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PALINDROME PRESENTATIONS OF RATIONAL KNOTS

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ABSTRACT

We give explicit palindrome presentations of the groups of rational knots, i.e. presentations with relators which read the same forwards and backwards. This answers a question posed by Hilden, Tejada and Toro in 2002. Using such presentations we obtain simple alternative proofs of some classical results concerning the Alexander polynomial of all rational knots and the character variety of certain rational knots. Finally, we derive a new recursive description of the $\mathrm{SL}(2,\mathbb{C})$ character variety of twist knots.

Keywords: Manifold; branched covering; rational knot; Alexander polynomial; cyclic presentation; split extension; character variety; palindrome; knot group.

Mathematics Subject Classification 2000: Primary 57M12, 57M25; Secondary 57M50

1. Introduction

The aim of this paper is to obtain special presentations of the groups of all rational knots whose relators are *palindromes*. This means that the relators read the same forwards and backwards as words in the generators. The existence of palindrome presentations for tunnel number one knots was already known by [14]. Let $K_{\alpha/\beta}$ denote the *rational knot* or 2*-bridge knot* determined by the pair of coprime odd integers (α, β) with $1 \leq \beta < \alpha$. The major contribution in this paper is to explicitly find the palindrome presentation for the group of $K_{\alpha/\beta}$ in terms of the parameters α and β. This answers a question posed by Hilden, Tejada and Toro in [14]. To state our main result, we need some notation. Set $e_i = (-1)^{[i\beta^{-1}/\alpha]}$, where β^{-1} is the inverse of β in $\mathbb{Z}_{2\alpha}$ and $[x]$ denotes the integral part of x. By [20, Chap. 6, Lemma 9.2], we have $e_{\alpha-j} = e_j$, for any $j = 1, \ldots, \alpha - 1$, hence the integer

$$s_{\alpha-1} = \sum_{j=1}^{\alpha-1} e_j = 2 \sum_{j=1}^{\frac{\alpha-1}{2}} e_j$$

is even. Furthermore, it was proved in [25] that $s_{\alpha-1}$ is the signature of $K_{\alpha/\beta}$. Through the paper, the letter $\mathbf{x}$ in the center of a relator has been made bold to make it clearer that the relator is palindrome.

Theorem 1.1. *The group of the rational knot $K_{\alpha/\beta}$, α and β odd and coprime, $1 \leq \beta < \alpha$, admits the palindrome presentation*

$$\pi(\alpha/\beta) = \langle \theta, x \colon\ u(\alpha/\beta) = 1 \rangle,$$

where

$$\begin{aligned} u(\alpha/\beta) = {} & (\theta^{-\frac{s_{\alpha-1}}{2}} x \theta^{e_1} x^{-1} \theta^{e_2} x \theta^{e_3} \cdots x^{-\epsilon} \theta^{e_{(\alpha-1)/2}}) \mathbf{x}^{\epsilon} \\ & \times (\theta^{e_{(\alpha-1)/2}} x^{-\epsilon} \cdots \theta^{e_3} x \theta^{e_2} x^{-1} \theta^{e_1} x \theta^{-\frac{s_{\alpha-1}}{2}}) \end{aligned}$$

and $\epsilon = +1$ (respectively, -1) if $(\alpha-1)/2$ is even (respectively, odd).

In the appendix, we give explicit presentations for particular families of rational knots.

Then, we use the palindrome presentation in Theorem 1.1 to give a very simple proof of a classical result of Minkus [20, Chap. 6, Lemma 11.1] on the Alexander polynomial of all rational knots. Furthermore, these presentations allow us to obtain a recursive description of the $\mathrm{SL}(2,\mathbb{C})$ character variety for twist knots. This confirms the utility of the palindromic presentations of knot groups. We remark that the term "character variety" is used in different forms in the literature. Here the character variety includes characters of both abelian and non-abelian representations of a knot group π in $\mathrm{SL}(2,\mathbb{C})$ (or $\mathrm{PSL}(2,\mathbb{C})$), and denote it by $X(\pi)$ (or $\hat{X}(\pi)$). Of course, there is a natural map from $X(\pi)$ to $\hat{X}(\pi)$ since $\mathrm{SL}(2,\mathbb{C})$ is a 2-fold covering of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$. We use the symbol $X^{\mathrm{irr}}(\pi)$ to denote the union of all algebraic components of the character variety $X(\pi)$ containing characters of irreducible (and hence non-abelian) representations. People working in hyperbolic geometry are interested in representations of a finitely generated group G in $\mathrm{PSL}(2,\mathbb{C})$ because this last group can be identified with the group of orientation preserving isometries of the hyperbolic 3-space. Gonzáles-Acuña and Montesinos explained in [10] under what conditions representations into $\mathrm{PSL}(2,\mathbb{C})$ lift to $\mathrm{SL}(2,\mathbb{C})$. They postulated the existence of a homomorphism ψ from a free product of free abelian groups S_i into G such that ψ induces an epimorphism on the second homology with $\mathbb{Z}_2$ coefficients. Then a homomorphism $\gamma : G \to \mathrm{PSL}(2, \mathbb{C})$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if, for each i, $\gamma \circ \psi(S_i)$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see [10, Lemma 2.3 and Corollary 2.4]). Examples of such G are the link groups, where the S_i can be taken to be peripheral subgroups and $\psi|_{S_i} : S_i \to G$ the inclusions. Furthermore, we recall that a 2-bridge knot group is torsion free (see [22, Lemma 3]). Recently, Hoste and Shanahan have obtained in [17] a recursive description of the $\mathrm{PSL}(2, \mathbb{C})$ character variety for twist knots. In Sec. 4, we shall point out how it is related with our description in Theorem 1.2, and in what sense they differ.

Theorem 1.2. *The character variety of the rational knot $K_{\alpha/\beta}$, where $\frac{\alpha}{\beta} = \frac{4n+1}{2n+1}$, $n \geq 1$, (which is equivalent to the twist knot K_{2n} defined by the fraction $\frac{\alpha}{\beta'} = \frac{4n+1}{2}$) is given by the equation $q_{2,n} = 0$ in $\mathbb{C}^2$. The polynomials $q_{0,n}$, $q_{1,n}$ and $q_{2,n}$ satisfy the following recursive formulas*:

$$\begin{aligned}
q_{0,n} &= X_0 q_{0,n-1} + X_1 q_{1,n-1} + X_2 q_{2,n-1},\\
q_{1,n} &= Y_0 q_{0,n-1} + Y_1 q_{1,n-1} + Y_2 q_{2,n-1},\\
q_{2,n} &= Z_0 q_{0,n-1} + Z_1 q_{1,n-1} + Z_2 q_{2,n-1},
\end{aligned}$$

where

$$\begin{cases}
X_0 = \frac{3}{2}y^2z^2 + \frac{1}{2}y^4 - 2y^2 - \frac{3}{2}yz^4 + z^4 - 2z^2 - y^3z^2 + 2yz^2 + \frac{1}{2}y^2z^4 + 1,\\
X_1 = \frac{5}{4}y^3z^2 + \frac{1}{4}y^5 - \frac{3}{2}y^3 + 2yz^4 - 4yz^2 - z^4 - \frac{5}{4}y^2z^4 + 2y + 2z^2 - \frac{1}{2}y^4z^2\\
\qquad + y^2z^2 + \frac{1}{4}y^3z^4,\\
X_2 = \frac{1}{4}y^2z^3 - \frac{1}{2}y^3z^3 + \frac{1}{4}y^4z - \frac{3}{2}y^2z + \frac{1}{2}y^3z - \frac{3}{4}yz^5 + \frac{1}{2}z^5 - 2z^3 + \frac{5}{2}yz^3\\
\qquad + 2z - 2yz + \frac{1}{4}y^2z^5,
\end{cases}$$

$$\begin{cases}
Y_0 = 2yz^2 + y^3 - 2y - 2y^2z^2 + 3z^2 + yz^4 - 2z^4,\\
Y_1 = 2y^2z^2 + \frac{1}{2}y^4 - 2y^2 + \frac{3}{2}yz^2 - 3z^2 - y^3z^2 - 2yz^4 + 2z^4 + \frac{1}{2}y^2z^4 + 1,\\
Y_2 = -y^2z^3 + \frac{1}{2}y^3z - 2yz + y^2z + \frac{7}{2}z^3 - 2z + \frac{1}{2}yz^5 - z^5,
\end{cases}$$

$$\begin{cases}
Z_0 = -yz^3 + y^2z + 2z^3 - 2z - yz,\\
Z_1 = 2yz^3 + \frac{1}{2}y^3z - 2yz - 2z^3 - \frac{1}{2}y^2z^3 - \frac{1}{2}y^2z + 2z,\\
Z_2 = \frac{1}{2}y^2z^2 - \frac{1}{2}yz^4 + z^4 - 3z^2 + \frac{1}{2}yz^2 + 1.
\end{cases}$$

Here we start with $q_{0,0} = (1/2)y$, $q_{1,0} = 1$ and $q_{2,0} = 0$.

For $n = 1$, we immediately get $q_{2,1} = z(y-2)(y^2 - y(z^2-1) + z^2 - 1) = 0$ which gives the equation of the character variety of the figure eight knot $K_{5/3}$, which is originally due to Whittemore:

Corollary 1.3 [27, Theorem 1]. *Let* $\pi = \pi(5/3) = \langle \theta, x \colon u(5/3) = 1 \rangle$ *with*

$$u(5/3) = x\theta^{-1}x^{-1}\theta\mathbf{x}\theta x^{-1}\theta^{-1}x$$

be a palindrome presentation of the group of Listing's knot (i.e. the figure-eight knot $K_{5/3}$*). Then, the character variety* $X^{\mathrm{irr}}(\pi)$ *is given by*

$$X^{\mathrm{irr}}(\pi) = \{(y, z) \in \mathbb{C}^2 \,:\, y^2 - y(z^2 - 1) + z^2 - 1 = 0\},$$

where y *and* z *are complex numbers such that* $y \neq 2$ *and* $y \neq z^2 - 2$.

For $n = 2$ we obtain the following result:

Corollary 1.4. *Let* $\pi = \pi(9/5) = \langle \theta, x \,:\, u(9/5) = 1 \rangle$ *with*

$$u(9/5) = (x\theta^{-1}x^{-1}\theta)^2\mathbf{x}(\theta x^{-1}\theta^{-1}x)^2$$

be a palindrome presentation of the group of $K_{9/5}$*. Then, the polynomial*

$$y^4 + y^3(1 - 2z^2) + y^2(z^2 - 1)(z^2 + 3) + y(z^2 - 1)(2 - 3z^2) + 2z^4 - 4z^2 + 1 = 0,$$

where y *and* z *are complex numbers such that* $y \neq 2$ *and* $y \neq z^2 - 2$*, is the defining polynomial for the character variety* $X^{\mathrm{irr}}(\pi)$ *of* $K_{9/5}$.

As a final application we give an alternative proof of a result obtained in [12] on the character variety of $K_{7/3}$.

Theorem 1.5 [12, Theorem 2.5, case $p = 7$]. *Let* $\pi = \pi(7/3) = \langle \theta, x : u(7/3) = 1 \rangle$ *with*

$$u(7/3) = \theta^{-1}x\theta x^{-1}\theta^{-1}x\theta\mathbf{x}^{-\mathbf{1}}\theta x\theta^{-1}x^{-1}\theta x\theta^{-1}$$

be a palindrome presentation of the group of the knot $K_{7/3}$*. Then, the polynomial*

$$y^3 - y^2(2z^2 - 1) + y(z^2 - 1)(z^2 + 2) - 2z^4 + 4z^2 - 1 = 0,$$

where y *and* z *are complex numbers such that* $y \neq 2$ *and* $y \neq z^2 - 2$*, is the defining polynomial for the character variety* $X^{\mathrm{irr}}(\pi)$ *of* $K_{7/3}$.

We now describe the contents of the next sections in this paper. In Sec. 2, we give the basic definitions on cyclic presentations of groups, and discuss their connection with the topology of closed 3-manifolds. Then, we recall in detail the Minkus construction of a polyhedral scheme for the cyclic covering $M_n(K_{\alpha/\beta})$ of the 3-sphere branched over the 2-bridge knot $K_{\alpha/\beta}$. Theorem 2.1 in this section states that the fundamental group of $M_n(K_{\alpha/\beta})$ admits a cyclic presentation whose associated polynomial is essentially the Alexander polynomial of $K_{\alpha/\beta}$. In Sec. 3, we introduce the split extension of a cyclically presented group, and prove Theorem 1.1. As an application, we reprove the formula for the Alexander polynomial of $K_{\alpha/\beta}$, first obtained by Minkus in [20]. Then, we compare the results with those given by Fukuhara in [9]. In Sec. 4, we consider character varieties for certain rational knots.

We prove Theorem 1.2 which gives a recursive description of the $\mathrm{SL}(2,\mathbb{C})$ character variety of the twist knots. Then, we write explicitly the equations of the character varieties of $K_{5/3}$ and $K_{9/5}$ as special cases. This reproves a classical result for the character variety of the figure eight knot, given by Whittemore in [27] (see also [11, 13, and 19, Chap. 4, Exercise 4.4.4]), and a result on the character variety of $K_{7/3}$, given by Hilden, Lozano and Montesinos in [12].

2. Preliminaries

2.1. *Cyclic presentations*

Cyclically presented groups constitute a class of groups which are very interesting from a topological point of view. There are many connections between such groups and cyclic branched coverings of knots (see, for example, [4] and references therein). Here we briefly recall some basic definitions. Let F_n denote the free group of rank n on free generators $x_1, \ldots, x_n$. Let $\theta : F_n \to F_n$ be the automorphism of order n such that $\theta(x_i) = x_{i+1}$, $i = 1, \ldots, n$, where the indices are taken mod n. For any reduced word w in F_n, let us consider the quotient group $G_n(w) = F_n/R$, where R denotes the normal closure of the set $\{w, \theta(w), \ldots, \theta^{n-1}(w)\}$ in F_n. A group G is said to have a *cyclic presentation* if G is isomorphic to $G_n(w)$ for some w and n. Of course, θ induces an automorphism of $G_n(w)$ which determines an action of the cyclic group $\mathbb{Z}_n = \langle \theta : \theta^n = 1\rangle$ on $G_n(w)$. The *polynomial* associated with $G_n(w)$ is defined to be $f_w(t) := \sum_{i=0}^{n-1} a_i t^i$, where a_i is the exponent sum of x_i in w. More information on cyclic presentations can be found for example in [18].

2.2. *Schubert diagrams of rational knots*

Let K be a knot in the oriented 3-sphere $\mathbb{S}^3$. A closed 3-manifold M is called an *n-fold cyclic branched covering* of K if M is the n-fold cyclic covering of $\mathbb{S}^3$ branched over K (see, for example, [23, Chap. 10, Sec. C]). We now recall some classical results on rational knots. As it is well-known, any *rational knot* or *2-bridge knot* $K_{\alpha/\beta}$ is completely determined by a pair of coprime odd integers (α, β) with $1 \le \beta < \alpha$. A standard diagram for $K_{\alpha/\beta}$ is given by the *Schubert normal form* (see [3, Chap. 12, Sec. A]) which can be obtained as follows (see Fig. 1): each bridge is divided into α segments, and the points are numbered from 0 to $2\alpha - 1$ modulo 2α. The point labeled 0 in the top bridge is joined to the crossing point labeled β in the bottom bridge.

Next, one meets the top bridge at 2β, and then meets the bottom bridge at 3β. This is to be repeated until one reaches the point numbered $\alpha\beta \equiv \alpha \pmod{2\alpha}$ of the bottom bridge. Similarly, one starts from the point labeled 0 in the bottom bridge and meets the top bridge at β, and next meets the bottom bridge at 2β, and then meets the top bridge at 3β. This is to be repeated until one reaches $\alpha\beta \equiv \alpha$ (mod 2α) of the top bridge. Of course, we complete the construction to obtain a planar regular projection of the knot. We assign a coefficient $e_j = e_j(\alpha, \beta)(= \pm 1)$

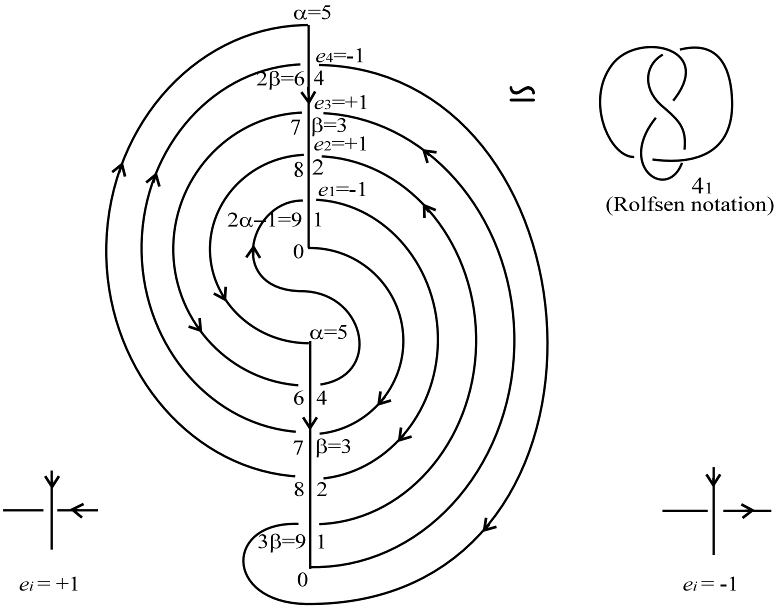

Fig. 1. The 2-bridge knot $K_{\alpha/\beta} = K_{5/3}$ (i.e. the figure eight-knot)

to each crossing point labeled j, $1 \leq j \leq \alpha - 1$, of the top bridge, according to the rule illustrated in Fig. 1. In particular, we have $e_j(\alpha, \beta) = (-1)^{[j\beta^{-1}/\alpha]}$, where β^{-1} is the inverse of the element β in $\mathbb{Z}_{2\alpha}$ and $[x]$ denotes the integral part of x (see [20, Chap. 6, Lemma 9.1]). The group $\pi(\alpha/\beta) = \pi_1(\mathbb{S}^3 \backslash K_{\alpha/\beta})$ of $K_{\alpha/\beta}$ has the finite presentation $\pi(\alpha/\beta) \cong \langle x, y : Ly = xL \rangle$, where $L = y^{e_1} x^{e_2} \cdots y^{e_{\alpha-2}} x^{e_{\alpha-1}}$ (see, for example [20, Chap. 6, Proof of Lemma 11.1]). The free calculus of Fox [8] (see also [6]) was used in [20] to compute the Alexander polynomial $\Delta_{\alpha/\beta}(t)$ of $K_{\alpha/\beta}$ from the above presentation. More precisely, we have

$$\Delta_{\alpha/\beta}(t) = \sum_{r=0}^{\alpha-1} (-1)^r t,$$

where $s_0 = 0$ and $s_r = s_r(\alpha, \beta) = \sum_{j=1}^{r} e_j(\alpha, \beta)$.

2.3. *The Minkus construction*

Let $M_n(K_{\alpha/\beta})$ denote the n-fold cyclic covering of $K_{\alpha/\beta}$. Then, the fundamental group of $M_n(K_{\alpha/\beta})$ has a cyclic presentation which depends on the coefficients s_r (see [20], for different cyclic presentations of such a class of manifolds, see also [1, 5, 21]). This was proved in [20] by using a polyhedral scheme which represents $M_n(K_{\alpha/\beta})$. A standard way to construct closed 3-manifolds consists of the pairwise identification of oppositely oriented boundary faces of a triangulated 3-ball. The quotient space is a closed 3-manifold if and only if its Euler characteristic vanishes (see [24]). A polyhedral representation of the n-fold cyclic branched coverings of two-bridge knots and links was given by Minkus in [20] (other combinatorial descriptions can be found in [1, 5, 21]). Now, we recall the Minkus construction of a polyhedral scheme for the manifold $M_n(K_{\alpha/\beta})$, where $K_{\alpha/\beta}$ is the 2-bridge knot of type (α, β), α and β odd and coprime, and $1 \leq \beta < \alpha$. Let us consider the unit 3-ball B^3 in $\mathbb{R}^3$, and set $\mathbb{S}^2 = \partial B^3$. On the bounding 2-sphere $\mathbb{S}^2$ draw n equally spaced great semicircles joining the north pole $N \equiv (0, 0, 1)$ to the south pole $S \equiv (0, 0, -1)$. This decomposes $\mathbb{S}^2$ into n congruent lunes. Subdivide each semicircle into α equal segments by drawing $\alpha - 1$ equally spaced vertices on each semicircle. Then, each lune can be looked at as a curvilinear 2α sided polygon on $\mathbb{S}^2$. Now, bisect each lune by drawing a great circle arc c_i inside the lune joining the vertex which is β segments down from N on each semicircle to the vertex β segments up from S on the next clockwise semicircle. The construction is illustrated in Fig. 2, where $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ and $S = \infty$. We have decomposed $\mathbb{S}^2$ into $2n$ congruent regions R_i and R'_i, $i = 1, \ldots, n$. Each region can be looked at as a curvilinear $\alpha + 1$ sided polygon. The regions R_i surround N in clockwise order, and the regions R'_i are obtained from R_i by moving counterclockwise to the adjacent lune and then shifting from the northern to the southern hemisphere of $\mathbb{S}^2$. The quotient 3-complex $M_n(K_{\alpha/\beta})$ is constructed from B^3 by pasting R_i to R'_i in such a way that $c_i \subset \partial R_i$

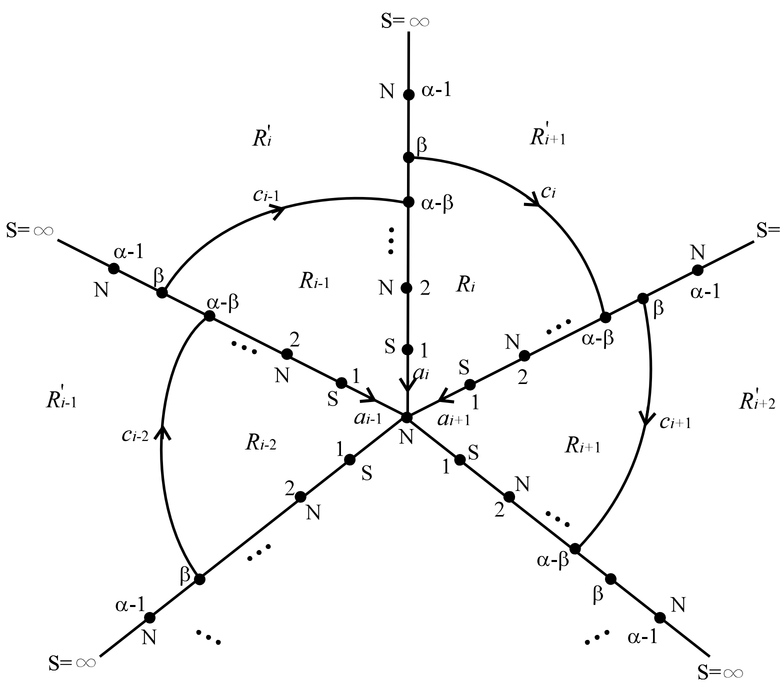

Fig. 2. The Minkus polyhedral schemata for the manifolds $M_n(K_{\alpha/\beta})$.

is identified with $c_{i-1} \subset \partial R_i'$. To show that $M_n(K_{\alpha/\beta})$ is a closed 3-manifold it suffices to calculate its Euler characteristic which must be zero. In fact, there are: one 3-cell, n faces (arising from the n pairs $R_i \equiv R_i'$), two vertices N and S, and $n+1$ 1-cells $a_1, \ldots, a_n, c$. Then we have $\chi(M_n(K_{\alpha/\beta})) = 2 - (n+1) + n - 1 = 0$, as requested. Here $a_1, \ldots, a_n$ (in clockwise order) denote n distinct oriented edges surrounding the central vertex N and all pointing toward the center of the diagram. Furthermore, the edge c in $M_n(K_{\alpha/\beta})$ arises from the identification of the equivalent edges c_i, $i = 1, \ldots, n$. Figure 2 can also be used to obtain a nice presentation for $\pi_1(M_n(K_{\alpha/\beta}))$ which corresponds to a spine (and hence to a Heegaard diagram) of the considered manifold.

If S is taken as the base point, then the closed paths $x_i = a_i c^{-1}$, $i = 1, \ldots, n$, give rise to a set of generators for $\pi_1(M_n(K_{\alpha/\beta}))$. Each 2-cell $R_i \equiv R_i'$ gives the following relation between the generators $x_1, \ldots, x_n$:

$$x_i x_{i+s_1}^{-1} x_{i+s_2} x_{i+s_3}^{-1} \cdots x_{i+s_{\alpha-2}}^{-1} x_{i+s_{\alpha-1}},$$

where $s_r = s_r(\alpha/\beta) = \sum_{i=1}^{r} e_i(\alpha, \beta)$ and subscripts are reduced mod n. This is a cyclic presentation with defining word

$$w = w(\alpha/\beta) := x_0 x_{s_1}^{-1} x_{s_2} x_{s_3}^{-1} \cdots x_{s_{\alpha-2}}^{-1} x_{s_{\alpha-1}},$$

hence $\pi_1(M_n(K_{\alpha/\beta})) \cong G_n(w) = G_n(\alpha/\beta)$. The polynomial associated with $G_n(\alpha/\beta)$ is precisely

$$f_{\alpha/\beta}(t) = 1 - t^{s_1} + t^{s_2} - t^{s_3} + \cdots - t^{s_{\alpha-2}} + t^{s_{\alpha-1}} = \sum_{r=0}^{\alpha-1} (-1)^r t^{s_r},$$

where $s_0 = 0$ by convention. Minkus shows in [20] that $f_{\alpha/\beta}(t)$ is in fact the Alexander polynomial of $K_{\alpha/\beta}$. The following result collects the statements of [20, Theorems 7, 10 and Lemma 11.1].

Theorem 2.1 (Minkus [20]). (a) *The manifold $M_n(K_{\alpha/\beta})$, constructed above, is the n-fold cyclic covering of the 3-sphere branched over the 2-bridge knot $K_{\alpha/\beta}$, α, β odd and coprime, and $1 \le \beta < \alpha$.*

(b) *The fundamental group of $M_n(K_{\alpha/\beta})$ admits the cyclic presentation*

$$G_n(\alpha/\beta) = \langle x_1, \ldots, x_n \colon x_i x_{i+s_1}^{-1} x_{i+s_2} x_{i+s_3}^{-1} \cdots x_{i+s_{\alpha-2}}^{-1} x_{i+s_{\alpha-1}} = 1 \\ (i = 1, \ldots, n; \textit{subscripts} \bmod n) \rangle$$

which corresponds to a spine of the manifold.

(c) *The polynomial $f_{\alpha/\beta}(t)$ associated with $G_n(\alpha/\beta)$ coincides, up to units in $\mathbb{Z}[t, t^{-1}]$, with the Alexander polynomial $\Delta_{\alpha/\beta}(t)$ of $K_{\alpha/\beta}$.*

3. Palindrome Presentations

Let $G_n(w)$ be a cyclically presented group with generators $x_1, \ldots, x_n$ and defining word $w = w(x_1, \ldots, x_n)$. The automorphism θ such that $\theta(x_i) = x_{i+1}$ (subscripts mod n) determines an action of the cyclic group $\mathbb{Z}_n = \langle \theta \colon \theta^n = 1 \rangle$ on $G_n(w)$. Let $H_n(v)$ denote the *split extension group* of $G_n(w)$ by $\mathbb{Z}_n = \langle \theta \colon \theta^n = 1 \rangle$. The group $H_n(v)$ has a presentation of the form $H_n(v) = \langle \theta, x \colon \theta^n = 1,\ v(\theta, x) = 1 \rangle$, where $v(\theta, x)$ is obtained from the word w by substituting any generator x_i with $\theta^i x \theta^{-i}$, and $x = x_0 = x_n$. We see that the group $G_n(\alpha/\beta)$ has the cyclic automorphism θ which sends x_i to x_{i+1} (subscripts mod n). This automorphism corresponds to the rotational symmetry (also denoted by θ) of the polyhedral scheme of $M_n(K_{\alpha/\beta})$ such that $\theta(R_i) = R_{i+1}$ and $\theta(R'_i) = R'_{i+1}$, where the indices are taken mod n. Let us consider the split extension group $H_n(\alpha/\beta)$ of $G_n(\alpha/\beta)$ by the cyclic group $\mathbb{Z}_n = \langle \theta : \theta^n = 1 \rangle$. Then $H_n(\alpha/\beta)$ has a finite presentation with generators θ and $x = x_0 = x_n$, and relations $\theta^n = 1$, and

$$\begin{aligned} v(\alpha/\beta) &= x\theta^{s_1} x^{-1} \theta^{s_2 - s_1} x \theta^{s_3 - s_2} x^{-1} \theta^{s_4 - s_3} \cdots \theta^{s_{\alpha-2} - s_{\alpha-3}} x^{-1} \theta^{s_{\alpha-1} - s_{\alpha-2}} x \theta^{-s_{\alpha-1}} \\ &= x\theta^{e_1} x^{-1} \theta^{e_2} x \theta^{e_3} x^{-1} \theta^{e_4} \cdots \theta^{e_{\alpha-2}} x^{-1} \theta^{e_{\alpha-1}} x \theta^{-s_{\alpha-1}} = 1 \end{aligned}$$

obtained from $w(\alpha/\beta)$ by substituting any x_i with $\theta^i x \theta^{-i}$. Then we have

Theorem 3.1. *Let $\mathcal{O}_n(\alpha/\beta)$ be the 3-dimensional orbifold whose underlying space is $\mathbb{S}^3$ and whose singular set is the 2-bridge knot $K_{\alpha/\beta}$ with branching index n. The split extension group $H_n(\alpha/\beta) = \langle \theta, x : \theta^n = 1, v(\alpha/\beta) = 1 \rangle$, where*

$$v(\alpha/\beta) = x\theta^{e_1} x^{-1} \theta^{e_2} x \theta^{e_3} x^{-1} \theta^{e_4} \cdots \theta^{e_{\alpha-2}} x^{-1} \theta^{e_{\alpha-1}} x \theta^{-s_{\alpha-1}}$$

is isomorphic to the fundamental group of the orbifold $\mathcal{O}_n(\alpha/\beta)$. The group presentation $\langle \theta, x : v(\alpha/\beta) = 1 \rangle$ defines the knot group $\pi(\alpha/\beta)$ of $K_{\alpha/\beta}$, and the generator θ corresponds to a meridian of the knot.

Proof. The group $G_n = G_n(\alpha/\beta)$ can be embedded as a normal subgroup of index n in $H_n = H_n(\alpha/\beta)$. The map $\phi \colon G_n \to H_n$, defined by $\phi(x_i) = \theta^i x \theta^{-i}$, gives the desired embedding. Furthermore, G_n is isomorphic to the normal closure of x in H_n, and there is a short exact sequence

$$1 \to G_n \xrightarrow{\phi} H_n \to \mathbb{Z}_n \to 1,$$

where $\mathbb{Z}_n = \langle \theta \colon \theta^n = 1 \rangle$. Let us consider the quotient space obtained from $M_n = M_n(K_{\alpha/\beta})$ under the action of the rotational symmetry (also denoted by θ) of the polyhedral scheme of M_n. This quotient space is a 3-dimensional orbifold, denoted by $\mathcal{O}_n = \mathcal{O}_n(\alpha/\beta)$, whose underlying space is $\mathbb{S}^3$ and whose singular set is the 2-bridge knot $K_{\alpha/\beta}$ with branching index n (use Theorem 2.1). The n-fold covering map $M_n \to \mathcal{O}_n$ induces a group embedding $G_n \triangleleft \Omega_n$, where $\Omega_n = \Omega_n(\alpha/\beta)$ denotes the fundamental group of $\mathcal{O}_n$. In particular, we have $[\Omega_n : G_n] = n$, and Ω_n fits in a short exact sequence $1 \to G_n \to \Omega_n \to \mathbb{Z}_n \to 1$, where $\mathbb{Z}_n$ is generated by the rotational symmetry θ. Now, Five Lemma implies the isomorphism $H_n \cong \Omega_n$. $\square$

Theorem 3.1 implies Theorem 1.1 because $e_{\alpha-j} = e_j$, for every $j = 1, \ldots, \alpha - 1$, and $s_{\alpha-1}$ is even.

Proof of Theorem 2.1(c). We will use the free calculus of Fox to compute the Alexander polynomial $\Delta_{\alpha/\beta}(t)$ of $K_{\alpha/\beta}$ from the presentation of $\pi(\alpha/\beta)$ given in Theorem 3.1. Let $\pi^{\mathrm{ab}} = \pi^{\mathrm{ab}}(\alpha/\beta)$ ($\cong \mathbb{Z}$) denote the abelianized group of $\pi = \pi(\alpha/\beta)$, and $\eta\colon \mathbb{Z}\pi \to \mathbb{Z}\pi^{ab} = \mathbb{Z}[t, t^{-1}]$ the abelianization map between the group rings. Then $\eta(x) = 0$ and $\eta(\theta) = t$. Recall that the free derivatives of Fox satisfy the characteristic properties $\frac{\partial(uv)}{\partial x} = \frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$ and $\frac{\partial u^{-1}}{\partial x} = -u^{-1}\frac{\partial u}{\partial x}$, for u and $v \in F_n$. In our case, the Alexander polynomial $\Delta_{\alpha/\beta}(t)$ of $K_{\alpha/\beta}$ is equal to $\eta(\frac{\partial v(\alpha/\beta)}{\partial x})$, where $v(\alpha/\beta)$ is the word defining $H_n(\alpha/\beta)$ in Theorem 3.1. So we get

$$\begin{aligned}
\frac{\partial v(\alpha/\beta)}{\partial x} &= 1 + x\frac{\partial}{\partial x}(\theta^{e_1}x^{-1}\theta^{e_2}x\theta^{e_3}x^{-1}\cdots\theta^{e_{\alpha-2}}x^{-1}\theta^{e_{\alpha-1}}x\theta^{-s_{\alpha-1}}) \\
&= 1 + x\theta^{s_1}\frac{\partial}{\partial x}(x^{-1}\theta^{e_2}x\theta^{e_3}x^{-1}\cdots\theta^{e_{\alpha-2}}x^{-1}\theta^{e_{\alpha-1}}x\theta^{-s_{\alpha-1}}) \\
&= 1 + x\theta^{s_1}\left(-x^{-1} + x^{-1}\frac{\partial}{\partial x}(\theta^{e_2}x\theta^{e_3}x^{-1}\cdots\theta^{e_{\alpha-2}}x^{-1}\theta^{e_{\alpha-1}}x\theta^{-s_{\alpha-1}})\right) \\
&= 1 - \theta^{s_1} + \theta^{s_2}\frac{\partial}{\partial x}(x\theta^{e_3}x^{-1}\cdots\theta^{e_{\alpha-2}}x^{-1}\theta^{e_{\alpha-1}}x\theta^{-s_{\alpha-1}}) \\
&\;\;\vdots \\
&= 1 - \theta^{s_1} + \theta^{s_2} - \cdots + \theta^{s_{\alpha-3}} - \theta^{s_{\alpha-2}} + \theta^{s_{\alpha-1}}\frac{\partial}{\partial x}(x\theta^{-s_{\alpha-1}}) \\
&= \sum_{r=0}^{\alpha-1}(-1)^r\theta^{s_r},
\end{aligned}$$

since $\frac{\partial}{\partial x}(x\theta^{-s_{\alpha-1}}) = 1$ and $s_0 = 0$. Thus we have $\Delta_{\alpha/\beta}(\theta) := \eta(\frac{\partial v(\alpha/\beta)}{\partial x})$ by [8], and hence $\Delta_{\alpha/\beta}(t) = f_{\alpha/\beta}(t)$, as required. $\square$

Now, we compare the Minkus formula for the Alexander polynomial of $K_{\alpha/\beta}$ with the results obtained by Fukuhara in [9]. There he introduced the following functions:

$$\begin{aligned}
\epsilon_i(\alpha,\beta) &:= (-1)^{[i\beta/\alpha]}, \\
\mu(\alpha,\beta) &:= \sum_{i=1}^{\alpha-1}\epsilon_i(\alpha,\beta), \\
\nu_k(\alpha,\beta) &:= 1 + \sum_{i=1}^{\alpha-1}\epsilon_{k+i}(\alpha,\beta),
\end{aligned}$$

for $0 < i, k < \alpha$. Then, Fukuhara gave an explicit formula for the normalized Alexander polynomial, $D = D_{\alpha/\beta}(t)$ say, of $K_{\alpha/\beta}$. This polynomial satisfies the properties $D(1) = 1$ and $D(t^{-1}) = D(t)$.

Theorem 3.2 [9, Theorem 1.2(1)]. *For the 2-bridge knot $K_{\alpha/\beta}$, we have*

$$D_{\alpha/\beta}(t) = \frac{1}{2}(t^{-\mu/2} + t^{\mu/2}) - \frac{1}{4}(t^{-1/2} - t^{1/2})\sum_{i=1}^{\alpha-1}(-1)^i\epsilon_i(t^{-\nu_i/2} - t^{\nu_i/2}).$$

The following result relates the Minkus formula to that of Fukuhara.

Theorem 3.3.

$$D_{\alpha/\beta}(t) = t^{-\frac{s_{\alpha-1}}{2}} \Delta_{\alpha/\beta}(t) = \sum_{r=0}^{\alpha-1} (-1)^r t^{(2s_r - s_{\alpha-1})/2}.$$

Proof. As usual α and β are relatively prime odd integers, $1 \leq \beta < \alpha$. Let β' be the unique solution of the congruence $\beta\beta' \equiv 1 \pmod{2\alpha}$. The fraction α/β gives the same 2-bridge knot as the fraction α/β'. Of course we have

$$\epsilon_i(\alpha, \beta') = (-1)^{[i\beta'/\alpha]} = (-1)^{[i\beta^{-1}/\alpha]} = e_i(\alpha, \beta)$$

and

$$\mu(\alpha, \beta') = \sum_{i=1}^{\alpha-1} \epsilon_i(\alpha, \beta') = \sum_{i=1}^{\alpha-1} e_i(\alpha, \beta) = s_{\alpha-1}(\alpha, \beta).$$

Using a 2-bridge diagram for $K_{\alpha/\beta'}$ we find that $\pi_1(\mathbb{S}^3 \backslash K_{\alpha/\beta'})$ has a Wirtinger presentation $\langle x, y \colon L_1 y = x L_1 \rangle$, where $L_1 = y^{\epsilon_1} x^{\epsilon_2} \cdots y^{\epsilon_{\alpha-2}} x^{\epsilon_{\alpha-1}}$. Therefore we have the Wirtinger presentation for $K_{\alpha/\beta}$: $\pi_1(\mathbb{S}^3 \backslash K_{\alpha/\beta}) \cong \langle x, y \colon Ly = xL \rangle$, where $L = L(\alpha, \beta) = y^{e_1} x^{e_2} \cdots y^{e_{\alpha-2}} x^{e_{\alpha-1}} = L_1(\alpha, \beta') = L_1$. Let $\delta \colon \mathbb{Z}F(x, y) \to \mathbb{Z}[t, t^{-1}]$ be an abelianization map such that $\delta(x) = t$ and $\delta(y) = t$. Since $\langle x, y \colon Ly = xL \rangle$ is the Wirtinger presentation of $K_{\alpha/\beta}$, we know that $\delta(\frac{\partial L^{-1} x L y^{-1}}{\partial x})$ is the Alexander polynomial of $K_{\alpha/\beta}$. By [20, Proof of Lemma 11.1] and [9, Proof of Lemma 4.1 and formula (4.2)], we obtain

$$\begin{aligned}
\Delta_{\alpha/\beta}(t) &= \delta\left(\frac{\partial L^{-1} x L y^{-1}}{\partial x}\right) = t^{-s_{\alpha-1}} \left(1 + (t-1)\delta\left(\frac{\partial L}{\partial x}\right)\right) \\
&= t^{-s_{\alpha-1}} \left(1 + (t-1)\delta\left(\frac{\partial L_1}{\partial x}\right)\right) = t^{-s_{\alpha-1}} t^{\mu/2} D_{\alpha/\beta}(t) \\
&= t^{-\frac{s_{\alpha-1}}{2}} \Delta_{\alpha/\beta}(t),
\end{aligned}$$

where $\mu = \mu(\alpha, \beta') = s_{\alpha-1}$. $\square$

4. Character Varieties

To study hyperbolic structures on the complement of a knot in $\mathbb{S}^3$, it is natural to consider the representations, up to conjugation, of the knot group in $SL(2, \mathbb{C})$. The set of conjugacy classes of both abelian and nonabelian representations turns out to be a closed algebraic set, called the *character variety* of the knot [7]. It is the set of roots of complex variable polynomials. For 2-bridge knots $K_{\alpha/\beta}$ the character variety is determined by a polynomial in two complex variables, and the computation was done by using a recursion procedure (see [2, 3, 26]). As observed in [12], it would be preferable to have an explicit formula in the variables α and β. Such a formula seems to exist if β is fixed. Explicit computations were done in [12] for

$\beta = 3$. Further results on the recursive calculation of the $\mathrm{PSL}(2,\mathbb{C})$ character variety and the A-polynomial of certain 2-bridge knots can be found in [15–17]. In this section, we shall make various computations, by using our palindrome presentation of a rational knot. For this, we need some results from [14]. Let $\mathcal{M}(2,\mathbb{C})$ be the vector space of all complex number square matrices of order 2. Let $\mathbf{i}$ denote the linear transformation of $\mathcal{M}(2,\mathbb{C})$ defined by $\mathbf{i}\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) := \left(\begin{smallmatrix} t & -y \\ -z & x \end{smallmatrix}\right)$. Then $\mathbf{i}$ is an anti-involution, and has eigenvalues 1 and -1 with corresponding eigenspaces V_1 (*scalars*) and V_2 (*trace zero matrices*) of complex dimension 1 and 3, respectively:

$$V_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{C} \right\} \cong \mathbb{C} \quad V_2 = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x, y, z \in \mathbb{C} \right\}.$$

Decompose $\mathcal{M}(2,\mathbb{C})$ into the direct sum of scalars plus trace zero matrices. Then for any $X \in \mathcal{M}(2,\mathbb{C})$, we can write $X = X^+ + X^-$, where $X^+ = \frac{1}{2}(X + \mathbf{i}(X))$, $X^- = \frac{1}{2}(X - \mathbf{i}(X))$, X^+ is scalar and X^- has trace zero. For $A, B \in \mathcal{M}(2,\mathbb{C})$ we set $A^+ = a$, $B^+ = b$, and $(A^-B^-)^+ = c$, where $a, b, c \in \mathbb{C}$. It is easy to verify the following identities (see [14]):

(1) $\mathbf{i}(A^-) = -A^-$.
(2) $A^-B^- + B^-A^- = 2c$.
(3) If $A, B \in SL(2,\mathbb{C})$, then $(A^-)^2 = a^2 - 1$ and $(B^-)^2 = b^2 - 1$.
(4) $\mathbf{i}(A) = a - A^-$.
(5) If $A \in SL(2,\mathbb{C})$, then $\mathbf{i}(A) = A^{-1} = a - A^-$.

For the traces, we have $\mathrm{tr}(A) = 2a$, $\mathrm{tr}(B) = 2b$, $\mathrm{tr}(AB) = 2(ab + c)$ and $\mathrm{tr}(AB^{-1}) = 2(ab - c)$.

Theorem 4.1 [14, Lemma 6.4 and Theorem 6.5]. *If A and B are any elements of $SL(2,\mathbb{C})$, and $W = W(A,B)$ is a word in A, B, A^{-1}, B^{-1}, then*

$$W = p_1 I_2 + p_2 A^- + p_3 B^- + p_4 A^- B^-,$$

where $p_i = p_i(a,b,c) \in \mathbb{Z}[a,b,c]$ for every $i = 1, \ldots, 4$. If W is a palindrome in $SL(2,\mathbb{C})$, then the term A^-B^- disappears, that is, $p_4(a,b,c) \equiv 0$.

Suppose that we have a knot group defined by a presentation $\pi = \langle \theta, x \colon w = 1 \rangle$, where w is a palindrome. To get a representation in $SL(2,\mathbb{C})$, we consider the correspondences $x \to A$, $\theta \to B$, and $w \to I_2$. Therefore, $W(A,B) = p_1 I_2 + p_2 A^- + p_3 B^- = I_2$ gives $p_1 = 1$ and $p_2 = p_3 = 0$. As it was shown in [14], $p_1 = 1$ follows from the other two equations. So we have:

Theorem 4.2 [14, Theorem 6.8]. *If π is a 2-generator knot group defined by a palindrome word, then its character variety of all nonabelian representations into $SL(2,\mathbb{C})$ is an affine algebraic subset of $\mathbb{C}^3$ given by*

$$\{(a,b,c) \in \mathbb{C}^3 \colon p_2(a,b,c) = 0,\ p_3(a,b,c) = 0,\ c^2 \neq (a^2-1)(b^2-1)\}.$$

For our computations, we need the following lemma.

Lemma 4.3. *If $W = W(A, B)$ is a palindrome of length n in $SL(2, \mathbb{C})$, espressed as*

$$W = q_0^{(n)} I_2 + q_1^{(n)} A^- + q_2^{(n)} B^-,$$

then $W' = A^\epsilon W A^\epsilon$ and $W'' = B^\epsilon W B^\epsilon$, $\epsilon = \pm 1$, are palindromes of length at most $n + 2$ with coefficients

$$W' \begin{cases} q_0^{(n+2)} = (2a^2 - 1)q_0^{(n)} + 2\epsilon a(a^2 - 1)q_1^{(n)} + 2\epsilon ac q_2^{(n)}, \\ q_1^{(n+2)} = 2\epsilon a q_0^{(n)} + (2a^2 - 1)q_1^{(n)} + 2c q_2^{(n)}, \\ q_2^{(n+2)} = q_2^{(n)}, \end{cases}$$

$$W'' \begin{cases} q_0^{(n+2)} = (2b^2 - 1)q_0^{(n)} + 2\epsilon bc q_1^{(n)} + 2\epsilon b(b^2 - 1)q_2^{(n)}, \\ q_1^{(n+2)} = q_1^{(n)}, \\ q_2^{(n+2)} = 2\epsilon b q_0^{(n)} + 2c q_1^{(n)} + (2b^2 - 1)q_2^{(n)}. \end{cases}$$

Proof. Replace A^ϵ, $\epsilon = \pm 1$, by $a + \epsilon A^-$, multiply out and use the identities above. □

Using the palindrome presentations for rational knots of the form $\alpha/\beta = \frac{4n+1}{2n+1}$, $n \geq 1$, we can now obtain a recursive formula which describes the SL$(2, \mathbb{C})$ character variety of such knots. First we note that these rational knots are the twist knots K_{2n} depicted in Fig. 3 (see, for example, [3, Chap. 15, Sec. A]).

In fact, the diagram in Fig. 3 determines the continued fraction $\alpha/\beta' = [2n; -2] = 2n + \frac{1}{2} = \frac{4n+1}{2}$. So, for example, $[2; -2]$ gives the figure eight knot $K_{5/2}$, and $[4; -2]$ represents the knot $K_{9/2}$. Recall that two 2-bridge knots with fractions α/β and α/β' are ambient isotopic if and only if $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$ (see, for example [3, Chap. 12, Theorem 12.6]). Thus, the fraction $\alpha/\beta' = \frac{4n+1}{2}$ gives the same 2-bridge knot as the fraction $\alpha/\beta = \frac{4n+1}{2n+1}$.

Application of Lemma 4.3 (use also $c = b(a - 1)$) yields Theorem 1.2 on the recursive calculation of the SL$(2, \mathbb{C})$ character variety of twist knots K_{2n}. Hoste and Shanahan obtained in [17] a recursive description of the PSL$(2, \mathbb{C})$ character

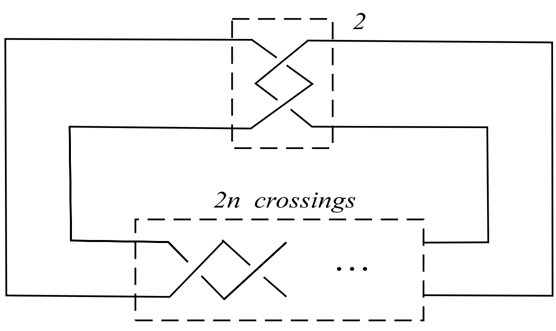

Fig. 3. The twist knot K_{2n}.

variety for the twist knot K_{2m}, $m < 0$, defined by the fraction $(4m-1)/2$ (that is, the mirror image of K_{2n}, $n = -m$, defined by $(4n+1)/2$). Their result is based on the Riley polynomials $r_m(x,z) \in \mathbb{Z}[x,z]$ which can be defined recursively by

$$r_{m+1}(x,z) - t(x,z)r_m(x,z) + r_{m-1}(x,z) = 0,$$

where $t(x,z) = 2 + 2x - 2z - xz + z^2$ and with initial conditions $r_0(x,z) = 1$ and $r_1(x,z) = z - 1$ (see also [22]). Now [17, Lemma 1] states that $(x-z)r_m(x,z)$ is the defining polynomial of the $\mathrm{PSL}(2,\mathbb{C})$ character variety of K_{2m}. In particular, for the figure-eight knot and $K_{9/5}$ one gets the polynomials $r_{-1}(x,z) = z^2-(x+3)z+2x+3$ and $r_{-2}(x,z) = z^4-(2x+5)z^3+(x^2+9x+11)-(4x^2+15x+12)z+4x^2+10x+5$, respectively. Our recursive description in Theorem 1.2 (see also Corollary 1.3 and Corollary 1.4) of the $\mathrm{SL}(2,\mathbb{C})$ character variety of K_{2n}, $n = -m > 0$, is significantly different from the $\mathrm{PSL}(2,\mathbb{C})$ version given in [17].

Proof of Theorem 1.2. A palindrome presentation for the rational knot of the form $\alpha/\beta = \frac{4n+1}{2n+1}$ has defining word $u(\alpha/\beta) = (x\theta^{-1}x^{-1}\theta)^n\mathbf{x}(\theta x^{-1}\theta^{-1}x)^n$. Sending x and θ to A and B, respectively, the palindrome $u(\alpha/\beta)$ gives the relation

$$R(A,B) = (AB^{-1}A^{-1}B)^nA(BA^{-1}B^{-1}A)^n = I_2$$

in $SL(2,\mathbb{C})$, which is equivalent to

$$AB^{-1} = (A^{-1}BAB^{-1})^nB^{-1}(A^{-1}BAB^{-1})^{-n}.$$

Thus, AB^{-1} is conjugate to B^{-1}, hence $c = b(a-1)$. For $n \geq 1$, we set

$$(AB^{-1}A^{-1}B)^nA(BA^{-1}B^{-1}A)^n = q_{0,n}I_2 + q_{1,n}A^- + q_{2,n}B^-,$$

where $q_{i,n} \in \mathbb{Z}[a,b,c]$, $i = 0,1,2$. This gives the recursive equation

$$\begin{aligned} &q_{0,n}I_2 + q_{1,n}A^- + q_{2,n}B^- \\ &\quad = AB^{-1}A^{-1}B(q_{0,n-1}I_2 + q_{1,n-1}A^- + q_{2,n-1}B^-)BA^{-1}B^{-1}A. \end{aligned}$$

Now the result follows by using Lemma 4.3, where $\mathrm{tr}(A) = y$ and $\mathrm{tr}(B) = z$. $\square$

To illustrate a special case of Theorem 1.2, we determine the character variety of the figure eight knot $K_{5/3}$, which is originally due to Whittemore [27]. For $n = 1$, Theorem 1.2 gives $q_{2,1} = z(y-2)(y^2-y(z^2-1)+z^2-1) = 0$ from which we obtain the equation in Corollary 1.3.

Proof of Corollary 1.3. Sending x and θ to A and B, respectively, gives the relation $R(A,B) = AB^{-1}A^{-1}BABA^{-1}B^{-1}A = I_2$ in $SL(2,\mathbb{C})$. Hence, we get $AB^{-1} = (A^{-1}BA)B^{-1}(A^{-1}BA)^{-1}$, that is, AB^{-1} is conjugate with B^{-1}. This implies that $\mathrm{tr}(AB^{-1}) = \mathrm{tr}(B^{-1})$, so $c = b(a-1)$. The relation $R(A,B) = I_2$ is equivalent to $A^{-1}BABA^{-1} = BA^{-2}B$ which has palindrome words on both sides. Starting from $A = a + A^-$ (hence $q_0^{(1)} = a$, $q_1^{(1)} = 1$ and $q_2^{(1)} = 0$) and applying Lemma 4.3 twice, we have

$$A^{-1}BABA^{-1} = q_0^{(5)}I_2 + q_1^{(5)}A^- + q_2^{(5)}B^-,$$

where

$$\begin{cases} q_0^{(5)} = -4a^3 + 3a - 8ab^2 + 2b^2 + 8a^2b^2, \\ q_1^{(5)} = 4a^2 - 8ab^2 + 4b^2 - 1, \\ q_2^{(5)} = 4ab - 2b. \end{cases}$$

Starting from $A^{-2} = (a - A^-)^2 = 2a^2 - 1 - 2aA^-$ (hence $\overline{q}_0^{(2)} = 2a^2 - 1$, $\overline{q}_1^{(2)} = -2a$ and $\overline{q}_2^{(2)} = 0$) and applying Lemma 4.3, we obtain

$$BA^{-2}B = \overline{q}_0^{(4)} I_2 + \overline{q}_1^{(4)} A^- + \overline{q}_2^{(4)} B^-,$$

where

$$\begin{cases} \overline{q}_0^{(4)} = 4ab^2 - 2a^2 - 2b^2 + 1, \\ \overline{q}_1^{(4)} = -2a, \\ \overline{q}_2^{(4)} = 4ab - 2b. \end{cases}$$

Thus, $q_2^{(5)} = \overline{q}_2^{(4)}$ is an identity, $\overline{q}_0^{(4)} - q_0^{(5)} = (a-1)(q_1^{(5)} - \overline{q}_1^{(4)})$, and $q_1^{(5)} = \overline{q}_1^{(4)}$ gives the equation $4a^2 - 8ab^2 + 4b^2 + 2a - 1 = 0$, which becomes the equation from the statement, by setting $2a = y$ and $2b = z$. The condition $c^2 \neq (a^2 - 1)(b^2 - 1)$, where $c = b(a-1)$, is a consequence of the irreducibility of the representation. It gives $a \neq 1$ and $a \neq 2b^2 - 1$, so $y \neq 2$ and $y \neq z^2 - 2$. □

Remark. Using a variable substitution of $\theta = u$ and $x = v^{-1}u$ (hence $v = \theta x^{-1}$) in the palindrome $u(5/3)$, one obtains the word $R(u,v) = v^{-1}u^{-1}vuv^{-1}uvu^{-1}v^{-1}u$ considered in [27]. The correspondences $x \to A$, $\theta \to B$, $u \to U$ and $v \to V$ in $SL(2,\mathbb{C})$ imply $\mathrm{tr}(B) = \mathrm{tr}(U) = z$ and $y = \mathrm{tr}(A) = \mathrm{tr}(V^{-1}U) = z^2 - w$, since $\mathrm{tr}(U) = \mathrm{tr}(V) = z$ and $\mathrm{tr}(UV) = w$ in [27]. Substituting the formula $y = z^2 - w$ in the above equation of $X^{\mathrm{irr}}(\pi)$ yields the equation $w^2 - w(z^2+1) + 2z^2 - 1 = 0$ as in [27, Theorem 1].

For $n = 2$, Theorem 1.2 gives the Corollary 1.4.

As a final application, we give a very quick proof of a result from [12, Theorem 2.5, case $p = 7$].

Proof of Theorem 1.5. By sending x and θ to A and B, respectively, gives the relation $R(A,B) = B^{-1}ABA^{-1}B^{-1}AB\mathbf{A}^{-1}BAB^{-1}A^{-1}BAB^{-1} = I_2$ in $SL(2,\mathbb{C})$. Hence, $AB^{-1} = (B^{-1}ABA^{-1}B^{-1}A)B^{-1}(B^{-1}ABA^{-1}B^{-1}A)^{-1}$, that is, AB^{-1} is conjugate to B^{-1}. So we obtain $c = b(a-1)$. The relation $R(A,B) = I_2$ is equivalent to $B^{-1}AB\mathbf{A}^{-1}BAB^{-1} = AB^{-1}A^{-1}\mathbf{B}^2A^{-1}B^{-1}A$ which has palindrome words on both sides. Starting from $A^{-1} = a - A^-$ (hence $q_0^{(1)} = a$, $q_1^{(1)} = -1$ and $q_2^{(1)} = 0$)

and applying Lemma 4.3 three times, we get

$$B^{-1}ABA^{-1}BAB^{-1} = q_0^{(7)} I_2 + q_1^{(7)} A^- + q_2^{(7)} B^-,$$

where

$$\begin{cases} q_0^{(7)} = -16b^4 + 8ab^2 - 16a^2b^2 + 4a^3 + 8b^2 - 3a + 16ab^4, \\ q_1^{(7)} = 8ab^2 - 4a^2 - 4b^2 + 1, \\ q_2^{(7)} = 16b^3 - 4ab - 16ab^3 + 8a^2b - 4b. \end{cases}$$

Starting from $B^2 = (b + B^-)^2 = 2b^2 - 1 + 2bB^-$ (hence $\overline{q}_0^{(2)} = 2b^2 - 1$, $\overline{q}_1^{(2)} = 0$ and $\overline{q}_2^{(2)} = 2b$) and applying Lemma 4.3 three times, we get

$$AB^{-1}A^{-1}B^2A^{-1}B^{-1}A = \overline{q}_0^{(8)} I_2 + \overline{q}_1^{(8)} A^- + \overline{q}_2^{(8)} B^-,$$

where

$$\begin{cases} \overline{q}_0^{(8)} = 24a^2b^2 + 8a^4 - 8a^2 - 48ab^4 + 16b^4 - 8b^2 - 32a^3b^2 + 16ab^2 \\ \qquad\quad + 32a^2b^4 + 1, \\ \overline{q}_1^{(8)} = 16ab^2 + 8a^3 - 4a + 12b^2 + 32ab^4 - 32a^2b^2 - 32b^4, \\ \overline{q}_2^{(8)} = -16ab^3 + 8a^2b + 16b^3 - 4b - 4ab. \end{cases}$$

Thus, $q_2^{(7)} = \overline{q}_2^{(8)}$ is an identity, $\overline{q}_0^{(8)} - q_0^{(7)} = (a-1)(-q_1^{(7)} + \overline{q}_1^{(8)})$, and $q_1^{(7)} = \overline{q}_1^{(8)}$ gives the equation $8ab^2 + 4a^2 + 16b^2 + 8a^3 - 4a + 32ab^4 - 32a^2b^2 - 32b^4 - 1 = 0$, which becomes the equation from the statement by setting $2a = y$ and $2b = z$. □

Remark. Using a variable substitution of $\theta = u$ and $x = v^{-1}u$ in the palindrome $u(7/3)$, one obtains the word $R(u, v) = \rho u \rho^{-1} v^{-1}$, where $\rho = uvu^{-1}v^{-1}uv$, which was considered in [12]. The correspondences $x \to A$, $\theta \to B$, $u \to U$ and $v \to V$ in $SL(2, \mathbb{C})$ imply $\mathrm{tr}(B) = \mathrm{tr}(U) = z$ and $y = \mathrm{tr}(A) = \mathrm{tr}(V^{-1}U) = z^2 - w$, since $\mathrm{tr}(U) = \mathrm{tr}(V) = z$ and $\mathrm{tr}(UV) = w$ in [12]. Substituting the formula $y = z^2 - w$ in the above equation of $X^{\mathrm{irr}}(\pi)$ yields the polynomial $w^3 - w^2 - 2w + 1 - z^2(w^2 - 3w + 2)$ obtained in [12].

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Appendix

As announced in Sec. 1, we write here explicit palindromic presentations for several rational knots.

α/β	palindrome $u(\alpha/\beta)$
$\frac{2n+1}{1}$	$\theta^{-n}(x\theta x^{-1}\theta)^{n/2}\mathbf{x}(\theta x^{-1}\theta x)^{n/2}\theta^{-n}$ $\quad n$ even
$\frac{2n+1}{1}$	$\theta^{-n}(x\theta x^{-1}\theta)^{\frac{n-1}{2}}x\theta\mathbf{x}^{-1}\theta x(\theta x^{-1}\theta x)^{\frac{n-1}{2}}\theta^{-n}$ $\quad n$ odd
$\frac{4n+1}{2n+1}$	$(x\theta^{-1}x^{-1}\theta)^n\mathbf{x}(\theta x^{-1}\theta^{-1}x)^n$
$\frac{4n-1}{4n-3}$	$(\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1})^{n/2}\mathbf{x}(x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1})^{n/2}$ $\quad n$ even
$\frac{4n-1}{4n-3}$	$(\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1})^{\frac{n-1}{2}}\theta^{-1}x\theta\mathbf{x}^{-1}$
	$\theta x\theta^{-1}(x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1})^{\frac{n-1}{2}}$ $\quad n$ odd
$\frac{6n+1}{3}$	$\theta^{-n}(x\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta)^{n/2}\mathbf{x}$
	$(\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta x)^{n/2}\theta^{-n}$ $\quad n$ even
$\frac{6n+1}{3}$	$\theta^{-n}(x\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta)^{\frac{n-1}{2}}x\theta x^{-1}\theta^{-1}x\theta\mathbf{x}^{-1}$
	$\theta x\theta^{-1}x^{-1}\theta x(\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta x)^{\frac{n-1}{2}}\theta^{-n}$ $\quad n$ odd
$\frac{6n-1}{3}$	$\theta^{-(n-1)}(x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta)^{n/2}\theta^{-1}\mathbf{x}$
	$\theta^{-1}(\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta^{-1}x)^{n/2}\theta^{-(n-1)}$ $\quad n$ even
$\frac{6n-1}{3}$	$\theta^{-(n-1)}(x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta x^{-1}\theta)^{\frac{n-1}{2}}x\theta^{-1}x^{-1}\theta\mathbf{x}$
	$\theta x^{-1}\theta^{-1}x(\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta^{-1}x)^{\frac{n-1}{2}}\theta^{-(n-1)}$ $\quad n$ odd
$\frac{10n+1}{5}$	$\theta^{-n}\big((x\theta x^{-1}\theta^{-1})^2x\theta x^{-1}\theta(x\theta^{-1}x^{-1}\theta)^2\big)^{n/2}\mathbf{x}$
	$\big((\theta x^{-1}\theta^{-1}x)^2\theta x^{-1}\theta x(\theta^{-1}x^{-1}\theta x)^2\big)^{n/2}\theta^{-n}$ $\quad n$ even
$\frac{10n+1}{5}$	$\theta^{-n}\big((x\theta x^{-1}\theta^{-1})^2x\theta x^{-1}\theta(x\theta^{-1}x^{-1}\theta)^2\big)^{\frac{n-1}{2}}(x\theta x^{-1}\theta^{-1})^2x\theta\mathbf{x}^{-1}$
	$\theta x(\theta^{-1}x^{-1}\theta x)^2\big((\theta x^{-1}\theta^{-1}x)^2\theta x^{-1}\theta x(\theta^{-1}x^{-1}\theta x)^2\big)^{\frac{n-1}{2}}\theta^{-n}$ $\quad n$ odd
$\frac{10n-1}{5}$	$\theta^{-(n-1)}\big((x\theta^{-1}x^{-1}\theta)^2(x\theta x^{-1}\theta^{-1})^2x\theta x^{-1}\theta\big)^{n/2}\theta^{-1}\mathbf{x}\theta^{-1}$
	$\big(\theta x^{-1}\theta x(\theta^{-1}x^{-1}\theta x)^2(\theta x^{-1}\theta^{-1}x)^2\big)^{n/2}\theta^{-(n-1)}$ $\quad n$ even
$\frac{10n-1}{5}$	$\theta^{-(n-1)}\big((x\theta^{-1}x^{-1}\theta)^2(x\theta x^{-1}\theta^{-1})^2x\theta x^{-1}\theta\big)^{\frac{n-1}{2}}(x\theta^{-1}x^{-1}\theta)^2\mathbf{x}$
	$(\theta x^{-1}\theta^{-1}x)^2\big(\theta x^{-1}\theta x(\theta^{-1}x^{-1}\theta x)^2(\theta x^{-1}\theta^{-1}x)^2\big)^{\frac{n-1}{2}}\theta^{-(n-1)}$ $\quad n$ odd
$\frac{10n-3}{5}$	$\theta^{-n}(x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta)^{n/2}\theta^{-1}x\theta\mathbf{x}^{-1}$
	$\theta x\theta^{-1}(\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x)^{n/2}\theta^{-n}$ $\quad n$ even
$\frac{10n-3}{5}$	$\theta^{-n}(x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta)^{\frac{n-1}{2}}x\theta x^{-1}\theta x\theta^{-1}\mathbf{x}^{-1}$
	$\theta^{-1}x\theta x^{-1}\theta x(\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x)^{\frac{n-1}{2}}\theta^{-n}$ $\quad n$ odd
$\frac{10n+3}{5}$	$\theta^{-(n-1)}(x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta)^{n/2}x\theta^{-1}\mathbf{x}^{-1}$
	$\theta^{-1}x(\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x)^{n/2}\theta^{-(n-1)}$ $\quad n$ even
$\frac{10n+3}{5}$	$\theta^{-(n-1)}(x\theta^{-1}x^{-1}\theta^{-1}x\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta)^{\frac{n+1}{2}}$
	$\theta^{-1}x\theta^{-1}x^{-1}\theta^{-1}x\theta\mathbf{x}^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x\theta^{-1}$
	$(\theta x^{-1}\theta x\theta x^{-1}\theta^{-1}x\theta^{-1}x^{-1}\theta x\theta x^{-1}\theta x\theta^{-1}x^{-1}\theta^{-1}x)^{\frac{n+1}{2}}\theta^{-(n-1)}$ $\quad n$ odd

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