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S^1-BOTT FUNCTIONS ON MANIFOLDS

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We study S^1-Bott functions on compact smooth manifolds. In particular, we investigate S^1-invariant Bott functions on manifolds with circle action.

1. Introduction

Let M^n be a compact closed manifold of dimension at least 3. We study the S^1-Bott functions on M^n. Separately, we investigate S^1-invariant Bott functions on M^{2n} with semifree circle action that have finitely many fixed points. The aim of this paper is to find the exact values of the minimum numbers of singular circles of some indices of S^1-invariant Bott functions on M^{2n}.

A more flexible object closely related to the S^1-Bott function on a manifold M^n is the decomposition of a round handle of M^n. In turn, to study the round-handle decomposition of M^n, we use a diagram, i.e., a graph that carries information on handles.

2. S^1-Bott Functions

Let M^n be a smooth manifold, let $f: M^n \to [0,1]$ be a smooth function, and let $x \in M^n$ be one of its critical points. Consider the Hessian $\Gamma_x(f): T_x \times T_x \to \mathbf{R}$ at this point. Recall that the index of the Hessian is the maximum dimension of T_x for which $\Gamma_x(f)$ is negative definite. The index of $\Gamma_x(f)$ is called the index of the critical point x, and the corank of $\Gamma_x(f)$ is called the corank of x. Suppose that the set of critical points of f forms a disjoint union of smooth submanifolds K_j^i whose dimensions do not exceed $n-1$. A connected critical submanifold $K_{j_0}^{i_0}$ is called *nondegenerate* if the Hessian is nondegenerate on subspaces orthogonal to $K_{j_0}^{i_0}$ (i.e., has the corank equal to $n - i_0$) at every point $x \in K_{j_0}^{i_0}$.

Definition 2.1. *A mapping $f: M^n \to [0,1]$ is called a Bott function if all critical points of it form nondegenerate critical submanifolds that do not intersect the boundary of M^n.*

Consider the following important example of Bott functions:

Definition 2.2. *A mapping $f: M^n \to [0,1]$ is called an S^1-Bott function if all critical points of it form nondegenerate critical circles.*

Note that S^1-Bott functions do not exist on any smooth manifold [12]. S^1-Bott functions were studied and used by many authors [1–7, 9, 11, 14]. The following theorem can be found in [8, 11]:

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Theorem 2.1. *Let M^n be a smooth closed manifold, let $f: M^n \to [0,1]$ be an S^1-Bott function, and let $\gamma \subset M^n$ be its critical circle. Then there is a system of coordinates in a neighborhood of γ of one of the following types:*

1. *Trivial $\nu: S^1 \times D^{n-1}(\varepsilon) \to M^n$, where $D^{n-1}(\varepsilon)$ is a disk of radius ε, $\nu(S^1 \times 0) = \gamma$, and*

$$f\big(\nu(\theta, x)\big) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{n-1}^2 \quad \textit{for} \quad (\theta, x) \in S^1 \times D^{n-1}(\varepsilon).$$

2. *Twisted $\tau: ([0,1] \times D^{n-1}(\varepsilon)/\sim) \to M^n$, where τ is a smooth imbedding such that $(\tau([0,1]) \times 0/\sim) = \gamma$ and*

$$f(\tau(t,x)) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{n-1}^2 \quad \textit{for} \quad (t,x) \in (\tau: [0,1] \times D^{n-1}(\varepsilon)/\sim).$$

Here, $([0,1] \times D^{n-1}(\varepsilon)/\sim)$ is diffeomorphic to $S^1 \times D^{n-1}(\varepsilon)$ on the identification of $0 \times D^{n-1}(\varepsilon)$ and $1 \times D^{n-1}(\varepsilon)$ by the mapping

$$(0, x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_{n-1}) \leftrightarrow (1, -x_1, \ldots, x_\lambda, -x_{\lambda+1}, \ldots, x_{n-1}).$$

The number λ is called the index of the critical circle γ.

Let M^n be a smooth manifold and let $f: M^n \to [0,n]$ be an S^1-Bott function. We say that f is a *nice* S^1-Bott function if the submanifold

$$M_i(f) = f^{-1}\left[0, i + \frac{1}{2}\right]$$

contains all closed orbits of index $\lambda \le i$. Every nice S^1-Bott function defines a filtration on the manifold $M^n: M_0(f) \subset M_1(f) \subset \ldots \subset M_{n-1}(f) \subset M^n$. It is well known [11] that the existence of a nice S^1-Bott function on a manifold is equivalent to the existence of a round-handle decomposition of the manifold. We recall some necessary definitions.

Definition 2.3. *We define an n-dimensional round handle R_λ of index λ as follows:*

$$R_\lambda = S^1 \times D^\lambda \times D^{n-\lambda-1},$$

where D^i is a disk of dimension i.

We define a twisted n-dimensional round handle TR_λ of index λ, $0 < \lambda < n-1$, as follows:

$$TR_\lambda = [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim,$$

where the identification is given by the mapping

$$(0, x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_{n-1}) \leftrightarrow (1, -x_1, \ldots, x_\lambda, -x_{\lambda+1}, \ldots, x_{n-1}).$$

Apparently, Thurston [15] was the first who noted that the existence of an S^1-Bott function on a manifold is equivalent to the existence of a handle decomposition of the manifold. We describe this fact in more detail.

Definition 2.4. *We say that a manifold M_λ^n is obtained from a smooth manifold M^n by attaching a round handle of index λ if*

$$M_\lambda^n = M^n \bigcup_\varphi S^1 \times D^\lambda \times D^{n-\lambda-1},$$

where $\varphi\colon S^1 \times \partial D^\lambda \times D^{n-\lambda-1} \to \partial M^n$ is a smooth imbedding.

A manifold M_λ^n is obtained from a smooth manifold M^n by gluing a twisted round handle of index λ if

$$M_\lambda^n = M^n \bigcup_\varphi [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim,$$

where $\varphi\colon ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\sim) \to M^n$ is a smooth imbedding.

Definition 2.5. *A round-handle decomposition of a smooth manifold M^n is a filtration*

$$\partial M^n \times [0,1] = M_0^n(R) \subset M_1^n(R) \subset \ldots \subset M_{n-1}^n(R) = M^n,$$

where the manifold $M_i^n(R)$ is obtained from the manifold $M_{i-1}^n(R)$ by gluing round handles and twisted round handles of index i. In the case where M^n is a closed manifold, the filtration begins with round handles of index 0.

In what follows, we recall the relationship between S^1 and the round-handle decomposition [11].

Theorem 2.2. *Let M^n be a smooth closed manifold. The following two conditions are equivalent:*

1. *On the manifold M^n, there is a nice S^1-Bott function with critical circles $\gamma_1, \ldots, \gamma_k$ of indices $\lambda_1, \ldots, \lambda_k$ with trivial coordinate systems and critical circles $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$ of indices $\mu_1, \ldots, \mu_l$ with twisted coordinate systems.*

2. *The manifold M^n admits a round-handle decomposition consisting of round handles $R_{\lambda_1}, \ldots, R_{\lambda_k}$ of indices $\lambda_1, \ldots, \lambda_k$ and twisted round handles $TR_{\mu_1}, \ldots, TR_{\mu_l}$ of indices $\mu_1, \ldots, \mu_l$ such that the critical circle γ_i corresponds to a round handle R_{λ_i}, $1 \le i \le k$, and the critical circle $\tilde{\gamma}_j$ corresponds to a twisted round handle TR_{μ_j}, $1 \le j \le l$.*

Thus, every nice S^1-Bott function on the manifold M^n generates a round-handle decomposition of M^n and vice versa.

The following result belongs to Asimov [5]:

Theorem 2.3. *Let M^n be a smooth closed manifold $(n > 3)$ with Euler characteristic $\chi(M^n) = 0$. Then M^n admits a round-handle decomposition.*

For three-dimensional manifolds, the situation is much more complicated [1, 12], and there are closed three-manifolds that do not admit a round-handle decomposition. The recent results on the three-dimensional Poincaré conjecture imply that a simply-connected three-dimensional manifold admits a round-handle decomposition.

We are interested in conditions under which an S^1-Bott function on M^n has the property that all its critical circles have a trivial coordinate system. We recall the necessary facts from [4].

By definition, an n-dimensional *handle* H_λ of index λ is $H_\lambda = D^\lambda \times D^{n-\lambda}$. We say that a smooth manifold M^n_λ is obtained from a smooth manifold M^n by attaching handles of index λ if

$$M^n_\lambda = M^n \bigcup_{\varphi} D^\lambda \times D^{n-\lambda},$$

where $\varphi\colon \partial D^\lambda \times D^{n-\lambda} \to \partial M^n$ is a smooth imbedding. $\partial D^\lambda \times 0$ $(D^\lambda \times 0)$ is called the core (disk) of the handle $D^\lambda \times D^{n-\lambda}$, and $\partial D^{n-\lambda} \times 0$ $(D^{n-\lambda} \times 0)$ is called its co-core sphere (disk).

A handle decomposition of a smooth manifold M^n is a filtration

$$\partial M^n \times [0,1] = M^n_0 \subset M^n_1 \subset \ldots \subset M^n_n = M^n,$$

where the manifold M^n_i is obtained from M^n_{i-1} by attaching handles of index i.

In the case where M^n is a closed manifold, the filtration begins with handles of index 0. There is a close relationship between the round-handle decomposition of a manifold and its handle decomposition; in [5], the following lemma was proved:

Lemma 2.1. *Let* $M^n = M^n_1 + H_\lambda + H_{\lambda+1}$ *be a smooth manifold obtained from the manifold with boundary* M^n_1 *by attaching handles of indices* λ *and* $\lambda+1$ *that do not intersect* $(n > 2)$. *If* $\lambda > 0$, *then the manifold* M^n *can be represented as* $M^n = M^n_1 + R_\lambda$, *where* R_λ *denotes the round handle of index* λ.

Lemma 2.2. *Let* M^n *be a smooth manifold* $(n > 2)$ *obtained from the manifold with boundary* M^n_1 *by attaching round (or twisted round) handles of index* $\lambda > 0$. *Then the manifold* M^n *can be represented as* $M^n = M^n_1 + H_\lambda + H_{\lambda+1}$. *If the round handle* R_λ *was glued, then the intersection index of* H_λ *and* $H_{\lambda+1}$ *is equal to* 0.

If the twisted handle TR_λ *is glued, then the intersection index of* H_λ *and* $H_{\lambda+1}$ *is equal to* ± 2.

Proof. The case where a handle is attached was proved in [4] (Lemma VIII.2). If the twisted handle TR_λ is glued to M^n_1, then the argument is the same. Let $\varphi\colon ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\!\sim) \to \partial M^n_1$ be a gluing mapping. We represent $\varphi([0,1] \times 0 \times 0/\!\sim)$ as the sum of two segments I_1 and I_2 such that $I_1 \cap I_2 = \partial I_1 = \partial I_2$ and $I_1 \cup I_2 = \big(\varphi([0,1] \times 0 \times 0/\!\sim\big)$. Consider the submanifold $H_\lambda = I_1 \times D^\lambda \times D^{n-\lambda-1}$. It can obviously be regarded as a handle of index λ that is attached to ∂M^n_1 along the set $\partial D^\lambda \times D^{n-\lambda-1} \times I_1$ with the restriction of φ. It is clear that the manifold

$$H_{\lambda+1} = \overline{TR_\lambda \setminus (I_1 \times D^\lambda \times D^{n-\lambda-1})} = I_2 \times D^\lambda \times D^{n-\lambda-1}$$

is a handle of index $\lambda+1$ that is attached to $\partial(M^n_1 \cup H_\lambda)$ along the set $(\partial I_2 \times D^\lambda \cup I_2 \times \partial D^\lambda) \times D^{n-\lambda-1}$.

By construction, the intersection index of these two handles is equal to ± 2.

Lemma 2.2 is proved.

Lemma 2.3. *Let* M^n *be a smooth closed manifold, let* $f\colon M^n \to [0,1]$ *be an* S^1*-Bott function, and let* c *be its critical value. Suppose that* $\varepsilon > 0$ *and there are no other critical values on the interval* $[c-\varepsilon, c+\varepsilon]$. *Assume that, on the surface level* $f^{-1}(c)$, *there are critical circles* $\gamma_1, \ldots, \gamma_k$ *of indices* $\lambda_1, \ldots, \lambda_k$ *with trivial coordinate systems and there are critical circles* $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$ *of indices* $\mu_1, \ldots, \mu_l$ *with twisted coordinate systems. Then the homology group* $H_*\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$ *is generated exactly by the handles that correspond to the critical circles* $\gamma_1, \ldots, \gamma_k, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$. *Each circle* γ_i *generates two subgroups that are isomorphic to* $\mathbf{Z}$, *a direct product in the homology group* $H_{\lambda_i}\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$ *and the other in the homology* *group* $H_{\lambda_{i+1}}\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$. *Each circle* $\tilde{\gamma}_j$ *generates a subgroup* $\mathbf{Z}_2$ *that is a direct product in a group* $H_{\mu_j}\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$.

Proof. Consider a function f associated with the decomposition of the manifold $f^{-1}[c-\varepsilon, c+\varepsilon]$ by round and twisted handles. Thus, the critical circles lie on the same level of the decomposition of round and twisted handles. We can choose handles so that they do not intersect each other. If we replace the round handles by the handles from the previous lemma, then each twisted round handle of index λ generates the homology of a subgroup isomorphic to $\mathbf{Z}_2$ in dimension λ, and each round handle of index λ generates the homology of two subgroups isomorphic to $\mathbf{Z}$ in dimensions λ and $\lambda+1$.

Lemma 2.3 is proved.

Corollary 2.1. *Let* M^n *be a smooth closed manifold, let* $f\colon M^n \to [0,1]$ *be an* S^1*-Bott function, and let* $c_1, \ldots, c_k$ *be its critical values. Suppose that* $\varepsilon_i > 0$, $1 \le i \le k$, *is such that the interval* $[c_i - \varepsilon_i, c_i + \varepsilon_i]$ *does not have other critical values. Then, on the level surface* $f^{-1}(c_i)$, *there are only critical circles with trivial coordinate systems if and only if the nonzero homology groups* $H_*\big(f^{-1}[c_i-\varepsilon_i, c_i+\varepsilon_i], f^{-1}(c_i-\varepsilon_i), \mathbf{Z}\big)$ *are free Abelian groups.*

Thus, we have a homological criterion for S^1-Bott functions to have no critical circles with twisted coordinate systems.

In the next section, we give another class of S^1-Bott functions that do not have critical circles with twisted coordinate systems.

3. Diagrams of S^1-Bott Functions and Their Applications

In this section, we explore S^1-Bott functions. Recall the definition of *partitions of diagrams* [4]. The partitions of diagrams represent the construction of S^1-Bott functions, especially for simply-connected manifolds.

Consider the decomposition of a closed smooth manifold M^n by handles $M_0^n \subset M_1^n \subset \ldots \subset M_n^n = M^n$, where the manifold M_i^n is obtained from the manifold M_{i-1}^n by attaching handles of index i. Assume that

$$C_i = H_i(M_i^n, M_{i-1}^n, \mathbf{Z}) \approx \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{k_i},$$

where k_i is the number of handles of index i. Mean disk handles of index i form a basis for the homology groups $H_i(M_i^n, M_{i-1}^n, \mathbf{Z})$. Using the exact homology sequence for the triple $M_{i-1}^n \subset M_i^n \subset M_{i+1}^n$, we can construct a chain complex of free Abelian groups, namely

$$(C, \partial)\colon C_0 \leftarrow \ldots \leftarrow C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1} \leftarrow \ldots \leftarrow C_n,$$

whose homology coincides with the homology of the manifold M^n. Suppose that the manifold M^n is oriented. The choice of orientation allows us to orient the medium and comedium spheres of the handle, which enables us to determine the homology indices λ and $\lambda+1$ in the manifolds ∂M_λ^n. Thus, the homomorphism ∂_λ is given by the matrix of indices of homologous intersections of the right-hand and left-hand spheres of handles in the submanifold ∂M_λ^n.

If each handle determines a vertex and we bridge the edges of the vertices for which the corresponding handles have a nonzero intersection, then we obtain a graph. Note that the structure of this graph can be complicated. However, it can be simplified.

It is known [4] that, by the addition of handles, all matrices of homomorphisms ∂_i, $0 \le i \le n$, can be made diagonal.

Suppose that M^n is a simply-connected manifold, $n > 5$, and there are no handles of indices 1 and $n-1$. Then, certainly, the homologies of the intersection indices of the right-hand and left-hand spheres coincide with their geometric intersection indices.

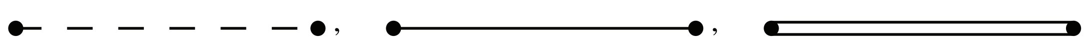
Thus, a pair of adjacent handles with indices λ and $\lambda + 1$ may either not intersect or have the intersection ± 1, ± 2, or $\pm m$, where $|m| > 2$. Since the Euler characteristic of a closed smooth manifold M^n that admits a round-handle decomposition is zero, it follows that, for the handle decomposition of M^n, we can introduce the following object (*diagram*): A diagram is a disconnected graph whose vertices correspond to handles and whose edges connect vertices if and only if the intersection of the handle is nonzero. A more precise definition is presented below.

Definition 3.1. *Ω_n is called a diagram of length n if the plane is given by $n+1$ sets of points $\left(a_0^1, \ldots, a_{k_0}^1; a_1^1, \ldots, a_{k_1}^1; \ldots; a_1^n, \ldots, a_{k_n}^n\right)$ that satisfy the following conditions:*

(1) for some i, the set $(a_1^i, \ldots, a_{k_i}^i)$ may be empty;

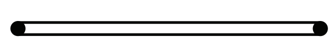
(2) $k_0 - k_1 + k_2 - \ldots + (-1)^n k_n = 0$;

(3) a point of the set $(a_1^i, \ldots, a_{k_i}^i)$, $1 < i < n-1$, can be connected either with only one point from the set $\left(a_1^{i-1}, \ldots, a_{k_{i-1}}^{i-1}\right)$ or with only one point from $(a_1^{i+1}, \ldots, a_{k_{i+1}}^{i+1})$ in one of the following three ways:

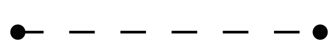


A set of points $a_1^0, \ldots, a_{k_0}^1; \ldots; a_1^i, \ldots, a_{k_i}^i$ is called an *i-skeleton diagram* Ω_n.

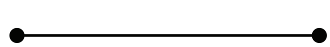
A point at which the chart is not linked to some other point is called *free*. If the chart has a fragment

a_j^i  a_t^{i+1},

then a_i^i is called a *semifree* point (the intersection of the handle is ± 2). If there is a fragment

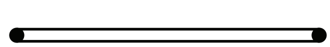
a_j^i  a_t^{i+1},

then a_i^i is called a *dependent* point (the intersection of the handle is $\pm m$). The fragment

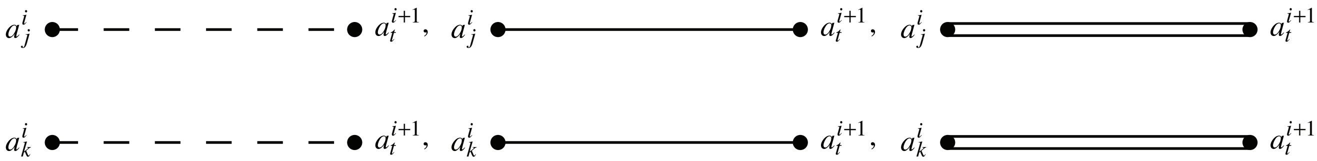
a_j^i  a_t^{i+1}

is called *inserted in dimension i* (the index of intersection of the corresponding handles is equal to ± 1).

Definition 3.2. *Two points of dimensions i and $i+1$ are independent in the dimension i if there is no connection between them or if they form the fragment*

 .

In what follows, we divide a chart into disjoint pairs of independent points. Let us introduce a restriction for the fragments of the diagram form, namely,



We do not allow breaking any of the fragments into a pair of the form (a_j^i, a_t^{i+1}), (a_k^i, a_l^{i+1}).

Definition 3.3. *If a chart Ω_n can be represented as a disjoint union of independent pairs of points, then it admits a partition. The two points (a_j^i, a_k^{i+1}) of this partition are called the vertices of the partition in dimension i.*

Let us fix a partition of the diagram Ω_n and denote it by $\Omega_n(\sigma)$. It is possible that the diagram $\Omega_n(\sigma)$ does not admit a partition because, in some dimensions, it may not have enough points for the formation of independent pairs.

Definition 3.4. *The base of the diagram Ω_n is the diagram $\overline{\Omega}_n$ obtained from Ω_n by eliminating all inserts.*

Definition 3.5. *A stabilization of the diagram Ω_n in dimension i is a diagram of the form $\Omega_n^{S(i)} = \Omega_n \underset{i}{\cup} A_i$, where A_i is a new insert in dimension i.*

Lemma 3.1. *For each chart Ω_n, there exists its stabilization in dimensions $i_1, \ldots, i_s$, denoted by $\Omega_n^{S(i_1,\ldots,i_s)}$, for which the diagram $\Omega_n^{S(i_1,\ldots,i_s)}$ admits a partition.*

Definition 3.6. *The number $\chi_i(\Omega_n) = k_i - k_{i-1} + \ldots + (-1)^{i+1}k_0$ is called the ith Euler characteristic of the diagram Ω_n.*

Obviously, the insertion of dimension i increases the ith Euler characteristic of $\Omega_n^{S(i)}$ by one and does not change the values of the remaining jth Euler characteristics $\chi_j\big(\Omega_n^{S(i)}\big) = \chi_j(\Omega_n)$ for $j \neq i$.

Lemma 3.2. *If the diagram Ω_n admits a partition, then the number of vertices of a partition of Ω_n in each dimension is the same for all of its possible partitions.*

Suppose that the diagram Ω_n admits a partition. Denote the number of vertices in dimension i of a partition of Ω_n by $m_i(\Omega_n)$ and let

$$M(\Omega_n) = \sum_{j=0}^{i} m_j(\Omega_n).$$

In view of the lemma, these numbers do not depend on the choice of a particular partition of the diagram Ω_n.

Definition 3.7. *The dimension λ of a chart Ω_n is called singular if $\chi_{\lambda-1}(\Omega_n) = \chi_{\lambda+1}(\Omega_n) = 0$, $\chi_\lambda(\Omega_n) = k > 0$, and the chart Ω_n in dimension λ does not consist of semifree fragments.*

In the process of decomposition of the diagram Ω_n into a pair of independent points, it is necessary in this situation to make one box of dimension $\lambda - 1$ or $\lambda + 1$, which leads to ambiguity. This will result in different numbers of pairs in dimension $\lambda + 1$ or $\lambda + 1$, depending on whether we have made insertions in any dimension.

Lemma 3.3. *The diagram* $\Omega_n = \left(a_0^1, \dots, a_{k_0}^1; a_1^1, \dots, a_{k_1}^1; \dots; a_1^n, \dots, a_{k_n}^n\right)$ *admits a partition if and only if it does not have negative* i*th Euler characteristics and singular dimensions. If the diagram* Ω_n *admits a partition, then the number of vertices in dimension* i *of a partition is equal to* $m_i(\Omega_n) = \chi_i(\Omega_n)$.

If the diagram Ω_n does not admit a partition, then there exists its stabilization Ω_n^S such that the diagram Ω_n^S admits a partition. There arises the question of the minimum possible number of vertices in dimension i among the stabilized diagrams $\Omega_n^{S_j}$ having a partition.

For diagram Ω_n, let $m_i^s(\Omega_n)$ denote the *minimum* possible number of vertices in dimension i among the stabilized diagrams $\Omega_n^{S_j}$ having a partition.

Let $\mathbf{N}$ be the set of integers. We put

$$\rho(n) = \frac{1}{2}\big(n + |n|\big),$$

where $n \in \mathbf{N}$.

Theorem 3.1. *Let* Ω_n *be an arbitrary diagram. Then* $m_i^s(\Omega_n)$ *of the diagram* Ω_n *is equal to* $m_i^s(\Omega_n) = \rho\big(\chi_i(\Omega_n)\big)$. *If* Ω_n^S *is a stabilization of the diagram* Ω_n, *then* $m_i^s(\Omega_n^S) \geq m_i^s(\Omega)$.

Definition 3.8. *For the diagram* Ω_n, *its* i*th Morse number* $M_i(\Omega_n)$ *is the number* $m_i^s(\overline{\Omega}_n)$, *where* $\overline{\Omega}_n$ *is the base of the diagram* Ω_n.

Definition 3.9. *A diagram* Ω_n *is called exact if there exists a stabilization* $\Omega_n^{S_*}$ *of* Ω_n *such that* $\Omega_n^{S_*}$ *admits a partition with the number of vertices in dimension* i *equal to* $m_i(\Omega_n^{S_*}) = M_i(\Omega_n)$ *simultaneously for all* i.

Theorem 3.2. *The diagram* Ω_n *is exact if and only if it does not have singular dimensions.*

A stabilization of the diagram Ω_n is called economical if

(1) for $\chi_i(\Omega_n) = k < 0$, one performs k insertions in the dimension i,

(2) in the case where i is a singular dimension, then one performs an insertion in the dimension $i-1$ or $i+1$.

We now describe how one can construct a diagram Ω_n on a round-handle decomposition of a smooth closed manifold M^n.

Let $M_0^n(R) \subset M_1^n(R) \subset \dots \subset M_{n-1}^n(R) = M^n$ be a round-handle decomposition of M^n. Using Lemma 2.2, we replace each handle of index λ by two ordinary handles of indices λ and $\lambda+1$. As a result, we obtain a handle decomposition of the manifold M^n, namely $M_0^n \subset M_1^n \subset \dots \subset M_n^n = M^n$. Using this handle decomposition of M^n, we can construct a chain complex of free abelian groups:

$$(C, \partial)\colon C_0 \leftarrow \dots \leftarrow C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1} \leftarrow \dots \leftarrow C_n.$$

Reducing the matrix of differentials to the diagonal form, we construct the diagram Ω_n.

The following statement is true:

Proposition 3.1. *Let* $M_0^n(R) \subset M_1^n(R) \subset \dots \subset M_{n-1}^n(R) = M^n$ *be a round-handle decomposition of the manifold* M^n *and let* Ω_n *be the diagram associated with this decomposition. Assume that the diagram* Ω_n *does not have semifree vertices. If the diagram* Ω_n *is the economical stabilization of its base* $\overline{\Omega}_n$, *then the original round-handle decomposition has missing twisted round handles.*

Proof. Indeed, in this case, the diagram Ω_n does not allow the insertion of a round twisted handle. All points of insertion are involved in the formation of vertices with other points of the diagram, and, by the condition of the proposition, there are no semifree vertices.

Remark 3.1. It is easy to construct a decomposition of the manifold M^n by round handles among which there are twisted round handles but, at the same time, no semifree vertices are associated with this decomposition diagram.

Definition 3.10. *Let M^n be a smooth closed manifold. The number*

$$\chi_i(M^n) = \mu\big(H_i(M^n, \mathbf{Z})\big) - \mu\big(H_{i-1}(M^n, \mathbf{Z})\big) + \ldots + (-1)^{i+1}\mu\big(H_0(M^n, \mathbf{Z})\big)$$

is called the ith Euler characteristic of M^n, where $\mu(H)$ is the minimal number of generators H.

Definition 3.11. *The dimension λ of a closed manifold M^n is called singular if $H_\lambda(M^n, \mathbf{Z})$ is a nonzero finite group distinct from $\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2$ and $\chi_{\lambda-1}(M^n) = \chi_{\lambda+1}(M^n) = 0$.*

Definition 3.12. *Let M^n be a smooth closed manifold. A round-handle decomposition is called quasiminimal if one of the following conditions is satisfied:*

(1) the number of round handles of index i is equal to $\rho(\chi_i(M^n)) + \varepsilon_i$, where $\varepsilon_i = 0$ if the dimension $i+1$ is nonsingular, and $\varepsilon_i = 1$ if the dimension $i+1$ is singular;

(2) the number of round handles of index i is equal to $\rho\big(\chi_i(M^n)\big)$; if the dimension $i+1$ is singular, then there is only one handle of index $i+2$.

In both cases, the number of round handles of index $i+1$ is equal to $\rho\big(\chi_{i+1}(M^n)\big)$.

A round-handle decomposition is called minimal if the number of round handles of index i is equal to $\rho\big(\chi_i(M^n)\big)$ for all i.

Using the handle decomposition of a manifold and the diagram technique, we can easily prove the following fact [4]:

Proposition 3.2. *Let M^n be a smooth, closed, simply-connected manifold $(n > 5)$. Then M^n admits a quasiminimal round-handle decomposition. If the manifold M^n does not have singular dimensions, then M^n admits a minimal round-handle decomposition.*

Definition 3.13. *Assume that the manifold M^n admits an S^1-Bott function. Then the S^1-Morse number $M_i^{S^1}(M^n)$ of index i is the minimum number of singular circles of index i taken over all S^1-Bott functions on M^n.*

Lemma 3.4. *Suppose that, on a closed manifold M^n, a smooth function $f\colon M^n \to \mathbb{R}$ exists such that each connected component of the singular set Σ_f of f is either a nondegenerate critical point p_i, $i = 1, \ldots, k$, or a nondegenerate critical circle S_j^1, $j = 1, \ldots, l$. Then the Euler characteristic of the manifold M^n is equal to*

$$\chi(M^n) = \sum_{i=1}^{k} (-1)^{\mathrm{index}(p_i)}.$$

Proof. It is known that, for any Morse function $g: M^n \to \mathbb{R}$ with critical points p_i, $i = 1, \dots, q$, on the manifold M^n, the following relation is true:

$$\chi(M^n) = \sum_{i=1}^{q} (-1)^{\operatorname{index}(p_i)}.$$

By a small perturbation of the function f, any nondegenerate critical circle S_j^1 of index λ can be replaced by nondegenerate critical points of indices λ and $\lambda + 1$ [1]. Therefore, the contribution to the formula for the Euler characteristic of these critical points is zero, and we obtain the desired formula.

4. Manifolds with Free S^1-Action

Assume that there is smooth free circle action on the smooth manifold M^n. Then, of course, the set M^n/S^1 is a manifold, and the natural projection $p: M^n \to M^n/S^1$ is a fiber bundle. Every smooth S^1-invariant function $f: M^n \to \mathbb{R}$ on a manifold M^n is called an *S^1-invariant Bott function* if each connected component of the singular set Σ_f is a nondegenerate critical circle.

It is clear that if f is an S^1-invariant Bott function on the manifold M^n, then its projection $\pi_*(f)$: $M^n/S^1 \to \mathbb{R}$ is a Morse function. Conversely, if $g: M^n/S^1 \to \mathbb{R}$ is a Morse function on the manifold M^n/S^1, then $\pi_*^{-1}(g) = g \circ \pi: M^n \to \mathbb{R}$ is an S^1-invariant Bott function on the manifold M^n. The critical point of index λ of the function g corresponds to the critical circle of index λ of the function $\pi_*^{-1}(g)$.

In this situation, for the manifold M^n, the *S^1-equivariant Morse number $M_i^{eqS^1}(M^n)$ of index i* is the minimum number of singular circles of index i taken over all S^1-invariant Bott functions on M^n.

For the manifold M^n/S^1, the *Morse number $M_i(M^n/S^1)$ of index i* is the minimum number of critical points of index i taken over all Morse functions on M^n/S^1.

Therefore, for the calculation of the S^1-equivariant Morse number of index i, it is possible to use Morse functions on the manifold M^n/S^1. The following fact is obvious:

Corollary 4.1. *Suppose that there is a smooth free circle action on the smooth manifold M^n. Then, for the manifold M^n, the S^1-equivariant Morse number of index i is equal to the Morse number of index i for the manifold M^n/S^1.*

A good example in this direction is the fiber bundle $p: S^{2n+1} \to \mathbb{CP}^n$. For this S^1-action, the S^1-equivariant Morse number is equal to 1 for even indices and to 0 for odd indices.

The next example shows that the S^1-equivariant Morse number of the manifold M^n depends on the circle action.

Let $p: S^3 \to S^2$ be a Hopf fiber bundle. Suppose that there is a trivial circle action on S^1. Using the Hopf fiber bundle and trivial circle action on S^1, we construct a new fiber bundle $p \times \mathrm{id}: S^3 \times S^1 \to S^2 \times S^1$. It is clear that, on the manifold $S^2 \times S^1$, there is a Morse function with one critical point of indices 0, 1, 2, and 3. Therefore, for this circle action on the manifold $S^2 \times S^1$, the S^1-equivariant Morse number is equal to 1 for any index.

On the other hand, assume that there is a trivial circle action on S^3 and q is a rotation on S^1. Consider the fiber bundle $\mathrm{id} \times q: S^3 \times S^1 \to S^3$. On S^3, there is a Morse function with one critical point of indices 0 and 3. Therefore, in this situation, for the manifold $S^3 \times S^1$, the S^1-equivariant Morse number is equal to 1 for the indices 0 and 3 and to 0 for the other indices.

Remark 4.1. This example shows that, for the manifold, the S^1-equivariant Morse number and the S^1-Morse number of some index may be different.

Definition 4.1. *Assume that there is a smooth free circle action on a smooth manifold M^n. This free circle action is minimal if, for all indices, the S^1-equivariant Morse number is equal to the S^1-Morse number for the manifold M^n.*

Corollary 4.2. *Suppose that there is a smooth minimal free circle action on a smooth simply-connected manifold M^n. Then the manifold M^n does not have singular dimensions.*

Proof. The corollary is obviously valid for dimension 3.

A manifold that admits a free circle action has Euler characteristic zero. If $n = 4$, then a free action does not exist on simply-connected manifolds M^4 because the Euler characteristic of a simply-connected four-dimensional manifold is always positive.

It follows from the structure of homology groups that a simply-connected manifold M^n, $8 \geq n \geq 5$, does not have singular dimensions.

Let $n \geq 9$. Suppose that there is a minimal smooth free circle action on M^n. It is obvious that

$$M_i(M^n/S^1) = M_i^{eqS^1}(M^n).$$

By the Smale theorem [10], on the manifold $M_i(M^n/S^1)$ there is a Morse function with the number of critical points of index i equal to $M_i(M^n/S^1)$ for all i simultaneously. Therefore, on the manifold M^n, there exists an S^1-invariant Bott function f with the number of critical circles of index i equal to $M_i(M^n/S^1) = M_i^{eqS^1}(M^n)$ for all i simultaneously. Since the free circle action is minimal, we have $M_i^{S^1}(M^n) = M_i^{eqS^1}(M^n)$. If the simply-connected manifold M^n has a singular dimension, then the S^1-Bott function on M^n cannot have the number of critical circles of index i equal to the ith S^1-Morse number $M_i^{S^1}(M^n)$ for all i simultaneously. Consequently, the manifold M^n does not have singular dimensions.

Corollary 4.2 is proved.

Theorem 4.1. *Suppose that there is a smooth free circle action on a smooth simply-connected manifold M^n. This circle action is minimal if and only if*

$$\mu(H_i(M^n/S^1, Z) + \mu(\text{Tors } H_{i-1}(M^n/S^1, Z) = \rho\big(\chi_i(M^n)\big) \quad \textit{for all } i.$$

Proof. It follows from the exact homotopy sequence of fibration that the manifold M^n/S^1 is simply-connected. Let $n = 3$. Using the results on the three-dimensional Poincaré conjecture [13], one can establish that $M^3 = S^3$ and $M^3/S^1 = S^2$, and we have the Hopf fiber bundle $p: S^3 \to S^2$. Therefore, Theorem 4.1 is proved.

If $n = 4$, then the free action does not exist on simply-connected manifolds M^4.

Let $n \geq 5$. *Necessity.* Suppose that there is a minimal smooth free circle action on M^n. If $n \geq 5$, then it follows from the results of Smale and Barden [3, 10] that the Morse number in dimension i of the manifold M^n/S^1 is equal to

$$M_i(M^n/S^1) = \mu(H_i\big(M^n/S^1, Z)\big) + \mu\big(\text{Tors } H_{i-1}(M^n/S^1, Z)\big).$$

We have $M_i(M^n/S^1) = M_i^{eqS^1}(M^n)$. By virtue of the condition of minimal free circle action, we get

$$M_i(M^n/S^1) = M_i^{eqS^1}(M^n) = M_i^{S^1}(M^n) = \rho\big(\chi_i(M^n)\big).$$

Sufficiency. On the manifold M^n/S^1, consider a Morse function with the number of critical points of index i equal to

$$M_i(M^n/S^1) = \mu(H_i(M^n/S^1, Z)) + \mu(\text{Tors } H_{i-1}(M^n/S^1, Z)).$$

By construction and the condition of the theorem, we have

$$M_i(M^n/S^1) = M_i^{eqS^1}(M^n) = \rho(\chi_i(M^n)).$$

However, $M_i^{S^1}(M^n) = \rho\big(\chi_i(M^n)\big)$, and, therefore, the free action of S^1 is minimal.

Theorem 4.1 is proved.

Corollary 4.3. *Suppose that there is a smooth free circle action on a smooth manifold M^n. Assume that the manifold M^n/S^1 is such that*

(a) $\pi_1(M^n/S^1) \approx \mathbb{Z}$, *or* $\pi_1(M^n/S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$, $n > 6$;

(b) $\pi_1(M^n/S^1)$ *is infinite,* $n > 8$.

Then the S^1-equivariant Morse number of index i for the manifold M^n is equal to

(a) $\widehat{S}^i_{(2)}(M^n/S^1) + \widehat{S}^{i+1}_{(2)}(M^n/S^1) + \dim_{N(Z[\pi])}\big(H^i_{(2)}(M^n/S^1, \mathbb{Z})\big)$;

(b) $\mathbb{D}^i(M^n/S^1) + \widehat{S}^i_{(2)}(M^n/S^1) + \widehat{S}^{i+1}_{(2)}(M^n/S^1) + \dim_{N(Z[\pi])}\big(H^i_{(2)}(M^n/S^1, \mathbb{Z})\big)$ *for* $3 < i < n-3$.

Proof. It follows from the results of [4, 14] that, on M^n, there are Morse functions with the number of critical points of index i equal to the Morse number of the manifold M^n/S^1.

5. Manifolds with Semifree S^1-Action

Let M^{2n} be a closed smooth manifold with semifree S^1-action that has only isolated fixed points. It is known that every isolated fixed point p of a semifree S^1-action has the following important property: near this point, the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point p, there exist an open invariant neighborhood U of p and a diffeomorphism h from U to an open unit disk D in $\mathbb{C}^n$ centered at the origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \ldots, 1)$. We will use both complex coordinates $(z_1, \ldots, z_n)$ and real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair (U, h) is called a *standard chart* at the point p. Let $f\colon M^{2n} \to \mathbb{R}$ be a smooth S^1-invariant function on the manifold M^{2n}. Denote by Σ_f the set of singular points of the function f. It is clear that the set of isolated singular points $\Sigma_f(p_j) \subset \Sigma_f$ of f coincides with the set of fixed points M^{S^1}.

For a nondegenerate critical point p_j, there exists a standard chart (U_j, h_j) such that, on U_j, the function f is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2.$$

Note that the index of a nondegenerate critical point p_j is always even.

Denote by $\Sigma_f(S^1)$ the set of singular points of the function f that are a disconnected union of circles. These circles will be called singular.

A circle $s \in \Sigma f_(S^1)$ is called nondegenerate if there is an S^1-invariant neighborhood U of s on which S^1 acts freely and such that the point $\pi(s)$ is nondegenerate for the function $\pi_*(f)\colon U/S^1 \to \mathbb{R}$ induced on U/S^1 by the natural mapping $\pi\colon U \to U/S^1$. An invariant version of the Morse lemma states that there exist an S^1-invariant neighborhood U of the circle s and coordinates $(x_1, \ldots, x_{2n-1})$ on U/S^1 such that the function $\pi_*(f)$ has the following representation:

$$\pi_*(f) = \pi_*\left(f(\pi(s))\right) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{2n-1}^2.$$

By definition, λ is the *index* of the singular circle s.

Definition 5.1. *A smooth S^1-invariant function $f\colon M^{2n} \to \mathbb{R}$ on a manifold M^{2n} with semifree circle action that has isolated fixed points is called an S^1_*-Bott function if every connected component of the singular set Σ_f is either a nondegenerate fixed point or a nondegenerate critical circle.*

Theorem 5.1. *Assume that M^{2n} is a closed manifold with smooth semifree circle action that has isolated fixed points $p_1, \ldots, p_k$. For any fixed point p_j, consider a standard chart (U_j, h_j) and the function*

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2$$

on U_j, where λ_j is an arbitrary integer from $0, 1, \ldots, n$.

Then there exists an S^1-invariant S^1_*-Bott function f on M^{2n} such that $f = f_j$ on U_j.

Proof. Consider the function f_j on U_j. Let $\pi_*(f_j)\colon U_j/S^1 \to \mathbb{R}$ be a continuous function induced on U_j/S^1 by the natural mapping $\pi\colon U_j \to U_j/S^1$. It is clear that the function $\pi_*(f_j)$ is smooth on the manifold $(U_j \setminus p_j)/S^1$. Denote by g the smooth extension of the functions $\pi_*(f_j)$ to M^{2n}/S^1. By a small deformation of the function g, which is fixed on U_j/S^1, we find a function g_1 on M^{2n}/S^1 such that g_1 is equal to $\pi_*(f_j)$ on U_j/S^1 and g_1 has only nondegenerate critical points on $M^{2n} \setminus \bigcup(U_j/S^1)$. Then the function $f = g_1 \circ p$ satisfies the conditions of the theorem.

Theorem 5.2. *The number of fixed points of any smooth semifree circle action on M^{2n} with isolated fixed points is always even and equal to the Euler characteristic of the manifold M^{2n}.*

Proof. First, we consider the functions

$$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2 \text{ on } U_1 \quad \text{and} \quad f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2 \text{ on } U_j,\ 2 \le j \le l$$

and extend them to an S^1-invariant Bott function f on the manifold $M^{2n} \setminus U_1 \bigcup U_2 \bigcup \ldots \bigcup U_l$. We assume that U_j is diffeomorphic to the open disk D^{2n} for any j. Consider the manifold $V^{2n} = W^{2n} \setminus \bigcup U_j$. The boundary of the manifold V^{2n} is the disconnected union of spheres S^{2n-1}. By the construction of the manifold V^{2n}, there is a free circle action. The boundary of the manifold V^{2n}/S^1 is the disconnected union of complex projective spaces $\mathbb{CP}^{n-1}$. If the number of boundary components of the manifold V^{2n}/S^1 is odd, then we glue the boundary components pairwise and obtain a compact smooth manifold with boundary $\mathbb{CP}^{n-1}$. The well-known fact that the manifold $\mathbb{CP}^{n-1}$ is noncobordant to zero implies that the number of fixed points of any smooth semifree circle action on M^{2n} with isolated fixed points is even. The value of the Euler characteristic $\chi(M^{2n}) = 2k$ follows from Lemma 3.4.

Definition 5.2. *Let f be an S^1-invariant S^1_*-Bott function for a smooth semifree circle action with isolated fixed points $p_1, \dots, p_{2k}$ on a closed manifold M^{2n}. Denote by λ_j the index of a critical point p_j of the function f. The state of the function f is defined as the collection of numbers $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2k})$ and is denoted by $St_f(\Lambda)$. It is clear that all numbers λ_j are even and $0 \le \lambda_j \le 2n$.*

Remark 5.1. It follows from Theorem 5.1 that, for every smooth semifree circle action on a closed manifold M^{2n} with isolated fixed points $p_1, \dots, p_{2k}$ and any collection of even numbers $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2k})$ such that $0 \le \lambda_j \le 2n$, there exists an S^1-invariant S^1_*-Bott function f on M^{2n} with state $St_f(\Lambda)$.

Definition 5.3. *Let M^{2n} be a closed smooth manifold with smooth semifree circle action that has finitely many fixed points $p_1, \dots, p_{2k}$. Fix any collection of even numbers $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2k})$ such that $0 \le \lambda_j \le 2n$.*

The S^1-Morse number $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$ of index i is the minimum number of singular circles of index i taken over all S^1-invariant S^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.

The following is an unsolved problem: For a manifold M^{2n} with a semifree circle action that has finitely many fixed points, find the exact values of the numbers $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$.

6. On S^1-Equivariant Morse Numbers $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$

Let M^{2n} be a compact closed manifold with semifree circle action that has finitely many fixed points $p_1, \dots, p_{2k}$. Denote the canonical map by $\pi\colon M^{2n} \to M^{2n}/S^1$. The set M^{2n}/S^1 is a manifold with singular points $\pi(p_1), \dots, \pi(p_{2k})$. It is clear that the neighborhood of any singular point is a cone over $\mathbb{CP}^{n-1}$. If $f\colon M^{2n} \to \mathbb{R}$ is a smooth S^1-invariant S^1_*-Bott function on the manifold M^{2n}, then $\pi_*(f)\colon M^{2n}/S^1 \to \mathbb{R}$ is a continuous function that is a Morse function on the smooth noncompact manifold

$$N^{2n-1} = M^{2n}/S^1 \setminus \bigcup_{j=1}^{2k} \pi(p_j).$$

We choose an invariant neighborhood U_i of the point p_j diffeomorphic to the open unit disk $D^{2n} \subset \mathbb{C}^n$ and set

$$U = \bigcup_{j=1}^{2k} U_j.$$

Consider the compact manifold $V^{2n-1} = (M^{2n} \setminus U)/S^1$. Its boundary is a disconnected union of complex projective spaces:

$$\partial V^{2n-1} = \mathbb{CP}_1^{n-1} \cup \ldots \cup \mathbb{CP}_{2k}^{n-1}.$$

It is clear that the manifolds $V^{2n-1} \setminus \partial V^{2n-1}$ and N^{2n-1} are diffeomorphic. We use the manifold V^{2n-1} for the investigation of S^1-invariant S^1_*-Bott functions on the manifold M^{2n} with state $St(\Lambda) = (0, \dots, 0, 2n, \dots, 2n)$. Let $\partial_0 V^{2n-1}$ be the part of the boundary of V^{2n-1} that consists of r components $\mathbb{C}P^{2n-2}$, $2k-1 \ge r \ge 1$, and let $\partial_1 V^{2n-1} = \partial V^{2n-1} \setminus \partial_0 V^{2n-1}$. On the manifold with boundary V^{2n-1}, we construct a Morse function $f\colon V \to [0,1]$ such that $f^{-1}(0) = \partial_0 V^{2n-1}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f, on the manifold M^{2n} we construct an S^1-equivariant S^1_*-Bott function F with state $St(0, \dots, 0, 2n, \dots, 2n)$ such that the restriction of $\pi_*(F)$ to V coincides with f. Therefore, the Morse number $M_i(V^{2n-1}, \partial_0 V^{2n-1})$ of index i for the manifold with boundary V^{2n-1} is equal to $\mathcal{M}_i^{S^1}(M^{2n}, St(0, \dots, 0, 2n, \dots, 2n)$.

Theorem 6.1. *Let M^{2n} $(2n > 8)$ be a closed smooth manifold that admits a smooth semifree circle action with isolated fixed points $p_1, \ldots, p_{2k}$. Then, for the manifold M^{2n} with state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$, one has*

$$\mathcal{M}_i^{S^1}(M^{2n}, St(\Lambda) = \mathbb{D}^i(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^{i+1}_{(2)}(V^{2n-1}, \partial_0 V^{2n-1})$$

$$+ \dim_{N(Z[\pi])}\left(H^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1})\right)$$

for $3 \le i \le 2n-4$.

Proof. We choose an invariant neighborhood U_i of the point p_i diffeomorphic to the unit disk $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Let f_i be the function on U_i equal to

$$f_i = |z_1|^2 + \ldots + |z_n|^2$$

and let f_j be the function on U_j equal to

$$f_j = 1 - |z_1|^2 - \ldots - |z_n|^2$$

for $i = 1, \ldots, r$ and $j = r+1, \ldots, 2k-r$. Consider the manifold $V^{2n} = (M^{2n} \setminus U)/S^1$. It is clear that its boundary is a disconnected union of complex projective spaces:

$$\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \ldots \cup \mathbb{C}P_{2k}^{2n-2}.$$

Let $\partial_0 V^{2n}$ be the part of the boundary of V^{2n} that consists of r components $\mathbb{C}P^{2n-2}$ that correspond to U_i and let $\partial_1 V^{2n}$ be the part of the boundary that consists of the component $\mathbb{C}P^{2n-2}$ that corresponds to U_j. On the manifold $V^{2n} = (M^{2n} \setminus U)/S^1$, we construct a Morse function $f\colon V \to [0,1]$ such that $f^{-1}(0) = \partial_0 V^{2n}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f, on the manifold M^{2n} we construct an S^1-equivariant S^1_*-Bott function F with state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$ such that the restriction of F to U_i coincides with f_i, the restriction of F to U_j coincides with f_j, and the restriction of $\pi_*(F)$ to V coincides with f. Therefore, the Morse number of the cobordism V is equal to $\mathcal{M}^{\lambda}_{S^1}\big(M^{2n}, St(\Lambda)\big)$. The value of the Morse number of a cobordism is given in [14].

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