

## Splitting along a submanifold pair

*Dedicated to the memory of Yu. P. Solovyeu*

by

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### Abstract

The paper introduces a group  $LSP$  of obstructions to splitting a homotopy equivalence along a pair of submanifolds. We develop exact sequences relating the  $LSP$ -groups with various surgery obstruction groups for a manifold triple and structure sets arising from a manifold triple. The natural map from the surgery obstruction group of the ambient manifold to the  $LSP$ -group provides an invariant when elements of the Wall group are not realized by normal maps of closed manifolds. Some  $LSP$ -groups are computed precisely.

*Key Words:* Splitting along submanifold, surgery of manifolds, surgery and splitting obstruction groups, surgery exact sequence, Browder-Livesay invariant, structure sets.

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### 1. Introduction

Consider a simple homotopy equivalence  $f : M \rightarrow X$  of closed  $n$ -dimensional oriented topological manifolds. Such a map is called an  $s$ -triangulation of the manifold  $X$ . Two  $s$ -triangulations

$$f_i : M_i \rightarrow X, \quad i = 1, 2$$

are said to be *equivalent* if there exists an orientation preserving homeomorphism  $h : M_1 \rightarrow M_2$  such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{h} & M_2 \\ & f_1 \searrow & \downarrow f_2 \\ & & X \end{array} \quad (1.1)$$

is homotopy commutative.

The set of equivalence classes of  $s$ -triangulations of the manifold  $X$  is denoted by  $\mathcal{S}(X) = \mathcal{S}^s(X)$  (see [18] and [21]). The computation of the structure set  $\mathcal{S}^s(X)$  for a manifold  $X$  is one of the main problems of geometric topology.

Let  $Y \subset X$  be a locally flat submanifold of codimension  $q$  in  $n$ -dimensional topological manifold  $X$ . A simple homotopy equivalence  $f : M \rightarrow X$  splits along the submanifold  $Y$  (see [18] and [21]) if it is homotopy equivalent to a map  $g$ , transversal to  $Y$  such that  $N = g^{-1}(Y)$  satisfies the following properties:

- i)  $g|_N : N \rightarrow Y$  is a simple homotopy equivalence
  - ii)  $g|_{(M \setminus N)} : M \setminus N \rightarrow X \setminus Y$  is a simple homotopy equivalence.
- (1.2)

A simple homotopy equivalence  $g : M \rightarrow X$  with the properties (1.2) is called an  $s$ -triangulation of the pair  $(X, Y)$ . The set of concordance classes of such  $s$ -triangulations is denoted by  $\mathcal{S}(X, Y, \xi)$  where  $\xi$  is the topological normal bundle of the submanifold  $Y$  in  $X$  (see [18, §7.2]).

Let  $U$  be a tubular neighborhood of the submanifold  $Y$  in  $X$ , and let  $\partial U$  denote the boundary of  $U$ . Denote by

$$F = \begin{pmatrix} \pi_1(\partial U) & \rightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \rightarrow & \pi_1(X) \end{pmatrix} \quad (1.3)$$

the push-out square of fundamental groups with orientations.

An obstruction to splitting the map  $f$  along the submanifold  $Y$  lies in the splitting obstruction group  $LS_{n-q}(F)$  which depends only on  $n - q \pmod{4}$  and on the push-out square  $F$ .

In fact, the obstruction to splitting defines correctly the map [18] that fits in the following exact sequence

$$\dots \rightarrow \mathcal{S}(X, Y, \xi) \rightarrow \mathcal{S}(X) \rightarrow LS_{n-q}(F). \quad (1.4)$$

The splitting obstruction groups are closely related to other obstruction groups which arise naturally for the manifold pair  $Y \subset X$  (see [1], [2], [13], [18], and [21]). The main relation among them is given by the following braid of exact

sequences (see [18] and [21])

$$\begin{array}{ccccccc}
 \rightarrow & L_n(\pi_1(X \setminus Y)) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & & \\
 & \nearrow & & \nearrow & & & \\
 & & & LP_{n-q}(F) & & & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\
 & \searrow & & \searrow & & & \\
 \rightarrow & LS_{n-q}(F) & \longrightarrow & L_{n-q}(\pi_1(Y)) & \longrightarrow & & \\
 & & & & & & LS_{n-q-1}(F) \rightarrow \\
 & & & & & & \nearrow & \searrow \\
 & & & & & & & \\
 & & & & & & \searrow & \nearrow \\
 & & & & & & \rightarrow & L_{n-1}(\pi_1(X \setminus Y)) \rightarrow \\
 & & & & & & & (1.5)
 \end{array}$$

where  $L_* = L_*^s$  denote the surgery obstruction groups and  $LP_*(F) = LP_*^s(F)$  denote the surgery obstruction groups of the manifold pair  $(X, Y)$ . The groups  $LP_*(F)$  also depend only on  $n - q \pmod 4$  and on the square  $F$ .

The main methods for computing the set  $\mathcal{S}(X)$  (for  $n \geq 4$ ) are based on the surgery exact sequence (see [17], [18], and [21])

$$\cdots \rightarrow L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}(X) \rightarrow [X, G/TOP] \xrightarrow{\sigma} L_n(\pi_1(X)) \quad (1.6)$$

where the set  $[X, G/TOP]$  is isomorphic to the set of concordance classes of topological normal maps to the manifold  $X$ .

The set  $\mathcal{S}(X, Y, \xi)$  fits into the surgery exact sequence [18, page 584] for the manifold pair  $(X, Y)$

$$\cdots \rightarrow LP_{n-q+1}(F) \rightarrow \mathcal{S}(X, Y, \xi) \rightarrow [X, G/TOP] \rightarrow LP_{n-q}(F). \quad (1.7)$$

The exact sequence (1.7) is the natural generalization of the exact sequence (1.6) to the case of a manifold pair.

The computation of the map  $\sigma$  in (1.6) is the basic step in investigating the surgery exact sequence. For manifolds with finite fundamental groups deep results in this direction were obtained in [5], [6], [9], [10], and [11]. The results of these papers are based on relations between the surgery exact sequence and the splitting problem for a one-sided submanifold.

Let

$$Z^{n-q-a'} \subset Y^{n-q} \subset X^n \quad (1.8)$$

be a triple of closed topological manifolds. We shall consider only locally flat topological submanifolds equipped with the structure of a normal topological

bundle (see [18, pages 562–563]). Such a triple of manifolds defines a stratified manifold  $\mathcal{X}$  in the sense of Browder and Quinn (see [4], [14], [15], [16], and [22]).

A simple homotopy equivalence  $f : M \rightarrow X$  is an *s-triangulation of the triple* if every pair of manifolds from this triple satisfies properties that are similar to (1.2) for the pair  $(X, Y)$  (see [4], [16], and [21]). The set of concordance classes of such *s-triangulations* is denoted by  $\mathcal{S}(\mathcal{X}) = \mathcal{S}(X, Y, Z)$ .

Surgery theory is applicable to stratified spaces, and we have the following exact sequence (see [4] and [22])

$$\cdots \rightarrow L_{n+1}^{BQ}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X}) \rightarrow [X, G/TOP] \rightarrow L_n^{BQ}(\mathcal{X}) \quad (1.9)$$

where  $L_*^{BQ}(\mathcal{X})$  are the Browder-Quinn surgery obstruction groups of the stratified space  $\mathcal{X}$ . For these groups we have isomorphisms

$$L_n^{BQ}(\mathcal{X}) = LT_{n-q-q'}(X, Y, Z)$$

with surgery obstruction groups  $LT_*$  of the manifold triple  $(X, Y, Z)$  (see [14] and [16]).

In the present paper we develop surgery theory for manifold triples in order to investigate splitting a homotopy equivalence along a submanifold pair. By definition, a simple homotopy equivalence  $f : M \rightarrow X$  *splits along the submanifold pair*  $(Z \subset Y)$  if it is concordant to an *s-triangulation*  $g$  of the triple  $Z \subset Y \subset X$ . We introduce groups  $LSP_*$  of obstructions to splitting a simple homotopy equivalence  $f : M \rightarrow X$  along a pair of embedded submanifolds  $(Z \subset Y) \subset X$  and describe their relations to classical obstruction groups in surgery theory. The group  $LSP_*$  is a natural straightforward generalization of the group  $LS_*$  if we consider a pair of submanifolds  $(Z \subset Y)$  instead of a submanifold  $Y$ . The  $LSP$ -groups give in a natural way an invariant for determining when elements of Wall groups are not realized by normal maps of closed manifolds. This invariant is equivalent to the pair of Hambleton's invariants  $(A$  and  $B)$  in paper [6].

The rest of the paper is organized as follows. In section 2, we recall notation, constructions and results from the literature, which will be needed in the current paper. In section 3, we construct the spectrum  $\mathbb{L}SP(X, Y, Z)$  and relate via exact sequences its homotopy groups  $LSP_*(X, Y, Z)$  to classical obstruction groups and structure sets arising from triples of manifolds. In section 4, we apply the above to obtain results when elements of Wall groups are not realized by normal maps of closed manifolds and compute some  $LSP_*$ -groups.

## 2. Preliminaries

The current paper will make significant use of constructions, ideas, and results in the papers [1], [2], [8], [13], [16], [17], [18], and [21]. This is an extension of work of A.Bak and Y.Muranov in [1] and [2]. A thread running through all of these articles is the use, due to Ranicki [17], [18], of spectra for developing the algebraic theory of surgery. In this section we recall some necessary definitions and results from these papers.

Consider a triple of topological manifolds (1.8). Let  $\xi$  denote the normal bundle of  $Y$  in  $X$  and  $F$  the square of fundamental groups in the splitting problem for the pair  $Y \subset X$ . Similarly we introduce the following bundles and squares:

- the bundle  $\eta$  and the square  $\Psi$  for the pair  $Z \subset Y$ ,
- the bundle  $\nu$  and the square  $\Phi$  for the pair  $Z \subset X$ .

Let  $U_\xi$  be the space of the normal bundle  $\xi$ . We shall assume that the space  $U_\nu$  of the normal bundle  $\nu$  is identified with the space  $V_\xi$  of the restriction of the bundle  $\xi$  to the space  $U_\eta$  of the normal bundle  $\eta$  so that  $\partial U_\nu = \partial U_\xi|_{U_\eta} \cup U_\xi|_{\partial U_\eta}$  (see [4], [15], [16], and [22]).

The conditions on the spaces of normal bundles for the manifold triple (1.8) yield a pair of manifolds with boundaries

$$(Y \setminus Z, \partial(Y \setminus Z)) \subset (X \setminus Z, \partial(X \setminus Z)) \tag{2.1}$$

where

$$\partial(Y \setminus Z) \subset \partial(X \setminus Z) \tag{2.2}$$

is a closed manifold pair. Denote by  $F_Z$  the square of fundamental groups in the splitting problem relative to boundary for the pair (2.1), and by  $F_U$  the square in the splitting problem for the pair (2.2).

For an arbitrary group  $\pi$  with orientation the surgery  $\Omega$ -spectrum  $\mathbb{L}(\pi) = \mathbb{L}(\mathbb{Z}\pi)$  is defined (see [8], [17], and [21]). Here  $\mathbb{Z}\pi$  denotes the integral group ring equipped with the involution

$$\Sigma a_g g \mapsto \Sigma a_g w(g)g^{-1}, \quad a_g \in \mathbb{Z}, g \in \pi$$

where  $w : \pi \rightarrow \{\pm 1\}$  is the orientation homomorphism. Recall that for this  $\Omega$ -spectrum we have

$$\pi_n(\mathbb{L}(\pi)) = L_n(\pi).$$

Let  $\mathbf{L}_\bullet$  denote the 1-connected cover of the spectrum  $\mathbb{L}(1)$  with  $\mathbf{L}_{\bullet 0} = G/TOP$ . For a topological space  $X$  we have the following cofibration (see [17] and [18])

$$X_+ \wedge \mathbf{L}_\bullet \rightarrow \mathbb{L}(\pi_1(X)) \rightarrow \mathbb{S}(X). \tag{2.3}$$

The homotopy long exact sequence of the cofibration (2.3) gives the algebraic surgery exact sequence of Ranicki [17]

$$\cdots \rightarrow L_{n+1}(\pi_1(X)) \rightarrow \mathcal{S}_{n+1}(X) \rightarrow H_n(X, \mathbf{L}_\bullet) \rightarrow L_n(\pi_1(X)) \rightarrow \cdots \quad (2.4)$$

with

$$\pi_{n+1}(\mathbb{S}(X)) = \mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X).$$

The left part of the exact sequence (2.4) is isomorphic to the exact sequence (1.6).

A similar result is valid for the exact sequences (1.4), (1.7), and (1.9). In particular, we have cofibrations of spectra

$$\mathbb{S}(X, Y, \xi) \rightarrow \mathbb{S}(X) \rightarrow \Sigma^{q+1}\mathbb{L}S(F), \quad (2.5)$$

$$X_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma^q\mathbb{L}P(F) \rightarrow \mathbb{S}(X, Y, \xi), \quad (2.6)$$

and

$$X_+ \wedge \mathbf{L}_\bullet \rightarrow \Sigma^{q+q'}\mathbb{L}T(X, Y, Z) \rightarrow \mathbb{S}(X, Y, Z), \quad (2.7)$$

where  $\Sigma$  denotes the suspension functor on the category of  $\Omega$ -spectra. These cofibrations generate exact sequences that contain parts which are isomorphic to the exact sequences (1.4), (1.7), and (1.9), respectively.

Recall that for an arbitrary pair  $(X, Y)$  of topological spaces equipped with orientation, a spectrum  $\mathbb{S}(X, Y)$  for the relative structure sets  $\mathcal{S}_*(X, Y)$  is defined (see [17] and [18]).

A homomorphism of oriented groups  $f : \pi \rightarrow \pi'$  induces a cofibration of  $\Omega$ -spectra

$$\mathbb{L}(\pi) \rightarrow \mathbb{L}(\pi') \rightarrow \mathbb{L}(f) \quad (2.8)$$

where  $\mathbb{L}(f)$  is the spectrum for relative  $L$ -groups of the map  $f$ .

For the manifold pair  $(X, Y)$  we have a homotopy commutative diagram of spectra (see [1], [2], [8], and [18])

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(\partial U) \rightarrow \pi_1(U)) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\ & \searrow & \downarrow & & \downarrow \\ & & \Sigma^{1-q}\mathbb{L}(\pi_1(\partial U)) & \rightarrow & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)), \end{array} \quad (2.9)$$

where the left maps are transfer maps on the spectra level, and the right horizontal maps are induced by inclusions.

The diagram (2.9) provides a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \rightarrow & \Sigma\mathbb{L}S(F) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \rightarrow & \Sigma^{1-q}\mathbb{L}(\pi_1(X \setminus Y)) & \rightarrow & \Sigma\mathbb{L}P(F) \end{array} \quad (2.10)$$

in which the horizontal rows are cofibrations. The homotopy long exact sequences of the maps from diagram (2.10) generate the diagram (1.5).

The triple (1.8) defines also on the spectra level the maps (see [16] and [22])

$$\mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{-q'+1}\mathbb{L}P(F_U) \rightarrow \Sigma^{-q'+1}\mathbb{L}P(F_Z) \quad (2.11)$$

where the first map is the transfer map, and the second map is induced by the inclusion in (2.1).

By [16] and [22] we have a cofibration

$$\mathbb{L}(\pi_1(Z)) \rightarrow \Sigma^{-q'+1}\mathbb{L}P(F_Z) \rightarrow \Sigma\mathbb{L}T(X, Y, Z) \quad (2.12)$$

where the first map is the composition of the maps in (2.11).

Consider the composition of the maps

$$\mathbb{L}P(F) \rightarrow \mathbb{L}(\pi_1(Y)) \rightarrow \mathbb{S}(Y) \rightarrow \Sigma^{q'+1}\mathbb{L}S(\Psi). \quad (2.13)$$

The first map in (2.13) follows from (2.10), the second is the map from (1.3) for the manifold  $Y$ , and the third map is the map from (2.5) for the pair  $(Y, Z)$ . By [14] and [16] we have the cofibration

$$\mathbb{L}P(F) \rightarrow \Sigma^{q'+1}\mathbb{L}S(\Psi) \rightarrow \Sigma^{q'+1}\mathbb{L}T(X, Y, Z). \quad (2.14)$$

From the cofibration (2.14), we obtain the homotopy pull-back square of spectra

$$\begin{array}{ccc} \mathbb{L}T(X, Y, Z) & \rightarrow & \Sigma^{-q'}\mathbb{L}P(F) \\ \downarrow & & \downarrow \\ \mathbb{L}P(\Psi) & \rightarrow & \Sigma^{-q'}\mathbb{L}(\pi_1(Y)) \end{array} \quad (2.15)$$

where the cofibres of the vertical maps are naturally homotopy equivalent to the spectrum  $\Sigma^{-q-q'+1}\mathbb{L}(\pi_1(X \setminus Y))$ .

Consider the commutative diagram of inclusions

$$\begin{array}{ccc} (Y \setminus Z) & \subset & (X \setminus Z) \\ \cap & & \cap \\ Y & \subset & X. \end{array} \quad (2.16)$$

The horizontal inclusions of submanifolds of codimension  $q$ , provide as in (2.10), the transfer maps fitting into the homotopy commutative diagram

$$\begin{array}{ccccccc} \mathbb{L}(\pi_1(Y \setminus Z)) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X \setminus Z)) & \rightarrow & \Sigma\mathbb{L}S(F_Z) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{L}(\pi_1(Y)) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow X) & \rightarrow & \Sigma\mathbb{L}S(F) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{L}(\pi_1(Y \setminus Z) \rightarrow (\pi_1(Y))) & \xrightarrow{trrel} & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Z) \rightarrow \pi_1(X)) & \rightarrow & \Sigma^{1+q'}\mathbb{L}NS & & \\ & & & & (2.17) & & \end{array}$$

in which the upper vertical maps are induced by the vertical maps from (2.16). The spectrum  $\mathbb{L}NS = \mathbb{L}NS(X, Y, Z)$  is the spectrum for the relative  $L$ -groups of the map  $tr^{rel}$  (see [7] and [15]) with the homotopy groups

$$LNS_n = LNS_n(X, Y, Z) = \pi_n(\mathbb{L}NS).$$

Note that the diagram (2.17) generates the following commutative diagram [15]

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \rightarrow & LS_{n-q}(FZ) & \rightarrow & L_{n-q}(\pi_1(Y \setminus Z)) & \rightarrow & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(W)) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \rightarrow & LS_{n-q}(F) & \rightarrow & L_{n-q}(\pi_1(Y)) & \rightarrow & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \rightarrow & LNS_k & \rightarrow & L_{n-q}(\pi_1(Y \setminus Z) \rightarrow \pi_1(Y)) & \rightarrow & L_n(\pi_1(W) \rightarrow \pi_1(X)) & \rightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
 & & & & & & (2.18)
 \end{array}$$

where  $k = n - q - q'$  and  $W = X \setminus Z$ .

### 3. Splitting a homotopy equivalence along a submanifold pair

For the triple of manifolds (1.8) we introduce below the spectrum  $\mathbb{L}SP(X, Y, Z)$  with homotopy groups

$$LSP_* = LSP_*(X, Y, Z) = \pi_n(\mathbb{L}SP(X, Y, Z)). \tag{3.1}$$

The groups  $LSP_*(X, Y, Z)$  are a natural straightforward generalization of the splitting obstruction groups  $LS_*(F)$  to the case when the manifold  $X$  contains a pair of embedded submanifolds  $(Z \subset Y) \subset X$  instead of first a single submanifold  $Y$ . We describe via exact sequences the relation of the groups  $LSP_*(X, Y, Z)$  to classical obstruction groups and structure sets which arise naturally for a triple of manifolds.

The bottom map in the diagram (2.15) and the commutative diagram (2.10) provide the homotopy commutative diagram of spectra

$$\begin{array}{ccc}
 \mathbb{L}P(\Psi) \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \\
 \downarrow = & \downarrow & (3.2) \\
 \mathbb{L}P(\Psi) \rightarrow & \Sigma^{1-q-q'}\mathbb{L}(\pi_1(X \setminus Y)) &
 \end{array}$$

in which the fiber of the bottom map is the spectrum  $\mathbb{L}T(X, Y, Z)$ . This follows from the pull-back property of the square (2.15). Denote by  $\mathbb{L}SP(X, Y, Z)$  the fiber



of the upper horizontal map in (3.2). We obtain the homotopy commutative diagram of spectra

$$\begin{array}{ccccc}
 \mathbb{L}P(\Psi) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \rightarrow & \Sigma\mathbb{L}SP(X, Y, Z) \\
 \downarrow = & & \downarrow & & \downarrow \\
 \mathbb{L}P(\Psi) & \rightarrow & \Sigma^{1-q-q'}\mathbb{L}(\pi_1(X \setminus Y)) & \rightarrow & \Sigma\mathbb{L}T(X, Y, Z)
 \end{array} \tag{3.3}$$

in which the right vertical map is induced by the two other vertical maps (see [20]). Note that the right square in (3.3) is a pull-back.

**Proposition 3.1** *The groups  $LSP_*(X, Y, Z)$  that are defined by (3.1) fit into the following braid of exact sequences*

$$\begin{array}{ccccccc}
 \rightarrow & L_n(C) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & LSP_{k-1} & \rightarrow \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & & LT_k(X, Y, Z) & & & \\
 & \searrow & & \searrow & & \searrow & \\
 \rightarrow & LSP_k & \longrightarrow & LP_k(\Psi) & \longrightarrow & L_{n-1}(C) & \rightarrow, \\
 & & & & & & \tag{3.4}
 \end{array}$$

where  $C = \pi_1(X \setminus Y)$ ,  $D = \pi_1(X)$ , and  $k = n - q - q'$ . The diagram (3.4) is realized on the spectra level.

*Proof:* The right square in the diagram (3.3) is a pull-back. The homotopy long exact sequences of this square provide the commutative braid of exact sequences (3.4).  $\square$

**Theorem 3.2** *There exists a commutative braid of exact sequences*

$$\begin{array}{ccccccc}
 \rightarrow & S_{n+1}(X, Y, Z) & \longrightarrow & H_n(X, \mathbf{L}_\bullet) & \longrightarrow & L_n(\pi_1(X)) & \rightarrow \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & & S_{n+1}(X) & & LT_{n-q-q'} & \\
 & \searrow & & \searrow^\alpha & & \searrow & \\
 \rightarrow & L_{n+1}(\pi_1(X)) & \longrightarrow & LSP_{n-q-q'} & \longrightarrow & S_n(X, Y, Z) & \rightarrow \\
 & & & & & & \tag{3.5}
 \end{array}$$

which is realized on the spectra level.

*Proof:* Consider the homotopy commutative square of spectra

$$\begin{array}{ccc}
 X_+ \wedge \mathbf{L}_\bullet & \rightarrow & \mathbb{L}(\pi_1(X)) \\
 \downarrow & & \downarrow = \\
 \Sigma^{q+q'}\mathbb{L}T & \rightarrow & \mathbb{L}(\pi_1(X))
 \end{array} \tag{3.6}$$

in which the upper horizontal map lies in (2.3), the left vertical map lies in (2.7), and the bottom horizontal map is the map from the diagram (3.4) on spectra level

(see [14] and [15]). The diagram (3.6) induces a map of the fibres of its horizontal maps. We obtain the homotopy commutative diagram of spectra

$$\begin{array}{ccccc}
 \Sigma^{-1}\mathbb{S}(X) & \rightarrow & X_+ \wedge \mathbf{L}\bullet & \rightarrow & \mathbb{L}(\pi_1(X)) \\
 \downarrow & & \downarrow & & \downarrow = \\
 \Sigma^{q+q'}\mathbb{L}SP & \rightarrow & \Sigma^{q+q'}\mathbb{L}T & \rightarrow & \mathbb{L}(\pi_1(X))
 \end{array}$$

in which the left square is a push-out. The homotopy long exact sequences of this square give the diagram (3.5). □

The commutative diagram (3.5) is a natural generalization of the diagram in [18, Proposition 7.2.6, iv ] to the case of a pair of submanifolds  $Z \subset Y$  in the manifold  $X$ . The map

$$\Sigma^{-1}\mathbb{S}(X) \rightarrow \Sigma^{q+q'}\mathbb{L}SP$$

induces a map

$$\alpha : \mathcal{S}_{n+1}(X) \rightarrow LSP_{n-q-q'}(X, Y, Z)$$

that on the algebraic level corresponds to taking the obstruction to splitting along the submanifold pair  $Z \subset Y$ .

Now we describe the relation of  $LSP_*$  to classical surgery obstruction groups for the triple  $(X, Y, Z)$  of manifolds (1.8).

**Theorem 3.3** *There exist braids of exact sequences*

$$\begin{array}{ccccccc}
 \rightarrow & LS_{n-q}(FZ) & \longrightarrow & LT_k(X, Y, Z) & \longrightarrow & L_n(\pi_1(X)) & \rightarrow \\
 & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\
 & & & LSP_k & & LP_k(\Phi) & \\
 & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
 \rightarrow & L_{n+1}(\pi_1(X)) & \longrightarrow & LS_k(\Phi) & \longrightarrow & LS_{n-q-1}(FZ) & \rightarrow, \\
 & & & & & & (3.7)
 \end{array}$$

$$\begin{array}{ccccccc}
 \rightarrow & LS_{n-q}(FZ) & \longrightarrow & LS_{n-q}(F) & \longrightarrow & LS_{k-1}(\Psi) & \rightarrow \\
 & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\
 & & & LSP_k & & LNS_k & \\
 & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
 \rightarrow & LS_k(\Psi) & \longrightarrow & LS_k(\Phi) & \longrightarrow & LS_{n-q-1}(FZ) & \rightarrow, \\
 & & & & & & (3.8)
 \end{array}$$

and

$$\begin{array}{ccccccc}
 \rightarrow & LS_{n-q+1}(F) & \longrightarrow & LS_k(\Psi) & \rightarrow & LT_k & \rightarrow \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & LP_{n-q+1}(F) & & & LSP_k & \\
 & \searrow & & \searrow & & \searrow & \\
 \rightarrow & LT_{k+1} & \longrightarrow & L_{n+1}(\pi_1(X)) & \longrightarrow & LS_{n-q}(F) & \rightarrow, \\
 & & & & & & (3.9)
 \end{array}$$

where  $k = n - q - q'$ . The braids (3.7), (3.8), and (3.9) are realized on the spectra level.

Proof: The natural forgetful maps (see [15] and [16])

$$\mathbb{L}T(X, Y, Z) \rightarrow \mathbb{L}P(\Phi) \rightarrow \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))$$

provide the homotopy commutative square

$$\begin{array}{ccc}
 \mathbb{L}T(X, Y, Z) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) \\
 \downarrow & & \downarrow = \\
 \mathbb{L}P(\Phi) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)).
 \end{array} \tag{3.10}$$

The square induces a map of the fibres of its horizontal maps (see [20]). Thus we obtain a homotopy commutative diagram

$$\begin{array}{ccccc}
 \mathbb{L}SP(X, Y, Z) & \rightarrow & \mathbb{L}T(X, Y, Z) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X)) \\
 \downarrow & & \downarrow & & \downarrow = \\
 \mathbb{L}S(\Phi) & \rightarrow & \mathbb{L}P(\Phi) & \rightarrow & \Sigma^{-q-q'}\mathbb{L}(\pi_1(X))
 \end{array}$$

in which the left square is a push-out (and hence a pull-back). Now, similarly to Proposition 3.1, we obtain the diagram (3.7).

The diagram (2.10) for the pair  $(Y, Z)$  provides a homotopy commutative pull-back square of spectra

$$\begin{array}{ccc}
 \mathbb{L}P(\Psi) & \rightarrow & \Sigma^{-q'}\mathbb{L}(\pi_1(Y)) \\
 \downarrow & & \downarrow \\
 \mathbb{L}(\pi_1(Z)) & \rightarrow & \Sigma^{-q'}\mathbb{L}(\pi_1(Y \setminus Z) \rightarrow \pi_1(Y)).
 \end{array} \tag{3.11}$$

There is a homotopy commutative pull-back square of spectra

$$\begin{array}{ccc}
 \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \xrightarrow{=} & \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\
 \downarrow & & \downarrow \\
 \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Z) \rightarrow \pi_1(X)) & \xrightarrow{=} & \Sigma^{-q'-q}\mathbb{L}(\pi_1(X \setminus Z) \rightarrow \pi_1(X))
 \end{array} \tag{3.12}$$

in which the vertical maps are induced by the natural inclusion. The transfer maps and diagrams (2.17) and (3.3) give the map of diagram (3.11) to diagram (3.12). The cofibers of this map of diagrams provide a homotopy commutative pull-back square of spectra (see [19])

$$\begin{array}{ccc} \Sigma \mathbb{L}SP & \rightarrow & \Sigma^{-q'+1} \mathbb{L}S(F) \\ \downarrow & & \downarrow \\ \Sigma \mathbb{L}S(\Phi) & \rightarrow & \Sigma \mathbb{L}NS. \end{array} \tag{3.13}$$

This follows from (2.9), (2.18), and (3.3). The diagram (3.8) follows from the square (3.13), similarly to the previous case.

The natural forgetful maps in the diagram (2.15)

$$\mathbb{L}T(X, Y, Z) \rightarrow \Sigma^{-q'} \mathbb{L}P(F) \rightarrow \Sigma^{-q-q'} \mathbb{L}(\pi_1(X))$$

provide the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbb{L}T(X, Y, Z) & \rightarrow & \Sigma^{-q'} \mathbb{L}P(F) & \rightarrow & \Sigma \mathbb{L}S(\Psi) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{L}T(X, Y, Z) & \rightarrow & \Sigma^{-q-q'} \mathbb{L}(\pi_1(X)) & \rightarrow & \Sigma \mathbb{L}SP(X, Y, Z), \end{array} \tag{3.14}$$

in which the rows are cofibrations, and the right vertical map is defined by [19]. Hence the right square in (3.14) is a pull-back. From this, the diagram (3.9) follows. □

**Corollary 3.4** *There exist exact sequences*

$$\begin{aligned} \dots &\rightarrow LSP_k \rightarrow LS_{n-q}(F) \rightarrow LS_{k-1}(\Psi) \rightarrow \dots, \\ \dots &\rightarrow LSP_k \rightarrow LS_k(\Phi) \rightarrow LS_{n-q-1}(FZ) \rightarrow \dots, \end{aligned}$$

and

$$\dots \rightarrow LSP_k \rightarrow LP_k(\Psi) \rightarrow L_{n-1}(\pi_1(X \setminus Y)) \rightarrow \pi_1(X) \rightarrow \dots,$$

in which the left maps are natural forgetful maps.

Now we describe some relations between the  $LSP_*$ -groups and various structure sets which arise for the triple of manifolds  $(X, Y, Z)$ .

**Theorem 3.5** *There exist braids of exact sequences*

$$\begin{array}{ccccccc} \rightarrow & \mathcal{S}_n(X) & \rightarrow & LSP_{k-1} & \rightarrow & \mathcal{S}_{l-1}(Y, Z, \eta) & \rightarrow \\ & \nearrow & & \nearrow & & \nearrow & \\ & \mathcal{S}_n(X, X \setminus Y) & & \mathcal{S}_{n-1}(X, Y, Z) & & \mathcal{S}_{l-1}(X, Y, Z) & \\ & \searrow & & \searrow & & \searrow & \\ \rightarrow & \mathcal{S}_l(Y, Z, \eta) & \rightarrow & \mathcal{S}_{n-1}(X \setminus Y) & \rightarrow & \mathcal{S}_{n-1}(X) & \rightarrow, \end{array} \tag{3.15}$$

$$\begin{array}{ccccccc}
 \rightarrow & H_l(Y, \mathbf{L}_\bullet) & \rightarrow & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) & \rightarrow & LSP_{k-1} & \rightarrow \\
 & \nearrow \searrow & & \nearrow \searrow & & \nearrow \searrow & \\
 & & LP_k(\Psi) & & \mathcal{S}_n(X, X \setminus Y) & & \\
 \rightarrow & LSP_k & \rightarrow & \mathcal{S}_l(Y, Z, \eta) & \rightarrow & H_{l-1}(Y, \mathbf{L}_\bullet) & \rightarrow, \\
 & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & \\
 & & & & & & (3.16)
 \end{array}$$

$$\begin{array}{ccccccc}
 \rightarrow & LS_l(F_Z) & \rightarrow & \mathcal{S}_n(X, Y, Z) & \rightarrow & \mathcal{S}_n(X) & \rightarrow \\
 & \nearrow \searrow & & \nearrow \searrow & & \nearrow \searrow & \\
 & & LSP_k & & \mathcal{S}_n(X, Z, v) & & (3.17) \\
 \rightarrow & \mathcal{S}_{n+1}(X) & \rightarrow & LS_k(\Phi) & \rightarrow & LS_{l-1}(F_Z) & \rightarrow, \\
 & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & \\
 & & & & & & 
 \end{array}$$

and

$$\begin{array}{ccccccc}
 \rightarrow & LS_{l+1}(F) & \rightarrow & LS_k(\Psi) & \rightarrow & \mathcal{S}_n(X, Y, Z) & \rightarrow \\
 & \nearrow \searrow & & \nearrow \searrow & & \nearrow \searrow & \\
 & & \mathcal{S}_{n+1}(X, Y, \xi) & & LSP_k & & (3.18) \\
 \rightarrow & \mathcal{S}_{n+1}(X, Y, Z) & \rightarrow & \mathcal{S}_{n+1}(X) & \rightarrow & LS_l(F) & \rightarrow, \\
 & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & \\
 & & & & & & 
 \end{array}$$

where  $l = n - q, k = n - q - q'$ . The diagrams (3.15)–(3.18) are realized on the spectra level.

Proof: The transfer map gives the commutative diagram (see [18])

$$\begin{array}{ccc}
 H_{n-q}(Y, \mathbf{L}_\bullet) & \xrightarrow{\cong} & H_n(X, X \setminus Y; \mathbf{L}_\bullet) \\
 \searrow & & \downarrow \\
 & & H_{n-1}(X \setminus Y; \mathbf{L}_\bullet).
 \end{array} \tag{3.19}$$

Consider the commutative triangle

$$\begin{array}{ccc}
 LP_{n-q-q'}(\Psi) & \rightarrow & L_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\
 \searrow & & \downarrow \\
 & & L_{n-1}(\pi_1(X \setminus Y))
 \end{array} \tag{3.20}$$

which lies in the commutative diagram (3.4).

The results of [18, Proposition 7.2.6] provide the maps from the groups in diagram (3.19) to the corresponding groups of diagram (3.20). On the spectra level the cofibres of this map give a homotopy commutative triangle of spectra

$$\begin{array}{ccc}
 \mathcal{S}(Y, Z, \eta) & \rightarrow & \Sigma^{-q} \mathcal{S}(X, X \setminus Y) \\
 \searrow & & \downarrow \\
 & & \Sigma^{-q+1} \mathcal{S}(X \setminus Y).
 \end{array} \tag{3.21}$$

By [20] the diagram (3.21) induces a homotopy commutative diagram

$$\begin{array}{ccccc} \mathbb{S}(Y, Z, \eta) & \rightarrow & \Sigma^{-q}\mathbb{S}(X, X \setminus Y) & \rightarrow & \Sigma^{q'+1}\mathbb{L}SP \\ \downarrow = & & \downarrow & & \downarrow \\ \mathbb{S}(Y, Z, \eta) & \rightarrow & \Sigma^{-q+1}\mathbb{S}(X \setminus Y) & \rightarrow & \Sigma^{-q+1}\mathbb{S}(X, Y, Z) \end{array}$$

in which the rows are cofibrations, and the right square is a pull-back. The homotopy long exact sequences of the maps of this square give the braid (3.15). In a similar way, the maps from (3.19) to (3.20) provide the pull-back square

$$\begin{array}{ccc} \Sigma^{q'}\mathbb{L}P(\Psi) & \rightarrow & \Sigma^{-q}\mathbb{L}(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) \\ \downarrow & & \downarrow \\ \mathbb{S}(Y, Z, \eta) & \rightarrow & \Sigma^{-q}\mathbb{S}(X, X \setminus Y) \end{array}$$

in which the cofibers of the vertical maps are homotopy equivalent to the spectrum  $Y_+ \wedge \mathbf{L}_\bullet$ . From this square we obtain the braid of exact sequences (3.16). The diagram (3.17) is obtained in a similar way if we consider on the spectra level the homotopy commutative triangle of the cofibers of the map from  $H_n(X, \mathbf{L}_\bullet)$  to the triangle of natural forgetful maps

$$\begin{array}{ccc} LT_{n-q-q'} & \rightarrow & LP_{n-q-q'}(\Phi) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)) \end{array} \tag{3.22}$$

which are obtained from square (3.10). We obtain diagram (3.18) in a way similar to that of diagram (3.17). To do this we have to consider the commutative triangle

$$\begin{array}{ccc} LT_{n-q-q'} & \rightarrow & LP_{n-q}(F) \\ & \searrow & \downarrow \\ & & L_n(\pi_1(X)) \end{array}$$

instead of the triangle (3.22). □

Let  $Y^{n-q} \subset X^n$  be a manifold pair with  $n - q \geq 5$  and  $q \geq 3$ . Then by [18] we have isomorphisms

$$LS_n(F) \cong L_n(\pi_1(Y)), LP_n(F) \cong L_{n+q}(\pi_1(X)) \oplus L_n(\pi_1(Y)). \tag{3.23}$$

Consider the triple of manifolds (1.8) with the conditions

$$n - q - q' \geq 5, q \geq 3, q' \geq 3. \tag{3.24}$$

By [14, Theorem 3] we have isomorphisms

$$LT_{n-q-q'} \cong L_n(\pi_1(X)) \oplus L_{n-q}(\pi_1(Y)) \oplus L_{n-q-q'}(\pi_1(Z)). \tag{3.25}$$

Next we obtain similar results for the  $LSP_*$ -groups.

**Theorem 3.6** *Suppose the triple of manifolds (1.8) satisfy the conditions (3.24). Then*

$$LSP_{n-q-q'}(X, Y, Z) \cong L_{n-q}(\pi_1(Y)) \oplus L_{n-q-q'}(\pi_1(Z)).$$

*Proof:* The result follows by considering the diagram (3.9) and using the isomorphisms (3.25) and (3.23).  $\square$

**Theorem 3.7** *Suppose the triple of manifolds (1.8) satisfy the conditions  $n-q-q' \geq 5$  and  $q \geq 3$ . Then*

$$LSP_{n-q-q'}(X, Y, Z) \cong LP_{n-q-q'}(\Psi).$$

*Proof:* We have isomorphisms

$$LS_n(FZ) \cong L_n(\pi_1(Y \setminus Z)), \quad LS_n(F) \cong L_n(\pi_1(Y)), \quad LS_n(\Phi) \cong L_n(\pi_1(Z)),$$

since  $q \geq 3$ . The isomorphism

$$LNS_n \cong L_{n+q'}(\pi_1(Y \setminus Z) \rightarrow \pi_1(Y))$$

follows from diagram (2.18), since  $q \geq 3$ . The assertion of the theorem follows now by chasing diagram (3.8).  $\square$

The Theorems 3.6 and 3.7 explain the geometrical meaning of the obstruction groups  $LSP_*$ . These groups provide obstructions to surgery on the submanifold pair  $(Y, Z)$  inside the ambient manifold  $X$ .

#### 4. Examples and applications

A pair of manifolds  $Y \subset X$  is called a *Browder-Livesay pair* if  $Y$  is an one-sided submanifold of codimension 1 and the horizontal maps in the square (1.3) are isomorphisms (see [3], [5], [6], [11], and [12]). In this case the splitting obstruction groups are denoted by

$$LN_n(\pi_1(X \setminus Y) \rightarrow \pi_1(X)) = LS_n(F).$$

Suppose the pairs of manifolds  $(X, Y)$  and  $(Y, Z)$  in the triple (1.8) are Browder-Livesay pairs. In this case  $q = q' = 1$ . Denote by  $r_p$  the map

$$L_n(\pi_1(X)) \rightarrow LSP_{n-3}(X, Y, Z)$$

in the braid (3.4). Let

$$r : L_n(\pi_1(X)) \rightarrow LS_{n-2}(F) = LN_{n-2}(\pi_1(X \setminus Y) \rightarrow \pi_1(X))$$

denote the map in the braid (1.5). The map  $r$  gives the Browder-Livesay invariant of an element  $x \in L_n(\pi_1(X))$ . If  $r(x) \neq 0$  then the element  $x$  is not realized by a normal map of closed manifolds [5].

In the paper [6] the invariants  $A$  and  $B$  were defined. The invariant  $A$  coincides with  $r$ , and the invariant  $B$  is defined on the kernel of the invariant  $A$ . The invariant  $B$  is called the second Browder-Livesay invariant [11]. It is proved in [6] that if  $B(x) \neq 0$  then the element  $x$  is not realized by a normal map of closed manifolds.

**Proposition 4.1** *Suppose the pairs of manifolds  $(X, Y)$  and  $(Y, Z)$  are Browder-Livesay pairs. Then  $r_p(x) \neq 0$  if and only if  $A(x) \neq 0$  or  $B(x) \neq 0$ .*

*Proof:* Consider the exact sequence fitting into the diagram (3.7)

$$\dots \rightarrow LT_{n-2}(X, Y, Z) \rightarrow L_n(\pi_1(X)) \xrightarrow{r_p} LSP_{n-3}(X, Y, Z) \rightarrow \dots$$

The proposition follows from this exact sequence and [15, Theorem 3]. □

**Corollary 4.2** *If  $r_p(x) \neq 0$  then the element  $x \in L_n(\pi_1(X))$  is not realized by a normal map of closed manifolds.*

Next we compute some  $LSP$ -groups. Consider the triple

$$(Z \subset Y \subset X) = (\mathbb{R}P^n \subset \mathbb{R}P^{n+1} \subset \mathbb{R}P^{n+2}) \tag{4.1}$$

of real projective spaces with  $n \geq 5$ . The orientation homomorphism

$$w : \pi_1(\mathbb{R}P^k) = \mathbb{Z}/2 \rightarrow \{\pm 1\}$$

is trivial for  $k$  odd and nontrivial for  $k$  even. We have the following table for surgery obstruction groups (see [12] and [21])

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$L_n(1)$	$\mathbb{Z}$	$0$	$\mathbb{Z}/2$	$0$
$L_n(\mathbb{Z}/2^+)$	$\mathbb{Z} \oplus \mathbb{Z}$	$0$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$L_n(\mathbb{Z}/2^-)$	$\mathbb{Z}/2$	$0$	$\mathbb{Z}/2$	$0$

The superscript "+" denotes the trivial orientation of the corresponding group and the superscript "-" denotes the nontrivial orientation. For the Browder-Livesay pairs in (4.1) we have the squares of fundamental groups

$$F^\pm = \left( \begin{array}{ccc} 1 & \rightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2^\mp & \rightarrow & \mathbb{Z}/2^\pm \end{array} \right).$$



Furthermore we have the isomorphisms (see [12, page 15] and [21])

$$LS_n(F^+) = LN_n(1 \rightarrow \mathbb{Z}/2^+) = BL_{n+1}(+) = L_{n+2}(1)$$

and

$$LS_n(F^-) = LN_n(1 \rightarrow \mathbb{Z}/2^-) = BL_{n+1}(-) = L_n(1).$$

We recall intermediate computations of the groups  $LP_*(F^\pm)$  and  $LT^*(X,Y,Z)$  from [14]. The computation of  $LP_*$ -groups for a pair  $Y \subset X$  use the braid of exact sequences (1.5) (see, also [19]). The natural map that forgets the manifold  $X$

$$LS_n(F^\pm) \rightarrow L_n(\mathbb{Z}/2^\mp)$$

coincides with the map

$$l_n : BL_n(\pm) \rightarrow L_{n-1}(\mathbb{Z}/2^\mp)$$

in [12, page 35]. Using this result and chasing the diagram (1.5) we obtain the computations (see also [13])

$$\begin{aligned} LP_n(F^+) &= \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}; \\ LP_n(F^-) &= \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2 \end{aligned}$$

for  $n = 0, 1, 2, 3 \pmod{4}$ , respectively.

Using connections between these groups and the  $LT_*(X,Y,Z)$ -groups, the following result was obtained in [14].

**Proposition 4.3** *Let  $M^{n-k}$  be a closed simply connected topological manifold. For the triple of manifolds*

$$(Z^n \subset Y^{n+1} \subset X^{n+2}) = (M^{n-k} \times \mathbb{R}P^k \subset M^{n-k} \times \mathbb{R}P^{k+1} \subset M^{n-k} \times \mathbb{R}P^{k+2})$$

with  $n \geq 5$ , we have the following results.

For  $k$  odd, the groups  $LT_n$  are isomorphic to

$$\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2$$

for  $n = 0, 1, 2, 3 \pmod{4}$ , respectively.

For  $k$  even,  $LT_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$  and  $LT_1 \cong \mathbb{Z}/2$ . The groups  $LT_3$  and  $LT_2$  fit into an exact sequence

$$0 \rightarrow LT_3 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow LT_2 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

We apply these results to compute  $LSP_*$ -groups in the situation above.

**Theorem 4.4** *Under assumptions of the Proposition 4.3 we have the following results.*

*For  $k$  odd, the groups  $LSP_n$  are isomorphic to*

$$\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2$$

*for  $n = 0, 1, 2, 3 \pmod{4}$ , respectively.*

*For  $k$  even, we have isomorphisms  $LSP_0 \cong LSP_1 \cong \mathbb{Z}/2$ . The groups  $LSP_3$  and  $LSP_2$  fit into an exact sequence*

$$0 \rightarrow LSP_3 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow LSP_2 \rightarrow 0.$$

*Proof:* Consider the case when  $k$  is odd. From [14] we conclude that all the maps  $LT_n \rightarrow LP_{n+1}(F^+)$  are epimorphisms. Now it is easy to describe the maps  $LP_n(F^+) \rightarrow L_{n+1}(\mathbb{Z}/2^+)$  in diagram (1.5). For  $n = 1 \pmod{4}$  and  $n = 2 \pmod{4}$  these maps are isomorphisms  $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ . For  $n = 0 \pmod{4}$  the map is trivial since the group  $L_1(\mathbb{Z}/2^+)$  is trivial. The map

$$\mathbb{Z} = LP_3(F^+) \rightarrow L_0(\mathbb{Z}/2^+) = \mathbb{Z} \oplus \mathbb{Z}$$

is an inclusion on a direct summand. The image of this map coincides with the image of the map  $L_0(1) \rightarrow L_0(\mathbb{Z}/2^+)$  that is induced by the inclusion  $1 \rightarrow \mathbb{Z}/2^+$ . This follows from the commutative triangle

$$\begin{array}{ccc} & \mathbb{Z} & \\ & \parallel & \\ & LP_3(F^+) & \\ & \cong \nearrow \quad \searrow & \\ L_0(1) & \xrightarrow{\text{mono}} & L_0(\mathbb{Z}/2^+) \\ \parallel & & \parallel \\ \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

in diagram (1.5). From diagram (3.4) we obtain an exact sequence

$$\dots \rightarrow LT_n \xrightarrow{\tau} L_{n+2}(\mathbb{Z}/2^+) \rightarrow LSP_{n-1} \rightarrow LT_{n-1} \rightarrow \dots$$

The map  $\tau$  is the composition

$$LT_n \rightarrow LP_{n+1}(F^+) \rightarrow L_{n+2}(\mathbb{Z}/2^+)$$

of maps that we already know. Now we can compute the map  $\tau$ . It is trivial for  $n = 3$ , an isomorphism  $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$  for  $n = 1$ , an epimorphism  $\mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$  with kernel  $\mathbb{Z}$  for  $n = 0$ , and a homomorphism  $\mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  with kernel  $\mathbb{Z}/2$  and cokernel  $\mathbb{Z}$  for  $n = 2$ . The result for  $k$  odd follows now from the exact sequence (4.24). The case of  $k$  even is obtained in a similar way.  $\square$

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