\frac{f(t_1)}{t_1^{2q-1}}t_2^{2q} - F(t_2) \\
&> \frac{1}{2q}f(t_2)t_2 - F(t_2).
\end{aligned}$$

Similarly, it is possible to prove that $t \mapsto \frac{1}{2q}f(t)t - F(t)$ is decreasing for $t \leqslant 0$.

Remark 2.2. Take $u \in \mathbb{E}$ with $u^{\pm} \neq 0$ and $\xi, \lambda \geqslant 0$, then

$$\left|\nabla\left(\xi u^+ + \lambda u^-\right)\right|^p = \left|\nabla\left(\xi u^+\right)\right|^p + \left|\nabla\left(\lambda u^-\right)\right|^p$$

and using the linearity of F and the positivity of K we also have

$$\int_{\mathbb{R}^3} K(x)F\left(\xi u^+ + \lambda u^-\right)dx = \int_{\mathbb{R}^3} K(x)\left(F\left(\xi u^+\right) + F\left(\lambda u^-\right)\right)dx.$$

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Hence, for any $u \in \mathbb{E}$ with $u^{\pm} \neq 0$ and $\xi, \lambda \geqslant 0$ we have

$$\mathcal{I}(\xi u^+ + \lambda u^-) = \mathcal{I}(\xi u^+) + \mathcal{I}(\lambda u^-). \tag{2.1}$$

Moreover,

$$\begin{aligned}\langle \mathcal{I}'(\xi u^+ + \lambda u^-), \xi u^+ \rangle = {} & \xi^p \int_{\mathbb{R}^3} (a|\nabla u^+|^p + V(x)|u|^p)\, dx + b\xi^{2p}\left(\int_{\mathbb{R}^3} |\nabla u^+|^p\, dx\right)^2 \\ & + \xi^q \int_{\mathbb{R}^3} (c|\nabla u^+|^q + V(x)|u|^q)\, dx + d\xi^{2q}\left(\int_{\mathbb{R}^3} |\nabla u^+|^q\, dx\right)^2 \\ & - \int_{\mathbb{R}^3} K(x) f(\xi u^+)\xi u^+\, dx\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{I}'(\xi u^+ + \lambda u^-), \xi u^+ \rangle = {} & \lambda^p \int_{\mathbb{R}^3} (a|\nabla u^-|^p + V(x)|u|^p)\, dx + b\lambda^{2p}\left(\int_{\mathbb{R}^3} |\nabla u^-|^p\, dx\right)^2 \\ & + \lambda^q \int_{\mathbb{R}^3} (c|\nabla u^-|^q + V(x)|u|^q)\, dx + d\lambda^{2q}\left(\int_{\mathbb{R}^3} |\nabla u^-|^q\, dx\right)^2 \\ & - \int_{\mathbb{R}^3} K(x) f(\lambda u^-)\lambda u^-\, dx.\end{aligned}$$

3. Preliminaries

The Nehari manifold associated with $\mathcal{I}$ is given by

$$\mathcal{N} = \{u \in \mathbb{E} \setminus \{0\} : \langle \mathcal{I}'(u), u \rangle = 0\}.$$

We denote by

$$\mathcal{M} = \{w \in \mathcal{N} : w^{\pm} \neq 0, \langle \mathcal{I}'(w), w^+ \rangle = \langle \mathcal{I}'(w), w^- \rangle = 0\},$$

and by $\mathbb{S}$ the unit sphere on $\mathbb{E}$. We note that $\mathcal{M} \subset \mathcal{N}$.

Once f is only continuous, the following results are crucial, since they allow us to overcome the non-differentiability of $\mathcal{N}$.

Lemma 3.1. *Suppose that $(V, K) \in \mathcal{K}$ and f satisfies conditions $(f_1) - (f_4)$. Then the following properties hold:*

(a) *For each $u \in \mathbb{E} \setminus \{0\}$, let $\varphi_u : \mathbb{R}_+ \to \mathbb{R}$ be defined by*

$$\varphi_u(t) = \mathcal{I}(tu).$$

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Then there is a unique $t_u > 0$ such that

$$\varphi_u'(t) > 0 \quad \text{for } t \in (0, t_u) \quad \text{and} \quad \varphi_u'(t) < 0 \quad \text{for } t \in (t_u, \infty);$$

(b) *There is $\tau > 0$, independent of u, such that $t_u \geqslant \tau$ for every $u \in \mathbb{S}$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}$, there is $C_{\mathbb{K}} > 0$ such that $t_u \leqslant C_{\mathbb{K}}$ for every $u \in \mathbb{K}$;*

(c) *The map $\hat{m} : \mathbb{E}\backslash\{0\} \to \mathcal{N}$ given by $\hat{m}(u) := t_u u$ is continuous and $m := \hat{m}|_{\mathbb{S}}$ is a homeomorphism between $\mathbb{S}$ and $\mathcal{N}$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|}$.*

Proof. (a) Let us assume that (VK_3) holds. Then, using assumptions (f_1)–(f_2) given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|f(t)\right| \leqslant \varepsilon|t|^{p-1} + C_\varepsilon|t|^{q^*-1}.$$

Therefore

$$\begin{aligned}
\mathcal{I}(tu) \geqslant{}& \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{c}{q}t^q\int_{\mathbb{R}^3}|\nabla u|^q\,dx + \int_{\mathbb{R}^3}V(x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)dx \\
&- \varepsilon t^p\int_{\mathbb{R}^3}K(x)|u|^p\,dx - C_\varepsilon t^{q^*}\int_{\mathbb{R}^3}K(x)|u|^{q^*}\,dx \\
\geqslant{}& \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{c}{q}t^q\int_{\mathbb{R}^3}|\nabla u|^q\,dx + \int_{\mathbb{R}^3}V(x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)dx \\
&- \varepsilon\left|\frac{K}{V}\right|_\infty t^p\int_{\mathbb{R}^3}V(x)|u|^p\,dx - C_\varepsilon'|K|_\infty t^{q^*}\|u\|^{q^*} \\
\geqslant{}& \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \left(\frac{1}{p} - \varepsilon\left|\frac{K}{V}\right|_\infty\right)t^p\int_{\mathbb{R}^3}V(x)|u|^p\,dx \\
&+ \frac{t^q}{q}\int_{\mathbb{R}^3}\left(c|\nabla u|^q + V(x)|u|^q\right)dx - C_\varepsilon'|K|_\infty t^{q^*}\|u\|^{q^*}.
\end{aligned}$$

Choosing $\varepsilon \in (0, (2p|\frac{K}{V}|_\infty)^{-1})$, we get $t_0 > 0$ sufficiently small such that

$$0 < \varphi_u(t) = \mathcal{I}(tu) \quad \text{for all } t \in (0, t_0).$$

Now, we assume that (VK_4) is true. Then, there exists a positive constant C_m such that, for each $\varepsilon \in (0, C_m)$ we get $R > 0$ such that for any $u \in \mathbb{E}$

$$\int_{\mathcal{B}_R^c(0)}K(x)|u|^m\,dx \leqslant \varepsilon\int_{\mathcal{B}_R^c(0)}\left(V(x)|u|^p + |u|^{q^*}\right)dx.$$

Using assumptions $(\tilde{f}_1)$ and (f_2), the Hölder and Sobolev inequality we get

$$\mathcal{I}(tu) \geqslant \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{c}{q}t^q\int_{\mathbb{R}^3}|\nabla u|^q\,dx + \int_{\mathbb{R}^3}V(x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)dx$$

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$$\begin{aligned}
&- C_1 t^m\left(\int_{\mathcal{B}_R(0)} K(x)|u|^m\,dx + \int_{\mathcal{B}^c_R(0)} K(x)|u|^m\,dx\right) - C_2 t^{q^*}\int_{\mathbb{R}^3} K(x)|u|^{q^*}\,dx\\
&\geqslant \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{c}{q}t^q\int_{\mathbb{R}^3}|\nabla u|^q\,dx + \int_{\mathbb{R}^3}V(x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)dx\\
&\quad - C_1 t^m|K|_{\frac{q^*}{q^*-m}}|u|^m_{q^*} - C_1 t^m\varepsilon\int_{\mathcal{B}^c_R(0)}\left(V(x)|u|^p + |u|^{q^*}\right)dx - C_2' t^{q^*}|K|_\infty\|u\|^{q^*}\\
&\geqslant \frac{a}{p}t^p\int_{\mathbb{R}^3}|\nabla u|^p\,dx + \frac{c}{q}t^q\int_{\mathbb{R}^3}|\nabla u|^q\,dx + \int_{\mathbb{R}^3}V(x)\left(\frac{t^p}{p}|u|^p + \frac{t^q}{q}|u|^q\right)dx\\
&\quad - C_2' t^{q^*}|K|_\infty\|u\|^{q^*} - \tilde{C}t^m\left[|K|_{\frac{q^*}{q^*-m}}|u|^m_{q^*} + \varepsilon\|u\|^p + \varepsilon\|u\|^{q^*}\right].
\end{aligned}$$

Therefore there exists $t_0 > 0$ sufficiently small such that

$$0 < \varphi_u(t) = \mathcal{I}(tu) \quad \text{for all } t \in (0, t_0).$$

Let $A \subset \operatorname{supp} u$ be a measurable set with finite and positive measure. From $F(t) \geqslant 0$ for any $t \in \mathbb{R}$, $1 < p < q$, and combining assumptions (f_3) together with Fatou's lemma, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\mathcal{I}(tu)}{\|tu\|^{2q}}\\
&\quad\leqslant \limsup_{t\to\infty}\left\{\frac{1}{p}\frac{1}{t^{2q-p}\|u\|^{2q-p}} + \frac{b}{2p}\frac{1}{t^{2(q-p)}\|u\|^{2(q-p)}} + \frac{1}{q}\frac{1}{t^q\|u\|^q} + \frac{d}{2q}\right.\\
&\qquad\left. - \int_A K(x)\frac{F(tu)}{(tu)^{2q}}\left(\frac{u}{\|u\|}\right)^{2q}dx\right\}\\
&\quad\leqslant \frac{d}{2q} - \liminf_{t\to\infty}\int_A K(x)\frac{F(tu)}{(tu)^{2q}}\left(\frac{u}{\|u\|}\right)^{2q}dx \leqslant -\infty.
\end{aligned}$$

Hence there exists $\bar{t} > 0$ large enough for which $\varphi_u(\bar{t}) < 0$. By virtue of the continuity of φ_u and using (f_4), there exists $t_u > 0$ which is a global maximum of φ_u with $t_u u \in \mathcal{N}$.

Next, we aim to prove that such t_u is the unique critical point of φ_u. Assume by contradiction that there exist $0 < t_1 < t_2$ which are critical points of φ_u. Then, from the definition of φ_u we get

$$\begin{aligned}
&\frac{1}{t_1^{2q-p}}\int_{\mathbb{R}^3}\left(a|\nabla u|^p + V(x)|u|^p\right)dx + \frac{b}{t_1^{2(q-p)}}\left(\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)^2 + \frac{1}{t_1^q}\int_{\mathbb{R}^3}\left(c|\nabla u|^q + V(x)|u|^q\right)dx\\
&\quad + d\left(\int_{\mathbb{R}^3}|\nabla u|^q\,dx\right)^2 - \int_{\mathbb{R}^3}K(x)\frac{f(t_1 u)}{(t_1 u)^{2q-1}}u^{2q}\,dx = 0
\end{aligned}$$

and

$$\frac{1}{t_2^{2q-p}}\int_{\mathbb{R}^3}\left(a|\nabla u|^p + V(x)|u|^p\right)dx + \frac{b}{t_2^{2(q-p)}}\left(\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)^2 + \frac{1}{t_2^q}\int_{\mathbb{R}^3}\left(c|\nabla u|^q + V(x)|u|^q\right)dx$$

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$$+ d\left(\int_{\mathbb{R}^3} |\nabla u|^q \, dx\right)^2 - \int_{\mathbb{R}^3} K(x) \frac{f(t_2 u)}{(t_2 u)^{2q-1}} u^{2q} \, dx = 0.$$

These equalities together with assumption (f_4) imply that

$$\begin{aligned} 0 &> \left(\frac{1}{t_2^{2q-p}} - \frac{1}{t_1^{2q-p}}\right) \int_{\mathbb{R}^3} \left(a|\nabla u|^p + V(x)|u|^p\right) dx + b\left(\frac{1}{t_2^{2(q-p)}} - \frac{1}{t_1^{2(q-p)}}\right)\left(\int_{\mathbb{R}^3} |\nabla u|^p \, dx\right)^2 \\ &\quad + \left(\frac{1}{t_2^q} - \frac{1}{t_1^q}\right) \int_{\mathbb{R}^3} \left(c|\nabla u|^q + V(x)|u|^q\right) dx \\ &= \int_{\mathbb{R}^3} K(x)\left(\frac{f(t_2 u)}{(t_2 u)^{2q-1}} - \frac{f(t_1 u)}{(t_1 u)^{2q-1}}\right) u^{2q} \, dx \geqslant 0 \end{aligned}$$

which leads to a contradiction.

(b) By (a) there exists $t_u > 0$ such that $\varphi_u'(t_u) = 0$, or equivalently $\langle \mathcal{I}'(t_u u), t_u u\rangle = 0$, and arguing as before, we find a positive τ independent of u such that $t_u \geqslant \tau$.

Now, let $\mathbb{K} \subset \mathbb{S}$ be a compact set and assume by contradiction that there exists $(u_n) \subset \mathbb{K}$ such that $t_{u_n} \to \infty$. Hence, there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathbb{E}$. Proceeding as in (a) we can prove that $\mathcal{I}(t_{u_n} u_n) \to -\infty$ in $\mathbb{R}$. Since $t_{u_n} u_n \in \mathcal{N}$, from Remark 2.1 and recalling that $1 < p < q$ we get

$$\begin{aligned} \mathcal{I}(t_{u_n} u_n) &= \mathcal{I}(t_{u_n} u_n) - \frac{1}{2q}\langle \mathcal{I}'(t_{u_n} u_n), t_{u_n} u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{2q}\right) t_{u_n}^p \int_{\mathbb{R}^3} \left(a|\nabla u_n|^p + V(x)|u_n|^p\right) dx + \frac{b}{2}\left(\frac{1}{p} - \frac{1}{q}\right) t_{u_n}^{2p} \left(\int_{\mathbb{R}^3} \left|\nabla(u_n)\right|^p dx\right)^2 \\ &\quad + \frac{1}{2q} t_{u_n}^q \int_{\mathbb{R}^3} \left(c|\nabla u_n|^q + V(x)|u_n|^q\right) dx \\ &\quad + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{2q} f(t_{u_n} u_n) t_{u_n} u_n - F(t_{u_n} u_n)\right] dx \geqslant 0, \end{aligned}$$

which leads to a contradiction.

(c) Note that $\hat{m}$, m and m^{-1} are well defined. In fact, from (a) we deduce that for each $u \in \mathbb{E} \setminus \{0\}$ there exists a unique $\hat{m}(u) \in \mathcal{N}$.

On the other hand, if $u \in \mathcal{N}$ then $u \neq 0$, and we deduce that $m^{-1}(u) = \frac{u}{\|u\|} \in \mathbb{S}$ and m^{-1} is well defined. We point out that

$$\begin{aligned} m^{-1}\big(m(u)\big) &= m^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|} = u \quad \text{for any } u \in \mathbb{S}, \\ m\big(m^{-1}(u)\big) &= m\left(\frac{u}{\|u\|}\right) = t_{\frac{u}{\|u\|}} \frac{u}{\|u\|} = u \quad \text{for any } u \in \mathcal{N}, \end{aligned}$$

so m is bijective with its inverse m^{-1} continuous.

Now, let $(u_n) \subset \mathbb{E}$ and $u \in \mathbb{E} \setminus \{0\}$ such that $u_n \to u$ in $\mathbb{E}$. Using (b) we can find $t_0 > 0$ such that $t_{u_n}\|u_n\| = t_{\frac{u}{\|u\|}} \to t_0$. Therefore $t_{u_n} \to \frac{t_0}{\|u\|}$. Using the fact that $t_{u_n} u_n \in \mathcal{N}$ and taking the limit as $n \to \infty$

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we deduce that $\frac{t_0}{\|u\|}u \in \mathcal{N}$ and $t_u = \frac{t_0}{\|u\|}$. This implies that $\hat{m}(u_n) \to \hat{m}(u)$, hence $\hat{m}$ and m are continuous functions. $\square$

Let us define the maps

$$\hat{\psi} : \mathbb{E} \to \mathbb{R} \quad \text{and} \quad \psi : \mathbb{S} \to \mathbb{R},$$

by $\hat{\psi}(u) := \mathcal{I}(\hat{m}(u))$ and $\psi := \hat{\psi}|_{\mathbb{S}}$.

The next result is a consequence of Lemma 2.1 (see [53]).

Proposition 3.1. *Suppose that $(V, K) \in \mathcal{K}$ and f fulfills $(f_1) - (f_4)$. Then the following properties hold:*

(a) $\hat{\psi} \in \mathcal{C}^1(\mathbb{E} \setminus \{0\}, \mathbb{R})$ *and*

$$\langle \hat{\psi}'(u), v \rangle = \frac{\|\hat{m}(u)\|}{\|u\|} \langle \mathcal{I}'(\hat{m}(u)), v \rangle \quad \textit{for all } u \in \mathbb{E} \setminus \{0\} \textit{ and } v \in \mathbb{E};$$

(b) $\psi \in \mathcal{C}^1(\mathbb{S}, \mathbb{R})$ *and* $\langle \psi'(u), v \rangle = \|m(u)\| \langle \mathcal{I}'(m(u)), v \rangle$, *for every* $v \in T_u\mathbb{S}$;

(c) *If (u_n) is a $(\mathrm{PS})_d$ sequence for ψ, then $\{m(u_n)\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_d$ sequence for $\mathcal{I}$. Moreover, if $(u_n) \subset \mathcal{N}$ is a bounded $(\mathrm{PS})_d$ sequence for $\mathcal{I}$, then $\{m^{-1}(u_n)\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_d$ sequence for the functional ψ;*

(d) *u is a critical point of ψ if and only if $m(u)$ is a nontrivial critical point for $\mathcal{I}$. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}} \psi(u) = \inf_{u\in\mathcal{N}} \mathcal{I}(u).$$

We notice that the following equalities hold:

$$d_\infty := \inf_{u\in\mathcal{N}} \mathcal{I}(u) = \inf_{u\in\mathbb{E}\setminus\{0\}} \max_{t>0} \mathcal{I}(tu) = \inf_{u\in\mathbb{S}} \max_{t>0} \mathcal{I}(tu). \tag{3.1}$$

In particular, from (a) of Lemma 2.1 and (3.1) it follows that

$$d_\infty > 0. \tag{3.2}$$

4. Technical lemmas

For each $u \in \mathbb{E}$ with $u^\pm \neq 0$, let us introduce the map $g_u : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$g_u(\xi, \lambda) = \mathcal{I}(\xi u^+ + \lambda u^-).$$

Lemma 4.1. *Suppose that $(V, K) \in \mathcal{K}$ and f fulfills (f_1)–(f_4). Then the following properties hold:*

(i) *The pair (ξ, λ) is a critical point of g_u with $\xi, \lambda > 0$ if and only if $\xi u^+ + \lambda u^- \in \mathcal{M}$.*

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(ii) *The map g_u has a unique critical point (ξ_+, λ_-), with $\xi_+ = \xi_+(u) > 0$ and $\lambda_- = \lambda_-(u) > 0$ which is the unique global maximum point of g_u.*

(iii) *The maps $a_+(r) := \frac{\partial g_u}{\partial \xi}(r, \lambda_-)$ and $a_-(r) := \frac{\partial g_u}{\partial \lambda}(\xi_+, r)r$ are such that $a_+(r) > 0$ if $r \in (0, \xi_+)$, $a_+(r) < 0$ if $r \in (\xi_+, \infty)$, $a_-(r) > 0$ if $r \in (0, \lambda_-)$ and $a_-(r) < 0$ if $r \in (\lambda_-, \infty)$.*

Proof. (i) Let us point out that the gradient of g_u is given by

$$\begin{aligned}\nabla g_u(\xi, \lambda) &= \left(\frac{\partial g_u}{\partial \xi}(\xi, \lambda), \frac{\partial g_u}{\partial \lambda}(\xi, \lambda)\right)\\ &= \left(\langle \mathcal{I}'(\xi u^+ + \lambda u^-), u^+\rangle, \langle \mathcal{I}'(\xi u^+ + \lambda u^-), u^-\rangle\right)\\ &= \left(\frac{1}{\xi}\langle \mathcal{I}'(\xi u^+ + \lambda u^-), \xi u^+\rangle, \frac{1}{\lambda}\langle \mathcal{I}'(\xi u^+ + \lambda u^-), \lambda u^-\rangle\right).\end{aligned}$$

Now, the pair (ξ, λ), with $\xi, \lambda > 0$, is a critical point of g_u if and only if

$$\langle \mathcal{I}'(\xi u^+ + \lambda u^-), \xi u^+\rangle = 0 \quad \text{and} \quad \langle \mathcal{I}'(\xi u^+ + \lambda u^-), \lambda u^-\rangle = 0$$

that is $\xi u^+ + \lambda u^- \in \mathcal{M}$.

(ii) First we prove that $\mathcal{M} \neq \emptyset$. For each $u \in \mathbb{E}$ with $u^\pm \neq 0$ and λ_0 fixed, let us define the function $g_1(\xi) : [0, \infty) \to [0, \infty)$ by $g_1(\xi) = g_u(\xi, \lambda_0)$.

As in Lemma 2.1, the map g_1 has a maximum positive point and furthermore there exists $\xi_0 = \xi_0(u, \lambda_0) > 0$ such that $g_1'(\xi) > 0$ for $\xi \in (0, \xi_0)$, $g_1'(\xi) < 0$ for $\xi \in (\xi_0, \infty)$ and $g_1'(\xi_0) = 0$.

Hence, it is well defined the function $\eta_1 : [0, \infty) \to [0, \infty)$ defined by $\eta_1(\lambda) := \xi(u, \lambda)$, where $\xi(u, \lambda)$ satisfies the properties just mentioned with λ in place of λ_0. Exploiting the definition of g_1, for all $\lambda \geqslant 0$ we get

$$g_1'(\eta_1(\lambda)) = \frac{\partial g_u}{\partial \xi}(\eta_1(\lambda), \lambda) = \langle \mathcal{I}'(\eta_1(\lambda)u^+ + \lambda u^-), \eta_1(\lambda)u^+\rangle = 0. \tag{4.1}$$

Note that, when $u^\pm \neq 0$ and the support of u^+ and u^- are disjoint in $\mathbb{R}^3$, it follows that (4.1) is equivalent to

$$\begin{aligned}&\eta_1(\lambda)^p \int_{\mathbb{R}^3} (a|\nabla u^+|^p + V(x)|u^+|^p)\,dx + b\eta_1(\lambda)^{2p}\left(\int_{\mathbb{R}^3} |\nabla u^+|^p\,dx\right)^2\\ &\quad + \eta_1(\lambda)^q \int_{\mathbb{R}^3} (c|\nabla u^+|^q + V(x)|u^+|^q)\,dx + d\eta_1(\lambda)^{2q}\left(\int_{\mathbb{R}^3} |\nabla u^+|^q\,dx\right)^2\\ &= \int_{\mathbb{R}^3} K(x) f(\eta_1(\lambda)u^+)\eta_1(\lambda)u^+\,dx.\end{aligned} \tag{4.2}$$

First we note that η_1 is a continuous map. Indeed, let (λ_n) be a sequence such that $\lambda_n \to \lambda_0$ as $n \to \infty$ in $\mathbb{R}$, and assume that $\eta_1(\lambda_n) \to \infty$ as $n \to \infty$. We aim to prove that $(\eta_1(\lambda_n))$ is bounded. By contradiction,

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let us suppose that there us a subsequence, still denoted by (λ_n), such that $\eta_1(\lambda_n) \to \infty$ as $n \to \infty$. In particular, for n sufficiently large we have that $\eta_1(\lambda_n) \geqslant \lambda_n$. From (4.2) we get

$$\begin{aligned} &\frac{1}{\eta_1(\lambda_n)^{2q-p}} \int_{\mathbb{R}^3} \left(a|\nabla u^+|^p + V(x)|u^+|^p\right) dx + \frac{b}{\eta_1(\lambda_n)^{2(q-p)}} \left(\int_{\mathbb{R}^3} |\nabla u^+|^p \, dx\right)^2 \\ &\quad + \frac{1}{\eta_1(\lambda_n)^q} \int_{\mathbb{R}^3} \left(c|\nabla u^+|^q + V(x)|u^+|^q\right) dx + d\left(\int_{\mathbb{R}^3} |\nabla u^+|^q \, dx\right)^2 \\ &= \int_{\mathbb{R}^3} K(x) \frac{f(\eta_1(\lambda_n)u^+)}{(\eta_1(\lambda_n)u^+)^{2q-1}} \left(u^+\right)^{2q} dx, \end{aligned}$$

recalling that $\eta_1(\lambda_n) \to \infty$ as $n \to \infty$, $\lambda_n \to \lambda_0$ as $n \to \infty$ and exploiting (f_3), (f_4) and Fatou's lemma, we get a contradiction. This shows that $(\eta_1(\lambda_n))$ is bounded. So there exists $\xi_0 \geqslant 0$ such that $\eta_1(\lambda_n) \to \xi_0$ as $n \to \infty$. Now, using (4.2) with $\lambda = \lambda_n$ and taking $n \to \infty$ we deduce

$$\begin{aligned} &\xi_0^p \int_{\mathbb{R}^3} \left(a|\nabla u^+|^p + V(x)|u^+|^p\right) dx + b\xi_0^{2p} \left(\int_{\mathbb{R}^3} |\nabla u^+|^p \, dx\right)^2 \\ &\quad + \xi_0^q \int_{\mathbb{R}^3} \left(c|\nabla u^+|^q + V(x)|u^+|^q\right) dx + d\xi_0^{2q} \left(\int_{\mathbb{R}^3} |\nabla u^+|^q \, dx\right)^2 \\ &= \int_{\mathbb{R}^3} K(x) f\left(\xi_0 u^+\right) \xi_0 u^+ \, dx \end{aligned}$$

that is

$$g_1'(\xi_0) = \frac{\partial g_u}{\partial \xi}(\xi_0, \lambda_0) = 0.$$

Hence, $\xi_0 = \eta_1(\lambda_0)$ which implies that η_1 is a continuous map.

Moreover, $\eta_1(0) > 0$. Indeed, if we suppose by contradiction that there exists a sequence (λ_n) such that $\eta_1(\lambda_n) \to 0^+$ and $\lambda_n \to 0$ as $n \to \infty$, then gathering (4.2) with (f_1) we get

$$d\left(\int_{\mathbb{R}^3} |\nabla u^+|^q \, dx\right)^2 \leqslant \int_{\mathbb{R}^3} K(x) \frac{f(\eta_1(\lambda_n)u^+)}{(\eta_1(\lambda_n)u^+)^{2q-1}} \left(u^+\right)^{2q} dx \to 0,$$

which gives a contradiction. Finally, we can also see that $\eta_1(\lambda) \leqslant s$ for s sufficiently large.

In a similar fashion, for each $\xi_0 \geqslant 0$ we define $g_2(\lambda) = g_u(\xi_0, \lambda)$, and we can introduce a map η_2 that satisfies the same properties as η_1. In particular, there exists a positive constant A_1 such that for each $\xi, \lambda \geqslant A_1$ it holds that $\eta_1(\lambda) \leqslant \lambda$ and $\eta_2(\xi) \leqslant \xi$.

Let $A_2 := \max\{\max_{\lambda \in [0,A_1]} \eta_1(\lambda), \max_{\xi \in [0,A_1]} \eta_2(\xi)\}$ and set $A := \max\{C_1, C_2\}$.

Next, we introduce the map $\Phi : [0, A] \times [0, A] \to \mathbb{R}^2$ defined by $\Phi(\xi, \lambda) = (\eta_1(\lambda), \eta_2(\xi))$. First we note that Φ is a continuous map due to the continuity of η_1 and η_2, moreover for every $s \in [0, A]$ we can see that

$$\text{if } \lambda \geqslant A_1 \text{ then } \eta_1(\lambda) \leqslant \lambda \leqslant A$$

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$$\text{if } \lambda \leqslant A_1 \text{ then } \eta_1(\lambda) \leqslant \max_{\lambda \in [0, A_1]} \eta_1(\lambda) \leqslant A_2 \leqslant A$$

and similarly

$$\begin{aligned}
&\text{if } \xi \geqslant A_1 \text{ then } \eta_2(\xi) \leqslant t \leqslant A, \\
&\text{if } \xi \leqslant A_1 \text{ then } \eta_2(\xi) \leqslant \max_{\xi \in [0, A_1]} \eta_1(\xi) \leqslant A_2 \leqslant A,
\end{aligned}$$

hence $\Phi([0, A] \times [0, A]) \subset [0, A] \times [0, A]$. Applying Brouwer's fixed point theorem there exists $(\xi_+, \lambda_-) \in [0, A] \times [0, A]$ such that

$$\Phi(\xi_+, \lambda_-) = \big(\eta_1(s_-), \eta_2(t_+)\big) = (\xi_+, \lambda_-).$$

Since η_1, η_2 are positive functions, $\xi_+, \lambda_- > 0$. In addition $\nabla g_u(\xi_+, \lambda_-) = 0$, hence (ξ_+, λ_-) is a critical point of g_u.

Next, we show the uniqueness of (ξ_+, λ_-). First, take $w \in \mathcal{M}$. By $w = w^+ + w^-$ and the definition of g_w it follows that $\nabla g_w(1, 1) = (0, 0)$, hence $(1, 1)$ is a critical point of g_w. Our aim is to show that $(1, 1)$ is the unique critical point of g_w with positive coordinates. With this goal, let (ξ_0, λ_0) be a critical point go g_w with $0 < \xi_0 \leqslant \lambda_0$. Using $\frac{\partial g_w}{\partial \xi}(\xi_0, \lambda_0) = 0$, which is equivalent to $\langle \mathcal{I}'(\xi_0 w^+ + \lambda_0 w^-), \xi_0 w^+ \rangle = 0$, we can see that

$$\begin{aligned}
&\frac{1}{\xi_0^{2q-p}} \int_{\mathbb{R}^3} \big(a|\nabla w^+|^p + V(x)|w^+|^p\big)\,dx + \frac{b}{\xi_0^{2(q-p)}}\bigg(\int_{\mathbb{R}^3} |\nabla w^+|^p\,dx\bigg)^2 \\
&\quad + \frac{1}{\xi_0^q} \int_{\mathbb{R}^3} \big(c|\nabla w^+|^q + V(x)|w^+|^q\big)\,dx + d\bigg(\int_{\mathbb{R}^3} |\nabla w^+|^q\,dx\bigg)^2 \\
&= \int_{\mathbb{R}^3} K(x)\frac{f(\xi_0 w^+)}{(\xi_0 w^+)^{2q-1}}\big(w^+\big)^{2q}\,dx.
\end{aligned}$$

Exploiting the fact that $w \in \mathcal{M}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(a|\nabla w^+|^p + V(x)|w^+|^p\big)\,dx + b\bigg(\int_{\mathbb{R}^3} |\nabla w^+|^p\,dx\bigg)^2 \\
&\quad + \int_{\mathbb{R}^3} \big(c|\nabla w^+|^q + V(x)|w^+|^q\big)\,dx + d\bigg(\int_{\mathbb{R}^3} |\nabla w^+|^q\,dx\bigg)^2 \\
&= \int_{\mathbb{R}^3} K(x)f\big(w^+\big)w^+\,dx
\end{aligned}$$

and subtracting we have

$$\bigg(\frac{1}{\xi_0^{2q-p}} - 1\bigg)\int_{\mathbb{R}^3} \big(a|\nabla w^+|^p + V(x)|w^+|^p\big)\,dx + b\bigg(\frac{1}{\xi_0^{2q-2p}} - 1\bigg)\bigg(\int_{\mathbb{R}^3} |\nabla w^+|^p\,dx\bigg)^2$$

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$$
\begin{aligned}
&+\left(\frac{1}{\xi_0^q}-1\right)\int_{\mathbb{R}^3}\left(c\left|\nabla w^+\right|^q+V(x)\left|w^+\right|^q\right)dx\\
&=\int_{\mathbb{R}^3}K(x)\left(\frac{f(\xi_0 w^+)}{(\xi_0 w^+)^{2q-1}}-\frac{f(w^+)}{(w^+)^{2q}}\right)\left(w^+\right)^{2q}dx.
\end{aligned}
\tag{4.3}
$$

Using (4.3) and (f_4) we get $\xi_0 \geqslant 1$.

Similarly, from $\frac{\partial g_w}{\partial \lambda}(\xi_0, \lambda_0) = 0$ we obtain

$$
\begin{aligned}
&\frac{1}{\lambda_0^{2q-p}}\int_{\mathbb{R}^3}\left(a\left|\nabla w^-\right|^p+V(x)\left|w^-\right|^p\right)dx+\frac{b}{\lambda_0^{2(q-p)}}\left(\int_{\mathbb{R}^3}\left|\nabla w^-\right|^p dx\right)^2\\
&\quad+\frac{1}{\lambda_0^q}\int_{\mathbb{R}^3}\left(c\left|\nabla w^-\right|^q+V(x)\left|w^-\right|^q\right)dx+d\left(\int_{\mathbb{R}^3}\left|\nabla w^-\right|^q dx\right)^2\\
&=\int_{\mathbb{R}^3}K(x)\frac{f(\lambda_0 w^-)}{(\lambda_0 w^-)^{2q-1}}\left(w^-\right)^{2q}dx.
\end{aligned}
$$

Note that from $w \in \mathcal{M}$ we also deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(a\left|\nabla w^-\right|^p+V(x)\left|w^-\right|^p\right)dx+b\left(\int_{\mathbb{R}^3}\left|\nabla w^-\right|^p dx\right)^2\\
&\quad+\int_{\mathbb{R}^3}\left(c\left|\nabla w^-\right|^q+V(x)\left|w^-\right|^q\right)dx+d\left(\int_{\mathbb{R}^3}\left|\nabla w^-\right|^q dx\right)^2\\
&=\int_{\mathbb{R}^3}K(x)f\left(w^-\right)w^-\,dx.
\end{aligned}
$$

Subtracting these last two equality and using assumption (f_4) we get $0 < \xi_0 \leqslant \lambda_0 \leqslant 1$. Hence $\xi_0 = \lambda_0 = 1$, this shows that $(1, 1)$ is the unique critical point of g_w with positive coordinates.

Next, take $u \in \mathbb{E}$ such that $u^\pm \neq 0$. Let (ξ_1, λ_1) and (ξ_2, λ_2) be two critical points of g_u such that $\xi_i, \lambda_i > 0$ for $i = 1, 2$. Define

$$
U_1 = \xi_1 u^+ + \lambda_1 u^- \quad \text{and} \quad U_2 = \xi_2 u^+ + \lambda_2 u^-.
$$

Then we have that $U_1, U_2 \in \mathcal{M}$ and $U_1^\pm \neq 0$. Furthermore, recalling that $\xi_1, \lambda_1 > 0$ we have

$$
\frac{\xi_2}{\xi_1}U_1^+ + \frac{\lambda_2}{\lambda_1}U_1^- = \frac{\xi_2}{\xi_1}\xi_1 u^+ + \frac{\lambda_2}{\lambda_1}\lambda_1 u^- = \xi_2 u^+ + \lambda_2 u^- = U_2 \in \mathcal{M}
$$

hence from (i) we deduce that $(\frac{\xi_2}{\xi_1}, \frac{\lambda_2}{\lambda_1})$ is a critical point of g_{U_1}. Due to the fact that $U_1 \in \mathcal{M}$ we infer that $\frac{\xi_2}{\xi_1} = \frac{\lambda_2}{\lambda_1} = 1$, that is $\xi_1 = \xi_2$ and $\lambda_1 = \lambda_2$, from which follows the uniqueness.

Now we prove that g_u has a maximum global point. Let $\Omega^+ \subset \operatorname{supp} u^+$ and $\Omega^- \subset \operatorname{supp} u^-$ be positive with finite measure. Gathering (2.1) with (f_3) and Fatou's lemma we get

$$
g_u(\xi, \lambda) = \mathcal{I}\left(\xi u^+\right) + \mathcal{I}\left(\lambda u^-\right)
$$

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$$
\begin{aligned}
&= \Big\{ \xi^p \int_{\mathbb{R}^3} \big(a|\nabla u^+|^p + V(x)|u^+|^p\big)\,dx + \frac{b}{2p}\xi^{2p}\Big(\int_{\mathbb{R}^3}|\nabla u^+|^p\,dx\Big)^2 \\
&\quad + \xi^q \int_{\mathbb{R}^3} \big(c|\nabla u^+|^q + V(x)|u^+|^q\big)\,dx + \frac{d}{2q}\xi^{2q}\Big(\int_{\mathbb{R}^3}|\nabla u^+|^q\,dx\Big)^2 \\
&\quad - \int_{\Omega^+} K(x)F\big(\xi u^+\big)\,dx \Big\} \\
&\quad + \Big\{ \lambda^p \int_{\mathbb{R}^3} \big(a|\nabla u^-|^p + V(x)|u^-|^p\big)\,dx + \frac{b}{2p}\lambda^{2p}\Big(\int_{\mathbb{R}^3}|\nabla u^-|^p\,dx\Big)^2 \\
&\quad + \lambda^q \int_{\mathbb{R}^3} \big(c|\nabla u^-|^q + V(x)|u^-|^q\big)\,dx + \frac{d}{2q}\lambda^{2q}\Big(\int_{\mathbb{R}^3}|\nabla u^-|^q\,dx\Big)^2 \\
&\quad - \int_{\Omega^-} K(x)F\big(\lambda u^-\big)\,dx \Big\} \\
&\leqslant \Big\{ \xi^p \int_{\mathbb{R}^3} \big(a|\nabla u^+|^p + V(x)|u^+|^p\big)\,dx + \frac{b}{2p}\xi^{2p}\Big(\int_{\mathbb{R}^3}|\nabla u^+|^p\,dx\Big)^2 \\
&\quad + \xi^q \int_{\mathbb{R}^3} \big(c|\nabla u^+|^q + V(x)|u^+|^q\big)\,dx + \frac{d}{2q}\xi^{2q}\Big(\int_{\mathbb{R}^3}|\nabla u^+|^q\,dx\Big)^2 \\
&\quad - C_1\xi^{2q} \int_{\Omega^+} K(x)\big(u^+\big)^{2q}\,dx \Big\} \\
&\quad + \Big\{ \lambda^p \int_{\mathbb{R}^3} \big(a|\nabla u^-|^p + V(x)|u^-|^p\big)\,dx + \frac{b}{2p}\lambda^{2p}\Big(\int_{\mathbb{R}^3}|\nabla u^-|^p\,dx\Big)^2 \\
&\quad + \lambda^q \int_{\mathbb{R}^3} \big(c|\nabla u^-|^q + V(x)|u^-|^q\big)\,dx + \frac{d}{2q}\lambda^{2q}\Big(\int_{\mathbb{R}^3}|\nabla u^-|^q\,dx\Big)^2 \\
&\quad - C_1\lambda^{2q} \int_{\Omega^-} K(x)\big(u^-\big)^{2q}\,dx \Big\} \\
&\quad + C_2|K|_\infty\big(|\Omega^+| + |\Omega^-|\big) \to -\infty \quad \text{as } |(\xi,\lambda)| \to \infty.
\end{aligned}
$$

Combining the fact that g_u is a continuous function with $g_u(\xi,\lambda) \to -\infty$ as $|(\xi,\lambda)| \to \infty$, we conclude that g_u assumes a global maximum in $(\bar{\xi},\bar{\lambda}) \in (0,\infty) \times (0,\infty)$. Using (2.1), for any $\xi,\lambda \geqslant 0$ we get

$$
g_u(\xi,0) + g_u(0,\lambda) = \mathcal{I}\big(\xi u^+\big) + \mathcal{I}\big(\lambda u^-\big) = \mathcal{I}\big(\xi u^+ + \lambda u^-\big) = g_u(\xi,\lambda),
$$

therefore

$$
0 < \max_{\xi \geqslant 0} g_u(\xi,0) < \max_{\xi,\lambda \geqslant 0} g_u(\xi,\lambda) = g_u(\bar{\xi},\bar{\lambda})
$$

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and

$$0 < \max_{\lambda \geqslant 0} g_u(0, \lambda) < \max_{\xi, \lambda \geqslant 0} g_u(\xi, \lambda) = g_u(\bar{\xi}, \bar{\lambda})$$

showing that $(\bar{\xi}, \bar{\lambda}) \in (0, \infty) \times (0, \infty)$. By virtue of the uniqueness of the critical point of g_u we have that $(\xi_+, \lambda_-) = (\bar{\xi}, \bar{\lambda})$, hence (ξ_+, λ_-) is the unique global maximum of g_u.

(iii) From Lemma 3.1(a) we get $\frac{\partial g_u}{\partial \xi}(r, \lambda_-) > 0$ if $r \in (0, \xi_+)$, $\frac{\partial g_u}{\partial \xi}(\xi_+, \lambda_-) = 0$ and $\frac{\partial g_u}{\partial \xi}(r, \lambda_-) > 0$ if $r \in (\xi_+, \infty)$. Similarly for $a_-(r)$. $\square$

Proceeding as in [38] we can prove the following lemma.

Lemma 4.2. *If $(u_n) \subset \mathcal{M}$ and $u_n \rightharpoonup u$ in $\mathbb{E}$, then $u \in \mathbb{E}$ and $u^{\pm} \neq 0$.*

Now, we denote by

$$c_\infty = \inf_{u \in \mathcal{M}} \mathcal{I}(u). \tag{4.4}$$

From $\mathcal{M} \subset \mathcal{N}$ it follows that $c_\infty \geqslant d_\infty > 0$.

5. Proof of Theorem 1.1

Let $(u_n) \subset \mathcal{M}$ be such that

$$\mathcal{I}(u_n) \to c_\infty \quad \text{in } \mathbb{R}. \tag{5.1}$$

First we show that (u_n) is bounded in $\mathbb{E}$. Suppose that there exists a subsequence still denoted by (u_n) such that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

Set $v_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Hence (v_n) is bounded in $\mathbb{E}$ so by Lemma 2.2 we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } \mathbb{E}, \\ &v_n \to v \quad \text{a.e. in } \mathbb{R}^3, \\ &v_n \to v \quad \text{in } L^r(\mathbb{R}^3) \text{ for } r \in (q, q^*). \end{aligned} \tag{5.2}$$

Now, from $u_n = \|u_n\| v_n$ it follows that

$$\|u_n\| v_n^+ + \|u_n\| v_n^- = \|u_n\| v_n = u_n \in \mathcal{M}$$

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and by Lemma 4.1 we have $\xi_+(v_n) = \lambda_-(v_n) = \|u_n\|$. Recalling that (ξ_+, λ_-) is the unique global maximum point of g_{v_n} with positive coordinates, for any $\xi > 0$ we infer

$$\begin{aligned}
\mathcal{I}(u_n) &= \mathcal{I}\big(\|u_n\| v_n\big) \\
&= \mathcal{I}\big(\xi_+(v_n)v_n^+ + \lambda_-(v_n)v_n^-\big) \\
&= g_{v_n}\big(\xi_+(v_n), \lambda_-(v_n)\big) \geqslant g_{v_n}(\xi, \xi) = \mathcal{I}(\xi v_n) \\
&= \frac{\xi^p}{p}\int_{\mathbb{R}^3}\big(a|\nabla v_n|^p + V(x)|v_n|^p\big)\,dx + \frac{b}{2p}\xi^{2p}\bigg(\int_{\mathbb{R}^3}|\nabla v_n|^p\,dx\bigg)^2 \\
&\quad + \frac{\xi^q}{q}\int_{\mathbb{R}^3}\big(c|\nabla v_n|^q + V(x)|v_n|^q\big)\,dx + \frac{d}{2q}\xi^{2q}\bigg(\int_{\mathbb{R}^3}|\nabla v_n|^q\,dx\bigg)^2 \\
&\quad - \int_{\mathbb{R}^3}K(x)F(\xi v_n)\,dx.
\end{aligned} \tag{5.3}$$

Note that $\|v_n\| = 1$, hence

$$\int_{\mathbb{R}^3}\big(a|\nabla v_n|^p + V(x)|v_n|^p\big)\,dx \leqslant 1 \quad \text{and} \quad \int_{\mathbb{R}^3}\big(c|\nabla v_n|^q + V(x)|v_n|^q\big)\,dx \leqslant 1.$$

Using $1 < p < q$, and assuming without loss of generality that $\xi > 1$ so that $\xi^q > \xi^p$, and exploiting the following inequality

$$a^q + b^q \geqslant C_q(a+b)^q \quad \text{for all } a, b \geqslant 0 \text{ and } q > 1$$

from (5.3) we deduce

$$\begin{aligned}
\mathcal{I}(u_n) &\geqslant \frac{\xi^p}{p}\int_{\mathbb{R}^3}\big(a|\nabla v_n|^p + V(x)|v_n|^p\big)\,dx + \frac{\xi^q}{q}\int_{\mathbb{R}^3}\big(c|\nabla v_n|^q + V(x)|v_n|^q\big)\,dx \\
&\quad - \int_{\mathbb{R}^3}K(x)F(\xi v_n)\,dx \\
&\geqslant \frac{\xi^p}{q}C_q\|v_n\|^q - \int_{\mathbb{R}^3}K(x)F(\xi v_n)\,dx \\
&= \frac{\xi^p}{q}C_q - \int_{\mathbb{R}^3}K(x)F(\xi v_n)\,dx.
\end{aligned} \tag{5.4}$$

Assume by contradiction that $v = 0$. From (5.2) and Lemma 2.3 we deduce that for any $\xi > 0$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}K(x)F(\xi v_n)\,dx = 0. \tag{5.5}$$

Taking the limit in (5.4), and using (5.1) and (5.5) we have

$$c_\infty \geqslant \frac{\xi^p}{q}C_q \quad \text{for any } \xi > 1,$$

which gives a contradiction. Therefore $v \neq 0$.

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On the other hand

$$\begin{aligned}
\frac{\mathcal{I}(u_n)}{\|u_n\|^{2q}} &= \frac{1}{p\|u_n\|^{2q}}\int_{\mathbb{R}^3}\left(a|\nabla u_n|^p + V(x)|u_n|^p\right)dx + \frac{b}{2p\|u_n\|^{2q}}\left(\int_{\mathbb{R}^3}|\nabla u_n|^p\,dx\right)^2 \\
&\quad + \frac{1}{q\|u_n\|^{2q}}\int_{\mathbb{R}^3}\left(c|\nabla u_n|^q + V(x)|u_n|^q\right)dx + \frac{d}{2q\|u_n\|^{2q}}\left(\int_{\mathbb{R}^3}|\nabla u_n|^q\,dx\right)^2 \\
&\quad - \int_{\mathbb{R}^3}K(x)\frac{F(u_n)}{\|u_n\|^{2q}}\,dx \\
&\leqslant \frac{1}{p\|u_n\|^{2q-p}} + \frac{b}{2p\|u_n\|^{2(q-p)}} + \frac{1}{q\|u_n\|^{q}} + \frac{d}{2q} \\
&\quad - \int_{\mathbb{R}^3}K(x)\frac{F(\|u_n\|v_n)}{(\|u_n\|v_n)^{2q}}\left(\frac{u_n}{\|u_n\|}\right)^{2q}dx.
\end{aligned} \tag{5.6}$$

Combining assumption (f_3) together with Fatou's lemma we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}K(x)\frac{F(\|u_n\|v_n)}{(\|u_n\|v_n)^{2q}}\left(\frac{u_n}{\|u_n\|}\right)^{2q}dx = +\infty \tag{5.7}$$

so taking the limit in (5.6) we get a contradiction in view of (5.1), $\|u_n\| \to \infty$ as $n \to \infty$ and (5.7). So (u_n) is a bounded sequence in $\mathbb{E}$ and there exists $u \in \mathbb{E}$ such that $u_n \rightharpoonup u$ in $\mathbb{E}$. From Lemma 4.2 we have $u^{\pm} \neq 0$ and by Lemma 4.1 there are $\xi_+, \lambda_- > 0$ such that $\xi_+ u^+ + \lambda_- u^- \in \mathcal{M}$, from which

$$\begin{aligned}
&\frac{1}{\lambda_-^{2q-p}}\int_{\mathbb{R}^3}\left(a|\nabla u^-|^p + V(x)|u^-|^p\right)dx + \frac{b}{\lambda_-^{2(q-p)}}\left(\int_{\mathbb{R}^3}|\nabla u^-|^p\,dx\right)^2 \\
&\quad + \frac{1}{\lambda_-^{q}}\int_{\mathbb{R}^3}\left(c|\nabla u^-|^q + V(x)|u^-|^q\right)dx + d\left(\int_{\mathbb{R}^3}|\nabla u^-|^q\,dx\right)^2 \\
&= \int_{\operatorname{supp} u^-}K(x)\frac{f(\lambda_- u^-)}{(\lambda_- u^-)^{2q-1}}(u^-)^{2q}\,dx.
\end{aligned} \tag{5.8}$$

Our aim is to prove that $\xi_+ = \lambda_- = 1$. Without loss of generality, let us suppose that $0 < \xi_+ \leqslant \lambda_-$. First we prove that $0 < \xi_+ \leqslant \lambda_- \leqslant 1$. Note that from $u_n \rightharpoonup u$ in $\mathbb{E}$ and exploiting Lemma 2.3 we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}K(x)f\left(u_n^{\pm}\right)u_n^{\pm}\,dx = \int_{\mathbb{R}^3}K(x)f\left(u^{\pm}\right)u^{\pm}\,dx \tag{5.9}$$

and also

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}K(x)F\left(u_n^{\pm}\right)dx = \int_{\mathbb{R}^3}K(x)F\left(u^{\pm}\right)dx, \tag{5.10}$$

and combining $(u_n) \subset \mathcal{M}$ with Fatou's lemma we get

$$\left\langle \mathcal{I}'(u), u^{\pm}\right\rangle \leqslant \liminf_{n\to\infty}\left\langle \mathcal{I}'(u_n), u_n^{\pm}\right\rangle = 0,$$

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which yields

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(a\left|\nabla u^-\right|^p + V(x)\left|u^-\right|^p\right) dx + b\left(\int_{\mathbb{R}^3} \left|\nabla u^-\right|^p dx\right)^2 \\
&\quad + \int_{\mathbb{R}^3} \left(c\left|\nabla u^-\right|^q + V(x)\left|u^-\right|^q\right) dx + d\left(\int_{\mathbb{R}^3} \left|\nabla u^-\right|^q dx\right)^2 \\
&\leqslant \int_{\operatorname{supp} u^-} K(x) f\left(u^-\right) u^- \, dx.
\end{aligned}
\tag{5.11}
$$

Subtracting (5.8) and (5.11) we obtain

$$
\begin{aligned}
&\left(\frac{1}{\lambda_-^{2q-p}} - 1\right) \int_{\mathbb{R}^3} \left(a\left|\nabla u^-\right|^p + V(x)\left|u^-\right|^p\right) dx + b\left(\frac{1}{\lambda_-^{2(q-p)}} - 1\right)\left(\int_{\mathbb{R}^3} \left|\nabla u^-\right|^p dx\right)^2 \\
&\quad + \left(\frac{1}{\lambda_-^{q}} - 1\right) \int_{\mathbb{R}^3} \left(c\left|\nabla u^-\right|^q + V(x)\left|u^-\right|^q\right) dx \\
&\geqslant \int_{\operatorname{supp} u^-} K(x)\left(\frac{f(\lambda_- u^-)}{(\lambda_- u^-)^{2q-1}} - \frac{u^-}{(u^-)^{2q-1}}\right)\left(u^-\right)^{2q} dx
\end{aligned}
$$

and using assumption (f_3) we deduce $0 < \lambda_- \leqslant 1$. Hence, $0 < \xi_+ \leqslant \lambda_- \leqslant 1$.

Next we show that

$$
\mathcal{I}\left(\xi_+ u^+ + \lambda_- u^-\right) = c_\infty. \tag{5.12}
$$

Now, from (4.4), $0 < \xi_+ \leqslant \lambda_- \leqslant 1$, assumption (f_4), (5.9) and (5.10) we obtain

$$
\begin{aligned}
c_\infty &\leqslant \mathcal{I}\left(\xi_+ u^+ + \lambda_- u^-\right) \\
&= \mathcal{I}\left(\xi_+ u^+ + \lambda_- u^-\right) - \frac{1}{2q}\left\langle \mathcal{I}'\left(\xi_+ u^+ + \lambda_- u^-\right), \xi_+ u^+ + \lambda_- u^-\right\rangle \\
&= \left(\frac{1}{p} - \frac{1}{2q}\right) \int_{\mathbb{R}^3} \left(a\left|\nabla\left(\xi_+ u^+ + \lambda_- u^-\right)\right|^p + V(x)\left|\xi_+ u^+ + \lambda_- u^-\right|^p\right) dx \\
&\quad + \frac{b}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\left(\int_{\mathbb{R}^3} \left|\nabla\left(\xi_+ u^+ + \lambda_- u^-\right)\right|^p dx\right)^2 \\
&\quad + \frac{1}{2q} \int_{\mathbb{R}^3} \left(c\left|\nabla\left(\xi_+ u^+ + \lambda_- u^-\right)\right|^q + V(x)\left|\xi_+ u^+ + \lambda_- u^-\right|^q\right) dx \\
&\quad + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{2q} f\left(\xi_+ u^+\right)\left(\xi_+ u^+\right) - F\left(\xi_+ u^+\right)\right) dx \\
&\quad + \int_{\mathbb{R}^3} K(x)\left(\frac{1}{2q} f\left(\lambda_- u^-\right)\left(\lambda_- u^-\right) - F\left(\lambda_- u^-\right)\right) dx
\end{aligned}
$$

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$$
\begin{aligned}
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\int_{\mathbb{R}^3}\big(a|\nabla u|^p + V(x)|u|^p\big)\,dx + \frac{b}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)^2 \\
&\quad + \frac{1}{2q}\int_{\mathbb{R}^3}\big(c|\nabla u|^q + V(x)|u|^q\big)\,dx \\
&\quad + \int_{\mathbb{R}^3}K(x)\left(\frac{1}{2q}f\big(u^+\big)u^+ - F\big(u^+\big)\right)dx + \int_{\mathbb{R}^3}K(x)\left(\frac{1}{2q}f\big(u^-\big)u^- - F\big(u^-\big)\right)dx \\
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\int_{\mathbb{R}^3}\big(a|\nabla u|^p + V(x)|u|^p\big)\,dx + \frac{b}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^3}|\nabla u|^p\,dx\right)^2 \\
&\quad + \frac{1}{2q}\int_{\mathbb{R}^3}\big(c|\nabla u|^q + V(x)|u|^q\big)\,dx + \int_{\mathbb{R}^3}K(x)\left(\frac{1}{2q}f(u)u - F(u)\right)dx \\
&\leqslant \liminf_{n\to\infty}\left\{\left(\frac{1}{p}-\frac{1}{2q}\right)\int_{\mathbb{R}^3}\big(a|\nabla u_n|^p + V(x)|u_n|^p\big)\,dx + \frac{b}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^3}|\nabla u_n|^p\,dx\right)^2\right. \\
&\quad \left. + \frac{1}{2q}\int_{\mathbb{R}^3}\big(c|\nabla u_n|^q + V(x)|u_n|^q\big)\,dx + \int_{\mathbb{R}^3}K(x)\left(\frac{1}{2q}f(u_n)u - F(u_n)\right)dx\right\} \\
&= \liminf_{n\to\infty}\left\{\mathcal{I}(u_n) - \frac{1}{2q}\big\langle\mathcal{I}'(u_n), u_n\big\rangle\right\} = c_\infty
\end{aligned}
$$

which implies that (5.12) holds true. In particular it follows that $\xi_+ = \lambda_- = 1$.

Next, we prove that the minimum point $u = u^+ + u^-$ is a critical point of $\mathcal{I}$. Assume by contradiction that $\mathcal{I}'(u) \neq 0$. Then, due to the continuity of $\mathcal{I}'$ we can find $\alpha, \beta > 0$ such that $\|\mathcal{I}'(v)\| \geqslant \beta$ for all $v \in \mathbb{E}$ with $\|v - u\| \leqslant 3\alpha$.

Define $D = [\frac{1}{2}, \frac{3}{2}]\times[\frac{1}{2}, \frac{3}{2}]$ and $\mathbb{E}^\pm = \{u \in \mathbb{E} : u^\pm \neq 0\}$, and let us consider the function $G_u : D \to \mathbb{E}^\pm$ defined by setting

$$
G_u(\xi, \lambda) = \xi u^+ + \lambda u^-.
$$

Using Lemma 4.1 we can see that $\mathcal{I}(G_u(1,1)) = c_\infty$ and $\mathcal{I}(G_u(\xi,\lambda)) < c_\infty$ in $D \setminus \{(1,1)\}$.

Define

$$
\tau = \max_{(\xi,\lambda)\in\partial D}\mathcal{I}\big(G_u(\xi,\lambda)\big),
$$

then $\tau < c_\infty$.

Set $\tilde{\mathcal{S}} = \{v \in \mathbb{E} : \|v - u\| \leqslant \alpha\}$ and choose $\varepsilon = \min\{\frac{1}{4}(c_\infty - \gamma), \frac{\alpha\beta}{8}\}$. By Theorem 2.3 in [54] there exists a deformation $\eta \in \mathcal{C}([0,1]\times\mathbb{E}, \mathbb{E})$ such that the following assertions hold:

(a) $\eta(\xi, v) = v$ if $v \notin \mathbb{E}^{-1}([c_\infty - 2\varepsilon, c_\infty + 2\varepsilon])$;
(b) $\mathcal{I}(\eta(1, v)) \leqslant c_\infty - \varepsilon$ for each $v \in \mathbb{E}$ with $\|v - u\| \leqslant \alpha$ and $\mathcal{I}(v) \leqslant c_\infty + \varepsilon$;
(c) $\mathcal{I}(\eta(1, v)) \leqslant \mathcal{I}(\eta(0, v)) = \mathcal{I}(v)$ for all $v \in \mathbb{E}$.

From (b) and (c) we get

$$
\max_{(\xi,\lambda)\in\partial D}\mathcal{I}\big(\eta\big(1, G_u(\xi,\lambda)\big)\big) < c_\infty. \tag{5.13}
$$

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Now we prove that

$$\eta\big(1, G_u(D)\big) \cap \mathcal{M} \neq \emptyset \tag{5.14}$$

because the definition of c_∞ and (5.14) contradict (5.13).

Let us define

$$\begin{aligned}
&\Phi_u(\xi,\lambda) = \eta\big(1, G_u(\xi,\lambda)\big),\\
&\psi_0(\xi,\lambda) = \big(\big\langle \mathcal{I}'\big(G_u(\xi,1)\big), \xi u^+\big\rangle, \big\langle \mathcal{I}'\big(G_u(1,\lambda)\big), \lambda u^-\big\rangle\big),\\
&\psi_1(\xi,\lambda) = \left(\frac{1}{\xi}\big\langle \mathcal{I}'\big(\Phi_u(\xi,1)\big), \Phi_u^+(\xi,1)\big\rangle, \frac{1}{\lambda}\big\langle \mathcal{I}'\big(\Phi_u(1,\lambda)\big), \Phi_u^-(1,\lambda)\big\rangle\right).
\end{aligned}$$

Exploiting Lemma 4.1(iii), the $\mathcal{C}^1$ function $\gamma_+(\xi) = g_u(\xi,1)$ has a unique global maximum point $\xi = 1$. By density, given $\varepsilon > 0$ small enough, there is $\gamma_{+,\varepsilon} \in \mathcal{C}^\infty([\frac{1}{2},\frac{3}{2}])$ such that $\|\gamma_+ - \gamma_{+,\varepsilon}\|_{\mathcal{C}^1([\frac{1}{2},\frac{3}{2}])} < \varepsilon$ with ξ_+ being the unique maximum global point of $\gamma_{+,\varepsilon}$ in $[\frac{1}{2},\frac{3}{2}]$. Hence, $\|\gamma'_+ - \gamma'_{+,\varepsilon}\|_{\mathcal{C}([\frac{1}{2},\frac{3}{2}])} < \varepsilon$, $\gamma'_{+,\varepsilon}(1) = 0$ and $\gamma''_{+,\varepsilon}(1) < 0$. Analogously, set $\gamma_-(\lambda) = g_u(1,\lambda)$, then there exists $\gamma_{-,\varepsilon} \in \mathcal{C}^\infty([\frac{1}{2},\frac{3}{2}])$ such that $\|\gamma'_- - \gamma'_{-,\varepsilon}\|_{\mathcal{C}([\frac{1}{2},\frac{3}{2}])} < \varepsilon$, $\gamma'_{+,\varepsilon}(1) = 0$ and $\gamma''_{+,\varepsilon}(1) < 0$.

Let us define $\psi_\varepsilon \in \mathcal{C}^\infty(D)$ by setting

$$\psi_\varepsilon(\xi,\lambda) = \big(\xi\gamma'_{+,\varepsilon}(\xi), \lambda\gamma'_{-,\varepsilon}(\lambda)\big).$$

We note that $\|\psi_\varepsilon - \psi_0\|_{\mathcal{C}(D)} < \frac{3\sqrt{2}}{2}\varepsilon$, $(0,0) \notin \psi_\varepsilon(\partial D)$, and $(0,0)$ is a regular value of ψ_ε in D.

Since $(1,1)$ is the unique solution of $\psi_\varepsilon(\xi,\lambda) = (0,0)$ in D, by the definition of Brouwer's degree, we can infer that, for ε small enough, it holds

$$\deg\big(\psi_0, D, (0,0)\big) = \deg\big(\psi_\varepsilon, D, (0,0)\big) = \operatorname{sgn} \operatorname{Jac}(\psi_\varepsilon)(1,1), \tag{5.15}$$

where $\operatorname{Jac}(\psi_\varepsilon)$ is the Jacobian determinant of ψ_ε and sgn denotes the sign function.

We note that

$$\operatorname{Jac}(\psi_\varepsilon)(1,1) = \big[\gamma'_{+,\varepsilon}(1) + \gamma''_{+,\varepsilon}(1)\big] \times \big[\gamma'_{-,\varepsilon}(1) + \gamma''_{-,\varepsilon}(1)\big] = \gamma''_{+,\varepsilon}(1) \times \gamma''_{-,\varepsilon}(1) > 0, \tag{5.16}$$

so combining (5.15) with (5.16) we find

$$\deg\big(\psi_0, D, (0,0)\big) = \operatorname{sgn}\big[\gamma''_{+,\varepsilon}(1) \times \gamma''_{-,\varepsilon}(1)\big] = 1.$$

By the definition of τ and the fact that $\varepsilon = \min\{\frac{1}{4}(c_\infty - \gamma), \frac{\alpha\beta}{8}\}$ we have that for any $(\xi,\lambda) \in \partial D$

$$\mathcal{I}\big(G_u(\xi,\lambda)\big) \leqslant \max_{(\xi,\lambda)\in\partial D} \mathcal{I}\big(G_u(\xi,\lambda)\big) < \frac{1}{2}(\tau + c_\infty) = c_\infty - 2\left(\frac{c_\infty - \tau}{4}\right) \leqslant c_\infty - 2\varepsilon.$$

This and (a)yields that $G_u = \Phi_u$ on ∂D. Therefore, $\psi_1 = \psi_0$ on ∂D and consequently

$$\deg\big(\psi_1, D, (0,0)\big) = \deg\big(\psi_0, D, (0,0)\big) = 1,$$

which shows that $\psi_1(\xi,\lambda) = (0,0)$ for some $(\xi,\lambda) \in D$.

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Now, in order to verify that (5.14) holds, we prove that

$$\psi_1(1,1) = \left(\left\langle \mathcal{I}'\left(\Phi_u(\xi,1)\right), \Phi_u(1,1)^+\right\rangle, \left\langle \mathcal{I}'\left(\Phi_u(1,1)\right), \Phi_u(1,1)^-\right\rangle\right) = 0. \tag{5.17}$$

As a matter of fact, (5.17) and the fact that $(1,1) \in D$, yield $\Phi_u(1,1) = \eta(1, G_u(1,1)) \in \mathcal{M}$.

We argue as follows. If the zero (ξ, λ) of ψ_1 obtained above is equal to $(1,1)$ there is nothing to prove. Otherwise, we take $0 < \delta_1 < \min\{|\xi - 1|, |\lambda - 1|\}$ and consider

$$D_1 = \left[1 - \frac{\delta_1}{2}, 1 + \frac{\delta_1}{2}\right] \times \left[1 - \frac{\delta_1}{2}, 1 + \frac{\delta_1}{2}\right].$$

Therefore $(\xi, \lambda) \in D \setminus D_1$. Hence, we can repeat for D_1 the same argument used for D, so that we can find a couple $(\xi_1, \lambda_1) \in D_1$ such that $\psi_1(\xi_1, \lambda_1) = 0$. If $(\xi_1, \lambda_1) = (1,1)$, there is nothing to prove. Otherwise, we can continue with this procedure and find in the n-th step that (5.17) holds, or produce a sequence $(\xi_n, \lambda_n) \in D_{n-1} \setminus D_n$ which converges to $(1,1)$ and such that

$$\psi_1(\xi_n, \lambda_n) = 0, \quad \text{for every } n \in \mathbb{N}. \tag{5.18}$$

Thus, taking the limit as $n \to \infty$ in (5.18) and using the continuity of ψ_1 we get (5.17). Therefore, $u = u^+ + u^-$ is a critical point of $\mathcal{I}$.

Finally, we consider the case when f is odd. Clearly, the functional ψ is even. In the light of (3.2) and $c_\infty \geqslant d_\infty > 0$ we can see that ψ is bounded from below in $\mathbb{S}$. Moreover, using Lemma 2.2 and Lemma 2.3, we deduce that ψ satisfies the Palais–Smale condition on $\mathbb{S}$. Hence, applying Proposition 3.1 and [52], we conclude that $\mathcal{I}$ has infinitely many critical points.

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References

[1] C.O. Alves, V. Ambrosio and T. Isernia, Existence, multiplicity and concentration for a class of fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$, *Commun. Pure Appl. Anal.* **18**(4) (2019), 2009–2045. doi:10.3934/cpaa.2019091.

[2] C.O. Alves and M.A.S. Souto, Existence of solutions for a class of elliptic equations in $\mathbb{R}^N$ with vanishing potentials, *J. Differential Equations* **252** (2012), 5555–5568. doi:10.1016/j.jde.2012.01.025.

[3] C.O. Alves and M.A.S. Souto, Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity, *J. Differential Equations* **254**(4) (2013), 1977–1991. doi:10.1016/j.jde.2012.11.013.

[4] C.O. Alves and M.A.S. Souto, Existence of least energy nodal solution for a Schrödinger–Poisson system in bounded domains, *Z. Angew. Math. Phys.* **65** (2014), 1153–1166. doi:10.1007/s00033-013-0376-3.

[5] A. Ambrosetti, V. Felli and A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, *J. Eur. Math. Soc. (JEMS)* **7**(1) (2005), 117–144. doi:10.4171/JEMS/24.

[6] A. Ambrosetti and Z.-Q. Wang, Nonlinear Schrödinger equations with vanishing and decaying potentials, *Differential Integral Equations* **18** (2005), 1321–1332.

[7] V. Ambrosio, Fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$ with critical growth, *Z. Anal. Anwend.* **39**(3) (2020), 289–314.

[8] V. Ambrosio, Concentrating solutions for a fractional Kirchhoff equation with critical growth, *Asymptot. Anal.* **116**(3–4) (2020), 249–278.

[9] V. Ambrosio and T. Isernia, A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^3$ with a general nonlinearity, *Commun. Contemp. Math.* **20**(5) (2018), 1750054.

[10] V. Ambrosio and T. Isernia, Concentration phenomena for a fractional Schrödinger–Kirchhoff type problem, *Math. Methods Appl. Sci.* **41**(2) (2018), 615–645.

[11] V. Ambrosio and T. Isernia, On a fractional $p\&q$ Laplacian problem with critical Sobolev–Hardy exponents, *Mediterr. J. Math.* **15**(6) (2018), 219. doi:10.1007/s00009-018-1259-9.

[12] V. Ambrosio and V.D. Rădulescu, Fractional double-phase patterns: Concentration and multiplicity of solutions, *J. Math. Pures Appl.* **142**(9) (2020), 101–145.

[13] A. Arosio and S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348**(1) (1996), 305–330. doi:10.1090/S0002-9947-96-01532-2.

[14] G. Autuori, A. Fiscella and P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, *Nonlinear Anal.* **125** (2015), 699–714. doi:10.1016/j.na.2015.06.014.

[15] G. Autuori, P. Pucci and M.C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Ration. Mech. Anal.* **196**(2) (2010), 489–516. doi:10.1007/s00205-009-0241-x.

[16] S. Barile and G.M. Figueiredo, Existence of a least energy nodal solution for a class of (p, q)-quasilinear elliptic equations, *Adv. Nonlinear Stud.* **14**(2) (2014), 511–530. doi:10.1515/ans-2014-0215.

[17] S. Barile and G.M. Figueiredo, Existence of least energy positive, negative and nodal solutions for a class of (p, q)-problems with potentials vanishing at infinity, *J. Math. Anal. Appl.* **427**(2) (2015), 1205–1233. doi:10.1016/j.jmaa.2015.02.086.

[18] V. Benci, C.R. Grisanti and A.M. Micheletti, Existence of solutions for the nonlinear Schrödinger equation with $V(\infty) = 0$, *Progr. Nonlinear Differential Equations Appl.* **66** (2005), 53–65. doi:10.1007/3-7643-7401-2_4.

[19] H. Berestycki and P.L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Ration. Mech. Anal.* **82** (1983), 313–346. doi:10.1007/BF00250555.

[20] S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles, *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **4** (1940), 17–26.

[21] M. Bhaktaa and D. Mukherjee, Multiplicity results for (p, q) fractional elliptic equations involving critical nonlinearities, preprint, arXiv:1801.09925.

[22] D. Bonheure and J. Van Schaftingen, Ground states for the nonlinear Schrödinger equation with potential vanishing at infinity, *Ann. Mat. Pura Appl. (4)* **189**(2) (2010), 273–301. doi:10.1007/s10231-009-0109-6.

[23] F. Cammaroto and L. Vilasi, On a Schrödinger–Kirchhoff-type equation involving the $p(x)$-Laplacian, *Nonlinear Anal.* **81** (2013), 42–53. doi:10.1016/j.na.2012.12.011.

[24] K. Chang and Q. Gao, Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in $\mathbb{R}^N$, 2017, arXiv:1701.03862.

[25] C. Chen and J. Bao, Existence, nonexistence, and multiplicity of solutions for the fractional $p\&q$-Laplacian equation in $\mathbb{R}^N$, *Bound, Value Probl.* (2016), 153. doi:10.1186/s13661-016-0661-0.

[26] S. Chen, X. Tang and F. Liao, Existence and asymptotic behavior of sign-changing solutions for fractional Kirchhoff-type problems in low dimensions, *NoDEA Nonlinear Differential Equations Appl.* **25**(5) (2018), 40.

[27] L. Cherfils and V. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with (p, q)-Laplacian, *Commun. Pure Appl. Anal.* **1**(4) (2004), 1–14.

[28] F. Colasuonno and P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.* **74**(17) (2011), 5962–5974. doi:10.1016/j.na.2011.05.073.

[29] C. De Filippis and G. Palatucci, Hölder regularity for nonlocal double phase equations, *J. Differ. Equ.* **267**(1) (2020), 547–586.

[30] Y.B. Deng, S.J. Peng and W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$, *J. Funct. Anal.* **269** (2015), 3500–3527. doi:10.1016/j.jfa.2015.09.012.

[31] G.M. Figueiredo, Existence of positive solutions for a class of (p, q) elliptic problems with critical growth on $\mathbb{R}^N$, *J. Math. Anal. Appl.* **378** (2011), 507–518. doi:10.1016/j.jmaa.2011.02.017.

[32] G.M. Figueiredo, Existence and multiplicity of solutions for a class of (p, q) elliptic problems with critical exponent, *Math. Nachr.* **286**(11–12) (2013), 1129–1141. doi:10.1002/mana.201100237.

[33] G.M. Figueiredo and R.G. Nascimento, Existence of a nodal solution with minimal energy for a Kirchhoff equation, *Math. Nachr.* **288**(1) (2015), 48–60. doi:10.1002/mana.201300195.

[34] G.M. Figueiredo and J.R. Santos Juńior, Existence of a least energy nodal solution for a Schrödinger–Kirchhoff equation with potential vanishing at infinity, *Journal of Mathematical Physics.* **56** (2015), 051506.

[35] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.* **94** (2014), 156–170. doi:10.1016/j.na.2013.08.011.

AUTHOR COPY

[36] D. Goel, D. Kumar and K. Sreenadh, Regularity and multiplicity results for fractional (p, q)-Laplacian equations, preprint, arXiv:1902.00395.
[37] Y. Guo and J. Nie, Existence and multiplicity of nontrivial solutions for p-Laplacian Schrödinger–Kirchhoff-type equations, *J. Math. Anal. Appl.* **428**(2) (2015), 1054–1069. doi:10.1016/j.jmaa.2015.03.064.
[38] X. Han, X. Ma and X. He, Existence of sign-changing solutions for a class of p-Laplacian Kirchhoff-type equations, *Complex Var. Elliptic Equ.* **64**(2) (2019), 181–203. doi:10.1080/17476933.2018.1427078.
[39] C. He and G. Li, The regularity of weak solutions to nonlinear scalar field elliptic equations containing (p, q)-Laplacians, *Ann. Acad. Sci. Fenn. Math.* **33**(2) (2008), 337–371.
[40] T. Isernia, Sign-changing solutions for a fractional Kirchhoff equation, *Nonlinear Anal.* **190** (2020), 111623. doi:10.1016/j.na.2019.111623.
[41] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
[42] D. Kumar, V. Rădulescu and K. Sreenadh, Singular elliptic problems with unbalanced growth and critical exponent, *Nonlinearity* **33**(7) (2020), 3336–3369. doi:10.1088/1361-6544/ab81ed.
[43] G. Li and Z. Guo, Multiple solutions for the (p, q)-Laplacian problem with critical exponent, *Acta Math. Sci. Ser. B Engl. Ed.* **29**(4) (2009), 903–918.
[44] G.B. Li and X. Liang, The existence of nontrivial solutions to nonlinear elliptic equation of p-q-Laplacian type on $\mathbb{R}^N$, *Nonlinear Anal.* **71** (2009), 2316–2334. doi:10.1016/j.na.2009.01.066.
[45] J.L. Lions, On some questions in boundary value problems of mathematical physics, in: *Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977)*, North-Holland Math. Stud., Vol. 30, North-Holland, Amsterdam-New York, 1978, pp. 284–346. doi:10.1016/S0304-0208(08)70870-3.
[46] E.S. Medeiros and K. Perera, Multiplicity of solutions for a quasilinear elliptic problem via the cohomoligical index, *Nonlinear Anal.* **71** (2009), 3654–3660. doi:10.1016/j.na.2009.02.013.
[47] X. Mingqi, V. Rădulescu and B. Zhang, Fractional Kirchhoff problems with critical Trudinger–Moser nonlinearity, *Calc. Var. Partial Differential Equations* **58**(2) (2019), 57.
[48] K. Perera and Z. Zhang, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.* **317** (2006), 456–463. doi:10.1016/j.jmaa.2005.06.102.
[49] K. Perera and Z.T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations* **221**(1) (2006), 246–255. doi:10.1016/j.jde.2005.03.006.
[50] S.I. Pohožaev, A certain class of quasilinear hyperbolic equations, *Mat. Sb.* **96** (1975), 152–166.
[51] P. Pucci and S. Saldi, Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators, *Rev. Mat. Iberoam.* **32**(1) (2016), 1–22. doi:10.4171/RMI/879.
[52] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, RI, 1986.
[53] A. Szulkin and T. Weth, The method of Nehari manifold, in: *Handbook of Nonconvex Analysis and Applications*, D.Y. Gao and D. Montreanu, eds, International Press, Boston, 2010, pp. 597–632.
[54] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.
[55] Z. Xiu, The existence of a nontrivial solutions for a p-Kirchhoff type elliptic equation in $\mathbb{R}^N$, *Abst Appl Anal.* (2013), Article ID 281949.

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