ISRAEL JOURNAL OF MATHEMATICS **194** (2013), 137–150
DOI: 10.1007/s11856-012-0120-2

ON THE CODIMENSION GROWTH OF ALMOST NILPOTENT LIE ALGEBRAS

BY

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ABSTRACT

We study codimension growth of infinite dimensional Lie algebras over a field of characteristic zero. We prove that if a Lie algebra L is an extension of a nilpotent algebra by a finite dimensional semisimple algebra then the PI-exponent of L exists and is a positive integer.

1. Introduction

We consider algebras over a field F of characteristic zero. Given an algebra A, we can associate to it the sequence of its codimensions $\{c_n(A)\}, n = 1, 2, \dots$. If A is an associative algebra with a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded [15] while $c_n(A) = n!$ if A is not PI.

* The first author was partially supported by the Slovenian Research Agency grants P1-0292-0101-04 and J1-2057-0101.

** The second author was partially supported by RFBR grant No 09-01-00303a.
Received April 17, 2011

For a Lie algebra L the sequence $\{c_n(L)\}$ is in general not exponentially bounded (see, for example, [13]). Nevertheless, a class of Lie algebras with exponentially bounded codimensions is quite wide. It includes, in particular, all finite-dimensional algebras [2], [8], Kac–Moody algebras [18], [19], infinite-dimensional simple Lie algebras of Cartan type [10], Virasoro algebra and many others.

When $\{c_n(A)\}$ is exponentially bounded, the upper and the lower limits of the sequence $\sqrt[n]{c_n(A)}$ exist and the natural question arises: does the ordinary limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exist? In 80's Amitsur conjectured that for any associative PI algebra such a limit exists and is a non-negative integer. This conjecture was confirmed in [5], [6]. For Lie algebras the series of positive results was obtained for finite-dimensional algebras [3], [4], [20], for algebras with nilpotent commutator subalgebras [12] and some other classes (see [11]).

On the other hand, it was shown in [21] that there exists a Lie algebra L with

$$3.1 < \liminf_{n\to\infty} \sqrt[n]{c_n(L)} \le \limsup_{n\to\infty} \sqrt[n]{c_n(AL)} < 3.9.$$

This algebra L is soluble and almost nilpotent, i.e. it contains a nilpotent ideal of finite codimension. Almost nilpotent Lie algebras are close in some sense to finite dimensional algebras. For instance, they have the Levi decomposition under some natural restrictions (see [1, Theorem 6.4.8]), satisfy the Capelli identity, have exponentially bounded codimension growth, etc. Almost nilpotent Lie algebras play an important role in the theory of codimension growth since all minimal soluble varieties of a finite basic rank with almost polynomial growth are generated by almost nilpotent Lie algebras. Two of them have exponential growth with ratio 2 and one is of exponential growth with ratio 3.

In the present paper we prove the following results.

Theorem 1: *Let L be an almost nilpotent Lie algebra over a field F of characteristic zero. If N is the maximal nilpotent ideal of L and L/N is semisimple, then the PI-exponent of L,*

$$\exp(L) = \lim_{n\to\infty} \sqrt[n]{c_n(L)},$$

exists and is a positive integer.

Recall that a Lie algebra L is said to be special (or SPI) if it is a Lie subalgebra of some associative PI-algebra.

Theorem 2: *Let L be an almost nilpotent soluble special Lie algebra over a field F of characteristic zero. Then the PI-exponent of L exists and is a positive integer.*

Note that the special condition in Theorem 2 is necessary since the counterexample constructed in [21] is a finitely generated almost nilpotent soluble Lie algebra satisfying the Capelli identity of low rank. Nevertheless, its PI-exponent $\exp(L)$ exists (see [16]) but is not an integer since $\exp(L) \approx 3,6$. Note also that Theorem 1 generalizes the main result of [4] and gives an alternative and easier proof of the integrality of the PI-exponent in the finite-dimensional case considered in [4].

2. Preliminaries

Let L be a Lie algebra over F. We shall omit Lie brackets in the product of elements of L and write ab instead of $[a,b]$. We shall also denote the right-normed product $a(b(c\cdots d)\dots)$ as $abc\cdots d$. One can find all basic notions of the theory of identities of Lie algebras in [1].

Let $\bar{F}$ be an extension of F and $\bar{L} = L \otimes_F \bar{F}$. It is not difficult to check that $c_n(\bar{L})$ over $\bar{F}$ coincides with $c_n(L)$ over F. Hence it is sufficient to prove our results only for algebras over an algebraically closed field.

Let now X be a countable set of indeterminates and let $Lie(X)$ be a free Lie algebra generated by X. Lie polynomial $f = f(x_1,\dots,x_n) \in Lie(X)$ is an identity of a Lie algebra L if $f(a_1,\dots,a_n) = 0$ for any $a_1,\dots,a_n \in L$. It is known that the set of all identities of L forms a T-ideal $Id(L)$ of $Lie(X)$, i.e. an ideal stable under all endomorphisms of $Lie(X)$. Denote by $P_n = P_n(x_1,\dots,x_n)$ the subspace of all multilinear polynomials in $x_1,\dots,x_n$ in $Lie(X)$. Then the intersection $P_n \cap Id(L)$ is the set of all multilinear in $x_1,\dots,x_n$ identities of L. Since $char\ F = 0$, the union $(P_1 \cap Id(L)) \cup (P_2 \cap Id(L)) \cup \cdots$ completely defines all identities of L.

An important numerical invariant of the set of all identities of L is the sequence of codimensions

$$c_n(L) = \dim P_n(L) \quad \text{where } P_n(L) = \frac{P_n}{P_n \cap Id(L)}, \quad n = 1,2,\dots .$$

If $\{c_n(L)\}$ is exponentially bounded, one can define the lower and the upper PI-exponents of L as

$$\underline{\exp}(L) = \liminf_{n\to\infty} \sqrt[n]{c_n(L)}, \quad \overline{\exp}(L) = \limsup_{n\to\infty} \sqrt[n]{c_n(L)},$$

and

$$\exp(L) = \overline{\exp}(L) = \underline{\exp}(L),$$

the (ordinary) PI-exponent of L, in case equality holds.

One of the main tools for studying asymptotics of $\{c_n(L)\}$ is the theory of representations of symmetric group S_n (see [9] for details). Given a multilinear polynomial $f = f(x_1, \dots, x_n) \in P_n$, one can define

$$\sigma f = f(x_1, \dots, x_n) = f = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{1}$$

Clearly, (1) induces S_n-action on P_n. Hence P_n is an FS_n-module and $P_n \cap Id(L)$ is its submodule. Then $P_n(L) = \frac{P_n}{P_n \cap Id(L)}$ is also an FS_n-module. Since F is of characteristic zero, $P_n(L)$ is completely reducible,

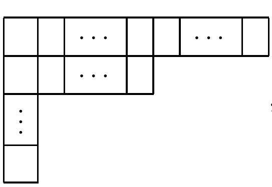
$$P_n(L) = M_1 \oplus \cdots \oplus M_t, \tag{2}$$

where $M_1, \dots, M_t$ are irreducible FS_n-modules and the number t of summands on the right-hand side of (2) is called the nth colength of L,

$$l_n(L) = t.$$

Recall that any irreducible FS_n-module is isomorphic to some minimal left ideal of group algebra FS_n which can be constructed as follows.

Let $\lambda \vdash n$ be a partition of n, i.e. $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k$ are positive integers and $\lambda_1 + \cdots + \lambda_k = n$. The Young diagram D_λ corresponding to λ is a tableau

$D_\lambda =$  ,

containing λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The Young tableau T_λ is the Young diagram D_λ with the integers $1, 2, \dots, n$ in the boxes. Given a Young tableau, denote by R_{T_λ} the row stabilizer of T_λ, i.e. the subgroup of all permutations $\sigma \in S_n$ permuting symbols only inside their rows. Similarly, C_{T_λ} is the column stabilizer of T_λ. Denote

$$R(T_\lambda) = \sum_{\sigma \in R_{T_\lambda}} \sigma, \quad C(T_\lambda) = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\ \tau)\tau, \quad \mathrm{e}_{\mathrm{T}_\lambda} = \mathrm{R}(\mathrm{T}_\lambda)\mathrm{C}(\mathrm{T}_\lambda).$$

Then e_{T_λ} is an essential idempotent of the group algebra FS_n, that is $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$ where $\alpha \in F$ is a non-zero scalar. It is known that $FS_n e_{T_\lambda}$ is an irreducible left FS_n-module. We denote its character by χ_λ. Moreover, if M is an FS_n-module with the character

$$\chi(M) = \sum_{\mu \vdash n} m_\mu \chi_\mu, \tag{3}$$

then $m_\lambda \neq 0$ in (3) for given $\lambda \vdash n$ if and only if $e_{T_\lambda} M \neq 0$.

If $M = P_n(L)$ for Lie algebra L, then the nth cocharacter of L is

$$\chi_n(L) = \chi(P_n(L)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{4}$$

and then

$$l_n(L) = \sum_{\lambda \vdash n} m_\lambda, \quad c_n(L) = \sum_{\lambda \vdash n} m_\lambda d_\lambda, \tag{5}$$

where m_λ are as in (4) and

$$d_\lambda = \deg \chi_\lambda = \dim FS_n e_{T_\lambda}.$$

Recall that Lie algebra L satisfies the Capelli identity of rank t if every multilinear polynomial $f(x_1, \ldots, x_n), n \geq t$, alternating on some $x_{i_1}, \ldots, x_{i_t}$, $\{i_1 \ldots, i_t\} \subseteq \{1, \ldots, n\}$ is an identity of L. It is known (see, for example, [7, Theorem 4.6.1]) that L satisfies the Capelli identity of rank $t+1$ if and only if all m_λ in (5) are zero as soon as D_λ has more than t rows, i.e. $\lambda_{t+1} \neq 0$.

A useful reduction in the proof of the existence of the PI-exponent is given by the following remark.

Lemma 1: *Let L be an almost nilpotent Lie algebra with the maximal nilpotent ideal N. Let $\dim L/N = p$ and let $N^q = 0$. Then:*

1. *L satisfies the Capelli identity of rank $p+q$; and*
2. *the colength $l_n(L)$ is a polynomially bounded function of n.*

Proof. Choose an arbitrary basis $e_1, \ldots, e_p$ of L modulo N and an arbitrary basis $\{b_\alpha\}$ of N. If $f = f(x_1, \ldots, x_n)$ is a multilinear polynomial, then f is an identity of L if and only if f vanishes under all evaluations $\{x_1, \ldots, x_n\} \to B = \{e_1, \ldots, e_p\} \cup \{b_\alpha\}$.

Suppose $n \geq p+q$ and f is alternating on $x_1, \ldots, x_{p+q}$. If $\varphi : \{x_1, \ldots, x_n\} \to B$ is an evaluation such that $\varphi(x_i) = \varphi(x_j)$ for some $1 \leq i < j \leq p+q$, then $\varphi(f) = 0$ since f is alternating on x_i, x_j. On the other hand, if any e_i appears among $y_1 = \varphi(x_1), \ldots, y_{p+q} = \varphi(x_{p+q})$ at most once, then $\{y_1, \ldots, y_{p+q}\}$ contains at least q basis elements from N. Hence $\varphi(f) = 0$ since $N^q = 0$ and we have proved the first claim of the lemma. The second assertion now follows from the results of [22]. ∎

As a consequence of Lemma 1 we get the following:

Lemma 2: *If L is an almost nilpotent Lie algebra, then the sequence $\{c_n(L)\}$ is exponentially bounded.*

Proof. By Lemma 1, there exist an integer t and a polynomial $f(n)$ such that $m_\lambda = 0$ in (5) for all $\lambda \vdash n$ with $\lambda_{t+1} \neq 0$ and $l_n(L) = \sum_{\lambda \vdash n} m_\lambda \leq f(n)$. It is well-known (see, for example, [7, Corollary 4.4.7]) that $d_\lambda = \deg \chi_\lambda \leq t^n$ if $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $k \leq n$. Hence we get from (5) an upper bound

$$c_n(L) \leq f(n) t^n$$

and the proof is completed. ∎

3. The upper bound for a PI-exponent

The exponential upper bound for codimensions obtained in Lemma 2 is not precise. In order to prove the existence and integrality of $\exp(L)$, we shall find a positive integer d such that $\overline{\exp}(L) \leq d$ and $\underline{\exp}(L) \geq d$.

Let L be a Lie algebra with a maximal nilpotent ideal N and a finite-dimensional semisimple factor-algebra $G = L/N$. Fix a decomposition of G into the sum of simple components

$$G = G_1 \oplus \cdots \oplus G_m$$

and denote by $\varphi_1, \ldots, \varphi_m$ the canonical projections of L to $G_1, \ldots, G_m$, respectively. Now let $g_1, \ldots, g_k$ be elements of L such that for some

$\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ one has

$$\varphi_{i_t}(g_t) \neq 0, \quad \varphi_{i_j}(g_t) = 0 \quad \text{for all } j \neq t,\ 1 \leq t \leq k.$$

For any non-zero product M of $g_1, \ldots, g_k$ and some $u_1, \ldots, u_t \in N$ we define the height of M as

$$ht(M) = \dim G_{i_1} + \cdots + \dim G_{i_k}.$$

Now we are ready to define a candidate to the PI-exponent of L as

$$d = d(L) = \max\{ht(M) | 0 \neq M \in L\}. \tag{6}$$

In order to get an upper bound for $\overline{\exp}(L)$ we define the following multialternating polynomials. Let $Q_{r,k}$ be the set of all polynomials f such that:

(1) f is multilinear, $n = \deg f \geq rk$,

$$f = f(x_1^1, \ldots, x_r^1, \ldots, x_1^k, \ldots, x_r^k, y_1, \ldots, y_s),$$

where $rk + s = n$; and

(2) f is alternating on each set $x_1^i, \ldots, x_r^i, 1 \leq i \leq k$.

We shall use the following lemma (see [20, Lemma 6]).

Lemma 3: *If $f \equiv 0$ is an identity of L for any $f \in Q_{d+1,k}$ for some d, k, then $\overline{\exp}(L) \leq d$.*

Note that Lemma 6 in [20] was proved for a finite-dimensional Lie algebra L. In fact, it is sufficient to assume that L satisfies the Capelli identity and that $l_n(L)$ is polynomially bounded.

Lemma 4: *Let L be an almost nilpotent Lie algebra and $d = d(L)$ as defined in (6). Then $\overline{\exp}(L) \leq d$.*

Proof. Let N be the maximal nilpotent ideal of L and let $N^p = 0$. We shall show that any polynomial from $Q_{d+1,p}$ is an identity of L and apply Lemma 3.

Given $1 \leq i \leq m$, we fix a basis B_i of L modulo

$$\widetilde{G}_1 + \cdots + \widetilde{G}_{i-1} + \widetilde{G}_{i+1} + \cdots + \widetilde{G}_m + N,$$

where $\widetilde{G}_j$ is the full preimage of G_j under the canonical homomorphism $L \to L/N$. In other words, $|B_i| = \dim G_i$, $\varphi_i(B_i)$ is a basis of G_i and $\varphi_j(B_i) = 0$ for any $j \neq i, 1 \leq j \leq m$.

Suppose $f = f(x_1^1, \ldots, x_{d+1}^1, \ldots, x_1^p, \ldots, x_{d+1}^p, y_1, \ldots, y_s) \in Q_{d+1,p}$ is multilinear and alternating on each set $\{x_1^i, \ldots, x_{d+1}^i\}$, $1 \le i \le p$. First consider an evaluation $\rho : X \to L$ such that $\rho(x_j^i) = b_j \in B_{t_j}$, $1 \le j \le d+1$, with

$$\dim G_{t_1} + \cdots + \dim G_{t_{d+1}} \ge d+1$$

in $L/N = G$. Then by definition of d any monomial in L containing factors $b_1, \ldots, b_{d+1}$ is zero, hence $\rho(f) = 0$. If $\dim G_{t_1} + \cdots + G_{t_{d+1}} \le d$, then $b_1, \ldots, b_{d+1}$ are linearly dependent modulo N, say, $b_{d+1} = \alpha_1 b_1 + \cdots + \alpha_d b_d + w$, $w \in N$. Then the value of $\rho(f)$ is the same as of $\rho' : X \to L$, where $\rho'(x_1^i) = \rho(x_1^i) = b_i, \ldots, \rho'(x_d^i) = \rho(x_d^i) = b_d$, $\rho'(x_{d+1}^i) = w$, since f is alternating on $x_1^i, \ldots, x_{d+1}^i$. It follows that for any evaluation $\rho : X \to L$ one should take at least one value $\rho(x_j^i)$ in N for any $i = 1, \ldots, p$, otherwise $\rho(f) = 0$. But in this case $\rho(f)$ is also zero since $\rho(f) \in N^p = 0$ and we have completed the proof. ∎

4. The lower bound for a PI-exponent

As in the previous Section let L be an almost nilpotent Lie algebra with the maximal nilpotent ideal N and suppose that a semisimple finite-dimensional factor-algebra $G = L/N = G_1 \oplus \cdots \oplus G_m$, where $G_1, \ldots, G_m$ are simple.

LEMMA 5: *Given an algebra L as above, there exist positive integers q and s such that for any $r = tq, t = 1, 2, \ldots$, and for any integer $j \ge s$ one can find a multilinear polynomial $h_t = h_t(x_1^1, \ldots, x_d^1, \ldots, x_1^r, \ldots, x_d^r, y_1, \ldots, y_{s+j})$ such that:*

1. *h_t is alternating on each set $\{x_1^i, \ldots, x_d^i\}$, $1 \le i \le r$; and*
2. *h_t is not an identity of L,*

where $d = d(L)$ is defined in (6).

Proof. Let $B_1, \ldots, B_m$ be as in Lemma 4. Then by the definition (up to reindexing of $G_1, \ldots, G_m$) there exist $b_1 \in B_1, \ldots, b_k \in B_k$, $a_1, \ldots, a_p \in L$ such that, for some multilinear monomial $w(z_1, \ldots, z_{k+p})$, the value

$$w(b_1, \ldots, b_k, a_1, \ldots, a_p)$$

is non-zero and $\dim G_1 + \cdots + \dim G_k = d$ in $G = L/N$.

Recall that for the adjoint representation of G_i there exists a central polynomial (see [14, Theorem 12.1]), i.e. an associative multilinear polynomial g_i which assumes only scalar values on $ad\ x_\alpha, x_\alpha \in G_i$. Moreover, g_i is not an identity of the adjoint representation of G_i and it depends on q disjoint alternating sets of variables of order $d_i = \dim G_i$. That is,

$$g_i = g_i(x^1_{1,d_i}, \dots, x^1_{d_i,d_i}, \dots, x^q_{1,d_i}, \dots, x^q_{d_i,d_i}) \tag{7}$$

is skew-symmetric on each $\{x^j_{1,d_i}, \dots, x^j_{d_i,d_i}\}$ and for some $a^1_{1,d_i}, \dots, a^q_{d_i,d_i} \in G_i$ the equality

$$g_i(ad\ a^1_{1,d_i}, \dots, ad\ a^q_{d_i,d_i})(c_i) = c_i$$

holds for any $c_i \in G_i$. If we evaluate (7) in L and take all $a^\alpha_{\beta\gamma}, c_i$ in B_i then we get

$$g_i(ad\ a^1_{1,d_i}, \dots, ad\ a^q_{d_i,d_i})(c_i) \equiv c_i (\text{mod } N). \tag{8}$$

On the other hand, if at least one of $a^\alpha_{\beta\gamma}$ lies in B_j, $j \neq i$, or in N, then

$$g_i(ad\ a^1_{1,d_i}, \dots, ad\ a^q_{d_i,d_i})(c_i) \equiv 0 \ (\text{mod } N). \tag{9}$$

Since we can apply g_i several times, the integer q can be taken to be the same for all $i = 1, \dots, k$. Moreover, it follows from (8), (9) that for any $t = 1, 2, \dots$ there exists a multilinear Lie polynomial

$$f^t_i = f^t_i(x^1_{1,d_i}, \dots, x^1_{d_i,d_i}, \dots, x^{tq}_{1,d_i}, \dots, x^{tq}_{d_i,d_i}, y_i)$$

alternating on each set $x^j_{1,d_i}, \dots, x^j_{d_i,d_i}$, $1 \le j \le tq$, such that

$$f^t_i(a^1_{1,d_i}, \dots, a^{tq}_{d_i,d_i}, c_i) \equiv c_i (\text{mod } N)$$

for some $a^1_{1,d_i}, \dots, a^{tq}_{d_i,d_i} \in B_i$ and for any $c_i \in B_i$.

Recall that the monomial $w = w(z_1, \dots, z_{k+q})$ has a non-zero evaluation

$$\bar{w} = w(b_1, \dots, b_k, a_1, \dots, a_p)$$

in L with $b_1 \in B_1, \dots, b_k \in B_k$. Replacing z_i by f^t_i in w and alternating the result, we obtain a polynomial

$$h_t = Alt\ w(f^t_1(x^1_{1,d_1}, \dots, x^{tq}_{d_1,d_1}, y_1), \dots, f^t_k(x^1_{1,d_k}, \dots, x^{tq}_{d_k,d_k}, y_k), z_{k+1}, \dots, z_p),$$

where $Alt = Alt_1 \cdots Alt_{tq}$ and Alt_j denotes the total alternation on variables

$$x^j_{1,d_1}, \dots, x^j_{d_1,d_1}, \dots, x^j_{1,d_k}, \dots, x^j_{d_k,d_k}.$$

Now if $\bar{w} = w(b_1, \dots, b_k, a_1, \dots, a_p) \in N^i \setminus N^{i+1}$ for some integer $i \ge 0$ in L, then according to (8), (9) we get

$$\rho(h_t) \equiv d_1! \cdots d_k! \bar{w} (\text{mod } N^{i+1}),$$

where $\rho : X \to L$ is an evaluation, $\rho(x^\alpha_{\beta\gamma}) = a^\alpha_{\beta\gamma}$, $\rho(y_j) = b_j$, $\rho(z_{k+j}) = a_j$. In particular, h_t is not an identity of L. Renaming variables

$$x^\alpha_{\beta\gamma}, y_1, \dots, y_k, z_{k+1}, \dots, z_{k+p}$$

we obtain the required polynomial h_t with $s = k + p$.

In order to get a similar multialternating polynomial h_t for $k + p + 1$ we replace the initial polynomial $w = w(z_1, \dots, z_{k+p})$ by $w' = w'(z_1, \dots, z_{k+p+1}) = w(z_1 z_{k+p+1}, z_2, \dots, z_{k+p})$. Since G_1 is simple we have $G_1^2 = G_1$. Hence there exists an element $a_{p+1} \in B_1$ such that

$$w'(b_1, \dots, b_k, a_1, \dots, a_{p+1}) = w(b_1 a_{p+1}, b_2, \dots, b_k, a_1, \dots, a_p) \neq 0.$$

Continuing this process we obtain similar h_t for all integers

$$k + p + 2, k + p + 3, \dots . \qquad ∎$$

Using multialternating polynomials constructed in the previous lemma we get the following lower bound for codimensions.

Lemma 6: *Let L, q and s be as in Lemma 5. Then there exists a constant $C > 0$ such that*

$$c_n(L) \geq \frac{1}{Cn^{2d}} \cdot d^n$$

for all $tq + s \leq n \leq tq + s + q - 1$ and for all $t = 1, 2, \dots$.

Proof. Given t and $s \leq s' \leq s + q - 1$, consider the polynomial h_t constructed in Lemma 5. Then $n = \deg h_t = tqd + s'$ and h_t depends on tq alternating sets of indeterminates of order d. Denote by M the FS_n-submodule of $P_n(L)$ generated by h_t. Let $n_0 = tqd$ and let the subgroup $S_{n_0} \subseteq S_n$ act on tqd alternating indeterminates $x^1_1, \dots, x^{tq}_d$. Then $M_0 = FS_{n_0} h_t$ is a non-zero subspace of M. Obviously,

$$c_n(L) \geq \dim M \geq \dim M_0. \tag{10}$$

Consider the character of M_0 and its decomposition onto irreducible components,

$$\chi(M_0) = \sum_{\lambda \vdash n_0} m_\lambda \chi_\lambda. \tag{11}$$

By Lemma 1, algebra L satisfies the Capelli identity of rank $d_0 \geq \dim L/N \geq d$. Hence $m_\lambda = 0$ in (11) as soon as the height $ht(\lambda)$ of λ, i.e. the number of rows in the Young diagram D_λ, is bigger than d_0.

Now we prove that for any multilinear polynomial $f = f(x_1, \dots, x_n)$ and for any partition $\lambda \vdash n_0$ with $\lambda_{d+1} \geq p$, where $N^p = 0$, the polynomial $e_{T_\lambda} f$ is an identity of L.

Since $e_{T_\lambda} = R(T_\lambda)C(T_\lambda)$ and $e^2_{T_\lambda} = \alpha e_{T_\lambda} \neq 0$, the product $e^*_{T_\lambda} = C(T_\lambda)R(T\lambda)$ is nonzero and generates minimal left ideal $FS_n e_{T_\lambda}$. Hence $e^*_{T_\lambda} f$ is an identity of L if and only if $e_{T_\lambda} f$ is an identity. On the other hand the set $\{x_1, \dots, x_{n_0}\}$ is a disjoint union

$$\{x_1, \dots, x_{n_0}\} = X_0 \cup X_1 \cup \dots \cup X_p,$$

where $|X_1|, \dots, |X_p| \geq d + 1$ and $e^*_{T_\lambda} f$ is alternating on each $X_1, \dots, X_p$, i.e. $e^*_{T_\lambda} f \in Q_{d+1,p}$. As it was shown in the proof of Lemma 4, $e^*_{T_\lambda} f$ is an identity of L.

It follows that $m_\lambda \neq 0$ in (11) for $\lambda \vdash n_0$ only if $\lambda_{d+1} < p$. In particular,

$$n_0 - (\lambda_1 + \dots + \lambda_d) \leq (d - d_0)p. \tag{12}$$

By the construction of essential idempotent e_{T_λ}, any polynomial $e_{T_\lambda} f(x_1, \dots, x_{n_0})$ is symmetric on λ_1 variables corresponding to the first row of T_λ. Since h_t depends on tq alternating sets of variables, it follows that $e_{T_\lambda} h_t = 0$ for any $\lambda \vdash n_0$ with $\lambda_1 \geq tq + 1$.

Denote $c_1 = (d - d_0)p$. If $m_\lambda \neq 0$ in (11) for $\lambda \vdash n_0$, $\lambda = (\lambda_1, \dots, \lambda_k)$, then $k \leq d_0$ and

$$\lambda_{d-1} \leq \dots \leq \lambda_1 \leq tq. \tag{13}$$

If $\lambda_d < tq - c_1$, then combining (12) and (13) we get

$$\lambda_{d+1} + \dots + \lambda_k = n_0 - (\lambda_1 + \dots + \lambda_d) \leq c_1$$

and

$$n_0 = (\lambda_1 + \dots + \lambda_{d-1}) + \lambda_d + (\lambda_{d+1} + \dots + \lambda_k) < tq(d-1) + tq - c_1 + c_1 = tqd = n_0,$$

a contradiction. Hence $\lambda_d \geq tq - c_1$ and the Young diagram D_λ contains a rectangular diagram D_μ where

$$\mu = (\underbrace{tq - c_1, \dots, tq - c_1}_{d})$$

is a partition of $n_1 = d(tq - c_1) = n_0 - c_1 d = n - s' - c_1 d \geq n - s - q - c_1 d + 1$ since $s' \leq s + q - 1$. From the Hook formula for dimensions of irreducible

S_n-representations (see [7, Proposition 2.2.8]) and from Stirling formula for factorials, it easily follows that

$$d_\mu = \deg \chi_\mu > \frac{d^{n_1}}{n_1^{2d}}$$

for all n sufficiently large. Since $\dim M_0 \geq d_\lambda \geq d_\mu$ and $n_1 \geq n - c_2$ for constant $c_2 = s + q + c_1 d - 1$, we conclude from (10) that

$$c_n(L) > \frac{d^n}{Cn^{2d}},$$

where $C = c_2^{2d}$ and we are done. ∎

5. Existence of PI-exponents

It follows from Lemma 6 that

$$\underline{\exp}(L) = \liminf_{n\to\infty} \sqrt[n]{c_n(L)} \geq d.$$

Combining this inequality with Lemma 4 we get the following

Theorem 1: *Let L be an almost nilpotent Lie algebra over a field F of characteristic zero. If N is the maximal nilpotent ideal of L and L/N is semisimple, then the PI-exponent of L,*

$$\exp(L) = \lim_{n\to\infty} \sqrt[n]{c_n(L)},$$

exists and is a positive integer.

Now consider the case of soluble almost nilpotent special Lie algebras.

Theorem 2: *Let L be an almost nilpotent soluble special Lie algebra over a field F of characteristic zero. Then the PI-exponent of L exists and is a positive integer.*

Proof. Let L be a special soluble Lie algebra with a nilpotent ideal N of a finite codimension. By Lemma 2, algebra L satisfies the Capelli identity of some rank. Then the variety $V = \mathrm{Var}\, L$ generated by L has a finite basis rank [17], that is L has the same identities as some k-generated Lie algebra $L_k \in V$. Clearly, $\underline{\exp}(L) = \underline{\exp}(L_k)$ and $\overline{\exp}(L) = \overline{\exp}(L_k)$. Since L is soluble, it follows that L_k is a finitely generated soluble Lie algebra from special variety V. By [1, Proposition 6.3.2, Theorem 6.4.6] we have $(L_k^2)^t = 0$ for some $t \geq 1$. In this case $\exp(L)$ exists and is a non-negative integer [12]. ∎

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