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Max–min measures on ultrametric spaces

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ABSTRACT

Ultrametrization of the set of all probability measures of compact support on the ultrametric spaces was first defined by Hartog and de Vink. In this paper we consider a similar construction for the so-called max–min measures on the ultrametric spaces. In particular, we prove that the functors of max–min measures and idempotent measures are isomorphic. However, we show that this is not the case for the monads generated by these functors.

1. Introduction

Ultrametric spaces naturally appear not only in different parts of mathematics, in particular, in real-valued analysis, number theory and general topology, but also have applications in biology, physics, theoretical computer science, etc. (see e.g. [6,11,14]).

Probability measures of compact support on ultrametric spaces were investigated by different authors. In particular, Hartog and de Vink [6] defined an ultrametric on the set of all such measures. The properties of the obtained construction were established in [7] and [14].

The aim of this paper is to find analogs of these results for other classes of measures. We define the so-called max–min measures, which play a similar role to that of probability measures in the idempotent mathematics, i.e., the part of mathematics which is obtained by replacing the usual arithmetic operations by idempotent operations (see [8,10]). The methods and results of idempotent mathematics have found numerous applications [1,2,4].

Note that max–min measures are non-additive. The class of non-additive measures has numerous applications, in particular, in mathematical economics, multicriteria decision making, image processing (see, e.g., [5]).

In the case of max–min measures, we start with such measures of finite supports; the general case (max–min measures of compact supports) is obtained by passing to the completions.

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One of our results shows that functors of max–min measures and idempotent measures in the category of ultrametric spaces and nonexpanding maps are isomorphic. However, we show that monads generated by these functors are not isomorphic.

2. Preliminaries

2.1. Max–min measures

By $\bar{\mathbb{R}}$ we denote the extended real line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Let $\wedge$ and $\vee$ denote the operations max and min in $\bar{\mathbb{R}}$, respectively. Following the traditions of idempotent mathematics we denote by $\odot$ the addition (convention $-\infty \odot x = x$ for all $x \in \bar{\mathbb{R}}$, $x < \infty$).

Let X be a topological space. As usual, by $C(X)$ we denote the linear space of (real-valued) continuous functions on X. The set $C(X)$ is a lattice with respect to the pointwise maximum and minimum and we preserve the notation $\wedge$ and $\vee$ for these operations.

Given $x \in X$, by δ_x we denote the Dirac measure in X concentrated at x. Given $x_i \in X$ and $\alpha_i \in \bar{\mathbb{R}}$, $i = 1, \dots, n$, such that $\bigwedge_{i=1}^n \alpha_i = \infty$, we denote by $\bigvee_{i=1}^n \alpha_i \wedge \delta_{x_i}$ the functional on $C(X)$ defined as follows:

$$\bigvee_{i=1}^n \alpha_i \wedge \delta_{x_i}(\varphi) = \bigvee_{i=1}^n \alpha_i \wedge \varphi(x_i).$$

Let us denote by $J_\omega(X)$ the set of all such functionals. We call the elements of $J_\omega(X)$ the *max–min measures* of finite support on X. The term ‘measure’ means nothing but the fact that $\mu = \bigvee_{i=1}^n \alpha_i \wedge \delta_{x_i} \in J_\omega(X)$ can also be interpreted as a set function with values in the extended real line: $\mu(A) = \bigvee\{\alpha_i \mid x_i \in A\}$, for any $A \subset X$.

The *support* of $\mu = \bigvee_{i=1}^n \alpha_i \wedge \delta_{x_i} \in J_\omega(X)$ is the set

$$\operatorname{supp}(\mu) = \{x_i \mid i = 1, \dots, n,\ \alpha_i > -\infty\} \subset X.$$

For any map $f : X \to Y$ of topological spaces, define the map $J_\omega(f) : J_\omega(X) \to J_\omega(Y)$ by the formula:

$$J_\omega(f)\left(\bigvee_{i=1}^n \alpha_i \wedge \delta_{x_i}\right) = \bigvee_{i=1}^n \alpha_i \wedge \delta_{f(x_i)}.$$

Let us recall that $I_\omega(X)$ denotes the set of functionals of the form $\bigvee_i \alpha_i \odot \delta_{x_i}$, where $\alpha_i \in \bar{\mathbb{R}}$ and $\bigvee_i \alpha_i = 0$. If $\varphi \in C(X)$, then $(\bigvee_i \alpha_i \odot \delta_{x_i})(\varphi) = \bigvee_i \alpha_i \odot \varphi(x_i)$. See e.g. [15], for the theory of spaces $I_\omega(X)$ (called spaces of idempotent measures of finite support) as well as related spaces $I(X)$ (called spaces of idempotent measures of compact support). Recall that the *support* of $\mu = \bigvee_{i=1}^n \alpha_i \odot \delta_{x_i} \in I_\omega(X)$ is the set

$$\operatorname{supp}(\mu) = \{x_i \mid i = 1, \dots, n,\ \alpha_i > -\infty\} \subset X.$$

Remark 2.1. We adopt the following conventions: $+\infty \wedge \delta_x = \delta_x$ in $J_\omega(X)$ and $0 \odot \delta_x = \delta_x$ in $I_\omega(X)$.

2.2. Ultrametric spaces

Recall that a metric d on a set X is said to be an *ultrametric* if the following strong triangle inequality holds:

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$.

By $O_r(A)$ we denote the r-neighborhood of a set A in a metric space. We write $O_r(x)$ if $A = \{x\}$. It is well known that in ultrametric spaces, for any $r > 0$, every two distinct elements of the family $\mathcal{O}_r = \{O_r(x) \mid x \in X\}$ are disjoint. We denote by $\mathcal{F}_r$ the set of all functions on X that are constant on the elements of the family $\mathcal{O}_r$. By $q_r : X \to X/\mathcal{O}_r$ we denote the quotient map. We endow the set $X/\mathcal{O}_r$ with the quotient metric, d_r. It is easy to see that $d_r(O_r(x), O_r(y)) = d(x, y)$, for any disjoint $O_r(x)$, $O_r(y)$, and the obtained metric is an ultrametric.

Recall that a map $f : X \to Y$, where (X, d) and (Y, ϱ) are metric spaces, is called *nonexpanding* if $\varrho(f(x), f(y)) \leqslant d(x, y)$, for every $x, y \in X$. Note that the quotient map $q_r : X \to X/\mathcal{O}_r$ is nonexpanding.

2.3. Hyperspaces and symmetric powers

By $\exp X$ we denote the set of all nonempty compact subsets in X endowed with the Hausdorff metric:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

We say that $\exp X$ is the *hyperspace* of X. For a continuous map $f : X \to Y$ the map $\exp f : \exp X \to \exp Y$ is defined as $(\exp f)(A) = f(A)$.

It is well known that $\exp f$ is a nonexpanding map if so is f. We denote by $s_X : X \to \exp X$ the singleton map, $s_X(x) = \{x\}$.

By S_n we denote the group of permutations of the set $\{1, 2, \ldots, n\}$. Every subgroup G of the group S_n acts on the n-th power X^n of the space X by the permutation of factors. Let $SP^n_G(X)$ denote the orbit space of this action. By $[x_1, \ldots, x_n]$ (or briefly $[x_i]$) we denote the orbit containing $(x_1, \ldots, x_n) \in X^n$.

If (X, d) is a metric space, then $SP^n_G(X)$ is endowed with the following metric $\tilde{d}$,

$$\tilde{d}\big([x_i], [y_i]\big) = \min\big\{\max\big\{d(x_i, y_{\sigma(i)}) \bigm| i = 1, \ldots, n\big\} \bigm| \sigma \in G\big\}.$$

It is well known that the space $(SP^n_G(X), \tilde{d})$ is ultrametric if such is also (X, d).

Define the map $\pi_G = \pi_{GX} : X^n \to SP^n_G(X)$ by the formula $\pi_G(x_1, \ldots, x_n) = [x_1, \ldots, x_n]$. It was shown in [7] (and it is easy to see) that the map π_G is nonexpanding.

2.4. Monads

We recall some necessary definitions from category theory; see, e.g., [3,9] for details. A *monad* $\mathbb{T} = (T, \eta, \mu)$ in the category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : 1_{\mathcal{E}} \to T$ (unity), $\mu : T^2 = T \circ T \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

Given two monads, $\mathbb{T} = (T, \eta, \mu)$ and $\mathbb{T}' = (T', \eta', \mu')$, we say that a natural transformation $\alpha : T \to T'$ is a *morphism* of $\mathbb{T}$ into $\mathbb{T}'$ if $\alpha\eta = \eta'$ and $\mu'\alpha_T T(\alpha) = \alpha\mu$.

We denote by UMET the category of ultrametric spaces and nonexpanding maps. One of examples of monads on the category UMET is the hyperspace monad $\mathbb{H} = (\exp, s, u)$. The singleton map $s_X : X \to \exp X$ is already defined and the map $u_X : \exp^2 X \to \exp X$ is the union map, $u_X(\mathcal{A}) = \bigcup \mathcal{A}$.

It is well known (and easy to prove) that the max-metric on the finite product of ultrametric spaces is an ultrametric. We will always endow the product with this ultrametric.

The Kleisli category of a monad $\mathbb{T}$ is a category $\mathcal{C}_{\mathbb{T}}$ defined by the conditions: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, T(Y))$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by the formula $g * f = \mu Z T(g) f$.

Define the functor $\Phi_{\mathbb{T}} : \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by

$$\Phi_{\mathbb{T}}(X) = X, \qquad \Phi_{\mathbb{T}}(f) = \eta_Y f, \quad X \in |\mathcal{C}|, \; f \in \mathcal{C}(X, Y).$$

A functor $\overline{F} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is called an *extension of the functor* $F : \mathcal{C} \to \mathcal{C}$ *on the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ if $\Phi_{\mathbb{T}} F = \overline{F} \Phi_{\mathbb{T}}$.

The proof of the following theorem can be found in [13].

Theorem 2.2. *There exists a bijective correspondence between the extensions of functor F onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of a monad $\mathbb{T}$ and the natural transformations $\xi : FT \to TF$ satisfying*

(1) $\xi F(\eta) = \eta_F$;
(2) $\mu_F T(\xi)\xi_T = \xi F(\mu)$.

3. Ultrametric on the set of max–min measures

Let (X, d) be an ultrametric space. For any $\mu, \nu \in J_\omega(X)$, let

$$\hat{d}(\mu, \nu) = \inf\big\{r > 0 \bigm| \mu(\varphi) = \nu(\varphi), \text{ for any } \varphi \in C(X)\big\}.$$

Since μ, ν are of finite support, it is easy to see that $\hat{d}$ is well defined.

Theorem 3.1. *The function $\hat{d}$ is an ultrametric on the set $J_\omega(X)$.*

Proof. We only have to check the strong triangle inequality. Suppose that $\mu, \nu, \tau \in J_\omega(X)$ and $\hat{d}(\mu, \tau) < r$, $\hat{d}(\nu, \tau) < r$. Then, for every $\varphi \in \mathcal{F}_r$, we have $\mu(\varphi) = \tau(\varphi) = \nu(\varphi)$, whence $\hat{d}(\mu, \nu) < r$. □

Proposition 3.2. *The map $x \mapsto \delta_x : X \to J_\omega(X)$ is an isometric embedding.*

Proof. Let $x, y \in X$ and $d(x, y) < r$. Then for every $\varphi \in \mathcal{F}_r(X)$, we have $\delta_x(\varphi) = \varphi(x) = \varphi(y) = \delta_y(\varphi)$, whence $\hat{d}(\delta_x, \delta_y) < r$. Therefore, $\hat{d}(\delta_x \delta_y) \leqslant d(x, y)$. The proof of the reverse inequality is simple as well. □

Proposition 3.3. *Let $f : X \to Y$ be a nonexpanding map of an ultrametric space (X, d) into an ultrametric space (Y, ϱ). Then the induced map $J_\omega(f)$ is also nonexpanding.*

Proof. Since the map f is nonexpanding, $\varphi f \in \mathcal{F}_r(X)$, for any $\varphi \in \mathcal{F}_r(Y)$.

If $\mu, \nu \in J_\omega(X)$ and $\hat{d}(\mu, \nu) < r$, then, for every $\varphi \in \mathcal{F}_r(Y)$, we have

$$J_\omega(f)(\mu)(\varphi) = \mu(\varphi f) = \nu(\varphi f) = J_\omega(f) = J_\omega(f)(\nu)(\varphi)$$

and therefore $\hat{\varrho}(J_\omega(f)(\mu), J_\omega(f)(\nu)) < r$. □

We therefore obtain a functor J_ω on the category UMET.

Proposition 3.4. *If $\mu, \nu \in J_\omega(X)$, then the following are equivalent*:

(1) $\hat{d}(\mu, \nu) < r$;
(2) $J_\omega(q_r)(\mu) = J_\omega(q_r)(\nu)$.

Proof. (1) ⇒ (2). For every $\varphi : X/\mathcal{O}_r \to \mathbb{R}$ we have $\varphi q_r \in \mathcal{F}_r$ and therefore

$$J_\omega(q_r)(\mu) = \mu(\varphi q_r) = \nu(\varphi q_r) = J_\omega(q_r)(\nu).$$

Thus, $J_\omega(q_r)(\mu) = J_\omega(q_r)(\nu)$.

(2) ⇒ (1). Let $\varphi \in \mathcal{F}_r$, then φ factors through q_r, i.e. there exists $\psi : X \to \mathbb{R}$ such that $\varphi = \psi q_r$. Then

$$\mu(\varphi) = \mu(\psi q_r) = J_\omega(q_r)(\mu)(\varphi) = J_\omega(q_r)(\nu)(\varphi) = \nu(\psi q_r) = \nu(\varphi).$$

Thus, $\hat{d}(\mu, \nu) < r$. □

In the sequel, given a metric space (X, d), we denote also by d the (extended, i.e. taking values in $[0, \infty]$) metric on the set of maps from a nonempty set Y into X defined by the formula: $d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$.

Proposition 3.5. *The functor J_ω is locally non-expansive, i.e., for every nonexpanding maps f, g of an ultrametric space (X, d) into an ultrametric space (Y, ϱ) we have $\hat{\varrho}(J_\omega(f), J_\omega(g)) \leqslant \varrho(f, g)$.*

Proof. If $\varrho(f, g) = \infty$, then there is nothing to prove. Suppose that $\varrho(f, g) < r < \infty$. Then $q_r f = q_r g$, where $q_r : Y \to Y/\mathcal{O}_r(Y)$ is the quotient map. For every $\mu \in J_\omega(X)$, we obtain

$$J_\omega(q_r) J_\omega(f)(\mu) = J_\omega(q_r f)(\mu) = J_\omega(q_r g)(\mu) J_\omega(q_r) J_\omega(g)(\mu)$$

and by Proposition 3.4, $\hat{\varrho}(J_\omega(f)(\mu), J_\omega(g)(\mu)) < r$. □

4. Categorical properties

Let (X, d) be an ultrametric space. Given a function $\varphi \in C(X)$, define $\bar{\varphi} : J_\omega(X) \to \mathbb{R}$ as follows: $\bar{\varphi}(\mu) = \mu(\varphi)$.

Proposition 4.1. *If $\varphi \in \mathcal{F}_r(X)$, then $\bar{\varphi} \in \mathcal{F}_r(J_\omega(X))$.*

Proof. Given $\mu, \nu \in J_\omega(X)$ with $\hat{d}(\mu, \nu) < r$, we see that $\bar{\varphi}(\mu) = \mu(\varphi) = \nu(\varphi) = \bar{\varphi}(\nu)$, whence $\bar{\varphi} \in \mathcal{F}_r(J_\omega(X))$. □

Let $M \in J^2_\omega(X)$. Define $\xi_X(M)$ by the condition $\xi_X(M)(\varphi) = M(\bar{\varphi})$, for any $\varphi \in C(X)$. If $M = \bigvee_i \alpha_i \wedge \delta_{\mu_i}$ and $\mu_i = \bigvee_j \beta_{ij} \wedge \delta_{x_{ij}}$, then

$$\xi_X(M) = \bigvee_i \bigvee_j \alpha_i \wedge \beta_{ij} \wedge \delta_{x_{ij}}.$$

Proposition 4.2. *The map ξ_X is nonexpanding.*

Proof. Let d denote the ultrametric on X, then $\hat{d}$ and $\hat{\hat{d}}$ denote the ultrametrics on $J_\omega(X)$ and $J^2_\omega(X)$ respectively. Let $M, N \in J^2_\omega(X)$ and $\hat{\hat{d}}(M, N) < r$, for some $r > 0$. Then, for every $\varphi \in \mathcal{F}(X)$ we obtain

$$\xi_X(M)(\varphi) = M(\bar{\varphi}) = N(\bar{\varphi}) = \xi_X(N)(\varphi)$$

and therefore $\hat{d}(\xi_X(M), \xi_X(N)) < r$. □

It is easy to verify that the maps ξ_X give rise to a natural transformation of the functor J^2_ω to the functor J_ω in the category UMET.

Theorem 4.3. *The triple $\mathbb{J}_\omega = (J_\omega, \delta, \xi)$ is a monad in the category* UMET.

Proof. Let $\mu = \bigvee_i \alpha_i \wedge \delta_{x_i} \in J_\omega(X)$. Then

$$\xi_X J_\omega(\delta_X)(\mu) = \xi_X\left(\bigvee_i \alpha_i \wedge \delta_{\delta_{x_i}}\right) = \bigvee_i \alpha_i \wedge \delta_{x_i} = \mu$$

and $\xi_X \delta_{J_\omega(X)}(\mu) = \xi_X(\delta_\mu) = \mu$. Therefore $\xi J_\omega(\delta) = 1_{J_\omega} = \xi \delta_{J_\omega}$.

Let $\mathfrak{M} = \bigvee_i \alpha_i \wedge \delta_{M_i} \in J^3_\omega(X)$, where $M_i = \bigvee_j \beta_{ij} \wedge \delta_{\mu_{ij}}$. Then

$$\xi_X J_\omega(\xi_X)(\mathfrak{M}) = \xi_X\left(\bigvee_i \alpha_i \wedge \delta_{\xi_X(M_i)}\right) = \xi_X\left(\bigvee_i \alpha_i \wedge \delta_{\bigvee_j \beta_{ij} \wedge \mu_{ij}}\right) = \bigvee_i \bigvee_j \alpha_i \wedge \beta_{ij} \wedge \mu_{ij}$$

$$= \bigvee_i \alpha_i \wedge \left(\bigvee_j \beta_{ij} \wedge \delta_{\mu_{ij}}\right) = \xi_X\left(\bigvee_i \alpha_i \wedge M_i\right) = \xi_X \xi_{J_\omega(X)}(\mathfrak{M})$$

and therefore $\xi J_\omega(\xi) = \xi \xi_{J_\omega}$. $\square$

Proposition 4.4. *The spaces $I_\omega(X)$ and $J_\omega(X)$ are isometric.*

Proof. Define a map $h = h_X : I_\omega(X) \to J_\omega(X)$ as follows. Let $\mu = \bigvee_i \alpha_i \odot \delta_{x_i} \in I_\omega(X)$. Define $h(\mu) = \bigvee_i -\ln(-\alpha_i) \wedge \delta_{x_i} \in J_\omega(X)$.

Suppose that $\hat{d}(\mu, \nu) < r$, where $\nu = \bigvee_j \beta_j \odot \delta_{y_j} \in I_\omega(X)$. For every $x \in X$ and $t \leqslant 0$, define $\varphi^x_t : X \to \mathbb{R}$ by the conditions: $\varphi^x_t(y) = 0$ if $y \in B_r(x)$ and $\varphi^x_t(y) = t$ otherwise.

Then

$$\max_{x_i \in B_r(x)} \alpha_i = \lim_{i \to -\infty} \mu\left(\varphi^x_t\right) = \lim_{i \to -\infty} \nu\left(\varphi^x_t\right) = \max_{y_j \in B_r(x)} \beta_j.$$

If $\varphi \in \mathcal{F}_r$, then

$$\mu(\varphi) = \bigvee_i \alpha_i \odot \varphi(x_i) = \bigvee_{x \in X} \bigvee_{x_i \in B_r(x)} \alpha_i \odot \varphi(x_i) = \bigvee_{x \in X} \bigvee_{y_j \in B_r(x)} \beta_j \odot \varphi(y_j)$$

and therefore

$$h(\mu)(\varphi) = \bigvee_i -\ln(-\alpha_i) \wedge \varphi(x_i) = \bigvee_{x \in X} \bigvee_{x_i \in B_r(x)} -\ln(-\alpha_i) \wedge \varphi(x_i) = \bigvee_{x \in X} \bigvee_{y_j \in B_r(x)} -\ln(-\beta_j) \wedge \varphi(y_j) = h(\nu)(\varphi).$$

Thus, $\hat{d}(h(\mu), h(\nu)) < r$ and we see that the map h is nonexpanding. One can similarly prove that the inverse map h^{-1} is also nonexpanding. $\square$

Proposition 4.5. *The class $\{h_X\}$ is a natural transformation of the functor I_ω to the functor J_ω.*

Proof. Let $f : X \to Y$ be a map and $\mu = \bigvee_i \alpha_i \odot \delta_{x_i} \in I_\omega(X)$. Then

$$J_\omega(f) h_X(\mu) = J_\omega(f)\left(\bigvee_i -\ln(-\alpha_i) \wedge \delta_{x_i}\right) = \bigvee_i -\ln(-\alpha_i) \wedge \delta_{f(x_i)} = h_Y\left(\bigvee_i \alpha_i \odot \delta_{f(x_i)}\right) = h_Y I_\omega(f)(\mu). \qquad \square$$

Corollary 4.6. *The functors I_ω and J_ω are isomorphic.*

Remark 4.7. Let $\alpha : [-\infty, 0] \to [-\infty, \infty]$ be an order-preserving bijection. Then the maps $g^\alpha_X : I_\omega(X) \to J_\omega(X)$ defined by the formula $g^\alpha_X(\bigvee_i t_i \odot \delta_{x_i}) = \bigvee_i \alpha(t_i) \wedge \delta_{x_i}$, determines an isomorphism of the functors I_ω and J_ω.

Proposition 4.8. *Every isomorphism of the functors I_ω and J_ω is of the form g^α (see Remark 4.7), for some order-preserving bijection $\alpha : [-\infty, 0] \to [-\infty, \infty]$.*

Proof. Let $k : I_\omega \to J_\omega$ be an isomorphism. Let $X = \{x, y, z\}$, where x, y, z are distinct points. Since the functor isomorphisms preserve the supports, we obtain

$$k_X(t \odot \delta_x \vee t \odot \delta_y \vee \delta_z) = \alpha(t) \wedge \delta_x \vee \alpha(t) \wedge \delta_y \vee \beta(t) \wedge \delta_z,$$

where $\alpha(t) \vee \beta(t) = +\infty$.

We are going to show that $\beta(t)=+\infty$, for every $t\in[-\infty,0]$. First note that $k_X(\delta_x\vee\delta_y\vee\delta_z)=\delta_x\vee\delta_y\vee\delta_z$. Suppose that, for some $t\in(-\infty,0)$, we have $\beta(t)<+\infty$. Denote by $r:X\to\{y,z\}$ the retraction that sends x to z. Then, since in this case $\alpha(t)=+\infty$, we obtain

$$k_{\{y,z\}}\big(I_\omega(r)(t\odot\delta_x\vee t\odot\delta_y\vee\delta_z)\big)=k_{\{y,z\}}(t\odot\delta_y\vee\delta_z)=\delta_y\vee\delta_z,$$

which is impossible, because the natural transformations preserve the symmetry with respect to the nontrivial permutation of $\{y,z\}$.

Thus,

$$k_X(t\odot\delta_x\vee t\odot\delta_y\vee\delta_z)=\alpha(t)\wedge\delta_x\vee\alpha(t)\wedge\delta_y\vee\delta_z$$

and identifying the points x and y we conclude that $k_{\{y,z\}}(t\odot\delta_y\vee\delta_z)=(\alpha(t)\wedge\delta_y\vee\delta_z)$. We see therefore that $k=g^\alpha$.

It is clear that α is a bijection of $[-\infty,0]$ onto $[-\infty,\infty]$. Suppose now that $X=\{x_1,x_2,\dots,x_n\}$, where $x_1,x_2,\dots,x_n$ are distinct points. Let $\mu=\bigvee_{i=1}^n t_i\odot\delta_{x_i}$ be such that $t_1=0$. Given $i>1$, consider a retraction $r_i:X\to\{x_1,x_i\}$ that sends every x_j, $j\neq i$, to x_1. Then, by what was proved above,

$$k_{\{x_1,x_i\}}I_\omega(r_i)(\mu)=k_{\{x_1,x_i\}}(\delta_{x_1}\vee t_i\odot\delta_{x_1})=\delta_{x_1}\vee\alpha(t_i)\wedge\delta_{x_1}=J_\omega(r_i)\big(k_X(\mu)\big)$$

and collecting the data for all $i>1$ we conclude that $k_X(\mu)=\bigvee_{i=1}^n\alpha(t_i)\wedge\delta_{x_i}$.

We are going to show that the map α is isotone. Again, let $X=\{x,y,z\}$, where x,y,z are distinct points. Suppose that $t_1,t_2\in[-\infty,0]$ and $t_1<t_2$. Then $k_X(t_1\odot\delta_x\vee t_2\odot\delta_y\vee\delta_z)=\alpha(t_1)\wedge\delta_x\vee\alpha(t_2)\wedge\delta_y\vee\delta_z$.

For a retraction $r:X\to\{y,z\}$ the retraction that sends x to y, we obtain

$$I_\omega(r)(t_1\odot\delta_x\vee t_2\odot\delta_y\vee\delta_z)=t_2\odot\delta_y\vee\delta_z$$

and therefore

$$I_\omega(r)\big(\alpha(t_1)\wedge\delta_x\vee\alpha(t_2)\wedge\delta_y\vee\delta_z\big)=\alpha(t_2)\wedge\delta_y\vee\delta_z,$$

whence we conclude that $\alpha(t_1)<\alpha(t_2)$. This finishes the proof of the proposition. □

Theorem 4.9. *The monads $\mathbb{I}_\omega$ and $\mathbb{J}_\omega$ are not isomorphic.*

Proof. Suppose the contrary and let a natural transformation $h:I_\omega\to J_\omega$ be an isomorphism of $\mathbb{I}_\omega$ and $\mathbb{J}_\omega$. Then, by Proposition 4.8, $h=g^\alpha$, for some order-preserving bijection $\alpha:[-\infty,0]\to[-\infty,\infty]$.

Let $X=\{a,b,c\}$. Suppose that $M=((-1)\odot\delta_\mu)\vee\delta_\nu\in I^2_\omega(X)$, where $\mu=(-2)\odot\delta_a\vee\delta_b$, $\nu=(-3)\odot\delta_b\vee\delta_c$.

Then

$$h_X\zeta_X(M)=h_X\big((-3)\odot\delta_a\vee(-3)\odot\delta_b\vee\delta_c\big)=\alpha(-3)\wedge\delta_a\vee\alpha(-3)\wedge\delta_b\vee\delta_c.$$

On the other hand,

$$\begin{aligned}\xi_X J_\omega(h_X)h_{I_\omega(X)}(M)&=\xi_X J_\omega(h_X)\big(\alpha(-1)\wedge\delta_\mu\vee\delta_\nu\big)\\&=\xi_X\big(\alpha(-1)\wedge\delta_{h_X(\mu)}\vee\delta_{h_X(\nu)}\big)=\xi_X\big(\alpha(-1)\big)\wedge\delta_{(\alpha(-2)\wedge\delta_a\vee\delta_c)}\vee\delta_{(\alpha(-3)\wedge\delta_b\vee\delta_c)}\\&=\big(\alpha(-2)\wedge\delta_a\vee\alpha(-3)\wedge\delta_b\vee\delta_c\big)\neq h_X\zeta_X(M).\qquad\square\end{aligned}$$

Let $\mu=\bigvee_i\alpha_i\wedge\delta_{x_i}\in J_\omega(X)$, $\nu=\bigvee_j\beta_j\wedge\delta_{y_j}\in J_\omega(Y)$. Define $\mu\otimes\nu\in J_\omega(X\times Y)$ by the formula:

$$\mu\otimes\nu=\bigvee_{ij}(\alpha_i\vee\beta_j)\wedge\delta_{(x_i,y_j)}.$$

Lemma 4.10. *The map*

$$(\mu,\nu)\mapsto\mu\otimes\nu:J_\omega(X)\times J_\omega(Y)\to J_\omega(X\times Y)$$

is nonexpanding.

Proof. Suppose that $\hat d((\mu,\nu),(\mu',\nu'))<r$. Then

$$J_\omega(q_r)(\mu\otimes\nu)=J_\omega(q_r)(\mu)\otimes J_\omega(q_r)(\nu)=J_\omega(q_r)\big(\mu'\big)\otimes J_\omega(q_r)\big(\nu'\big)=J_\omega(q_r)\big(\mu'\otimes\nu'\big)$$

and we conclude that

$$\hat d\big(J_\omega(q_r)(\mu\otimes\nu),J_\omega(q_r)\big(\mu'\otimes\nu'\big)\big)<r.$$

Therefore, the mentioned map is nonexpanding. □

Remark 4.11. The results concerning the operation $\otimes$ can be easily extended to the products of an arbitrary number of factors.

Theorem 4.12. *There exists an extension of the symmetric power functor SP^n onto the category of ultrametric spaces and nonexpanding maps with values that are max–min measures of finite supports.*

Proof. Let X be an ultrametric space. Define a map $\theta_X : SP^n_G(J_\omega(X)) \to J_\omega(SP^n_G(X))$ by the formula:

$$\theta_X[\mu_1, \dots, \mu_n] = J_\omega(p_G)(\mu_1 \otimes \dots \otimes \mu_n).$$

First, we observe that θ_X is well-defined. Indeed, if $[\mu_1, \dots, \mu_n] = [\nu_1, \dots, \nu_n]$, then there is a permutation $\sigma \in G$ such that $\nu_i = \mu_{\sigma(i)}$, for every $i \in \{1, \dots, n\}$. Denote by $h_\sigma : X^n \to X^n$ the map that sends $(x_1, \dots, x_n)$ to $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$, then

$$J_\omega(p_G)(\mu_1 \otimes \dots \otimes \mu_n) = J_\omega(p_G h_\sigma)(\mu_1 \otimes \dots \otimes \mu_n) = J_\omega(p_G) J_\omega(h_\sigma)(\mu_1 \otimes \dots \otimes \mu_n) = J_\omega(p_G)(\nu_1 \otimes \dots \otimes \nu_n).$$

Next, note that θ_X is nonexpanding, i.e., a morphism of the category UMET. This easily follows from Lemma 4.10 and the fact that the map π_G is nonexpanding.

Let $(x_1, \dots, x_n) \in X^n$. Then

$$\theta_X SP^n_G(\delta_X)(x_1, \dots, x_n) = J_\omega(p_G)(\delta_{x_1} \otimes \dots \otimes \delta_{x_n}) = J_\omega(p_G)(\delta_{(x_1, \dots, x_n)}) = \delta_{p_G(x_1, \dots, x_n)} = \delta_{[x_1, \dots, x_n]}.$$

Now let $M_1, \dots, M_n \in J^2_\omega(X)$ and $M_i = \bigvee \alpha_{ik} \wedge \delta_{\mu_{ik}}$, where $\mu_{ik} \in J_\omega(X)$. Then

$$\begin{aligned}
&\xi_X J_\omega(\theta_X)\theta_{J_\omega(X)}\big([M_1, \dots, M_n]\big) \\
&\quad = \xi_X J_\omega(\theta_X) J_\omega(\pi_{G J_\omega(X)})(M_1 \otimes \dots \otimes M_n) = J_\omega(\theta_X) J_\omega(\pi_{G J_\omega(X)})\Big(\bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge \delta_{(\mu_{1i_1}, \dots, \mu_{ni_n})}\Big) \\
&\quad = \mu_X J_\omega(\theta_X)\Big(\bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge \delta_{[\mu_{1i_1}, \dots, \mu_{ni_n}]}\Big) = \xi_X\Big(\bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge \delta_{\theta_X([\mu_{1i_1}, \dots, \mu_{ni_n}])}\Big) \\
&\quad = \bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge \theta_X\big([\mu_{1i_1}, \dots, \mu_{ni_n}]\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\theta_X SP^n_G(\xi_X)\big([M_1, \dots, M_n]\big) \\
&\quad = \theta_X\big(\big[\theta_X(M_1), \dots, \theta_X(M_n)\big]\big) = \theta_X\big([\vee\alpha_{1i_1} \wedge \mu_{1i_1}, \dots, \vee\alpha_{1i_1} \wedge \mu_{ni_n}]\big) \\
&\quad = J_\omega(\pi_G)\big((\vee\alpha_{1i_1} \wedge \mu_{1i_1}) \otimes \dots \otimes (\vee\alpha_{1i_1} \wedge \mu_{ni_n})\big) = J_\omega(\pi_G)\Big(\bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge (\mu_{1i_1} \otimes \dots \otimes \mu_{ni_n})\Big) \\
&\quad = \bigvee (\alpha_{1i_1} \wedge \dots \wedge \alpha_{ni_n}) \wedge J_\omega(\pi_G)(\mu_{1i_1} \otimes \dots \otimes \mu_{ni_n}),
\end{aligned}$$

i.e., $\xi_X J_\omega(\theta_X)\theta_{J_\omega(X)} = \theta_X SP^n_G(\xi_X)$. Applying Theorem 2.2 we obtain that the functor SP^n_G admits an extension onto the Kleisli category of the monad $\mathbb{J}_\omega$. □

Proposition 4.13. *The class of maps* $\mathrm{supp} = (\mathrm{supp}_X) : J_\omega(X) \to \exp X$ *is a morphism of the monad* $\mathbb{J}_\omega$ *into the hyperspace monad* $\mathbb{H}$.

Proof. Clearly, for every $x \in X$, where X is an ultrametric space, we have $s_X(x) = \{x\} = \mathrm{supp}(\delta_x)$.

Now let $M \in J^2_\omega(X)$, $M = \bigvee_{i=1}^n \alpha_i \wedge \mu_i$. We may assume that $\alpha_i > -\infty$, for all i. Let also $\mu_i = \bigvee_{j=1}^{m_i} \beta_{ij} \wedge \delta_{x_{ij}}$, where $\beta_{ij} > -\infty$, for all i, j.

Then $\xi_X(M) = \bigvee_{ij} \alpha_i \wedge \beta_{ij} \wedge \delta_{x_{ij}}$ and

$$\begin{aligned}
&u_X \exp(\mathrm{supp}_X)\, \mathrm{supp}_{J_\omega(X)}(M) \\
&\quad = u_X \exp(\mathrm{supp}_X)\big(\{\mu_1, \dots, \mu_n\}\big) = u_X\big\{\{x_{i1}, \dots, x_{im_i}\} \bigm| i = 1, \dots, n\big\} \\
&\quad = \{x_{ij} \mid i = 1, \dots, n,\ j = 1, \dots, m_i\} = \mathrm{supp}\big(\xi_X(M)\big). \qquad \square
\end{aligned}$$

5. Completion

Denote by CUMET the category of complete ultrametric spaces and nonexpanding maps. Given a complete ultrametric space (X, d), denote by $J(X)$ the completion of the space $J_\omega X$.

For any morphism $f : X \to Y$ of the category UMET there exists a unique morphism $J(F) : J(X) \to J(Y)$ that extends $J_\omega(f)$. We therefore obtain a functor $J :$ CUMET $\to$ CUMET.

The results of the previous section have their analogs also for the functor J. In particular, we have the following result.

Proposition 5.1. *The functors I and J are isomorphic.*

We keep the notation δ_X for the natural embedding $x \mapsto \delta_x : X \to J(X)$. Also, for any complete X, the set $J^2_\omega(X)$ is dense in $J^2(X)$ and therefore the nonexpanding map $\xi_X : J^2_\omega(X) \to J_\omega(X)$ can be uniquely extended to a nonexpanding map $J^2(X) \to J(X)$. We keep the notation ξ_X for the latter map.

Theorem 5.2. *The triple $\mathbb{J} = (J, \delta, \xi)$ is a monad in the category* CUMET.

Proof. Follows from the proof of Theorem 4.3. □

The monad $\mathbb{J}$ is called the *max–min measure monad* in the category CUMET. The support map

$$\bigvee_{i=1}^{n} \alpha_i \wedge \delta_{x_i} \mapsto \{x_1, \dots, x_n\} : J_\omega(X) \to \exp X$$

can be extended to the map $\operatorname{supp} : J(X) \to \exp X$, which we also call the support map.

Theorem 5.3. *The class of support maps $J_\omega(X) \to \exp X$ is a morphism of the max–min measure monad to the hyperspace monad in the category* CUMET.

Theorem 5.4. *There exists an extension of the symmetric power functor SP^n onto the Kleisli category of the monad $\mathbb{J}$.*

Proof. Similar to the proof of Theorem 4.12. □

The category mentioned in the above theorem is nothing but the category of ultrametric spaces and nonexpanding max–min measure-valued maps.

Theorem 5.5. *The monads $\mathbb{I}$ and $\mathbb{J}$ are not isomorphic.*

Proof. This follows from the fact that every morphism of monads generates a morphisms of submonads generated by the subfunctors of finite support. □

6. Open problems

Define the max–min measures for the compact Hausdorff spaces in the spirit of [15]. Is the extension of the symmetric power functor SP^n onto the category of ultrametric spaces and max–min measure-valued maps unique? This is known to be valid for the case of probability measures.

The class of K-ultrametric spaces was recently defined and investigated by Savchenko. Can analogs of the results of this paper be proved for the K-ultrametric spaces? See [12] where analogous questions are considered.

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