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On a PDE involving the $\mathcal{A}_{p(\cdot)}$-Laplace operator

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ABSTRACT

This paper establishes the existence of solutions for a partial differential equation in which a differential operator involving variable exponent growth conditions is present. This operator represents a generalization of the $p(\cdot)$-Laplace operator, i.e. $\Delta_{p(\cdot)}u = \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$, where $p(\cdot)$ is a continuous function. The proof of the main result is based on Schauder's fixed point theorem combined with adequate variational arguments. The function space setting used here makes appeal to the variable exponent Lebesgue and Sobolev spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. Let $p(\cdot) : \overline{\Omega} \to (2, \infty)$ be a continuous function such that

$$\lambda_1 := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} > 0. \tag{1}$$

We point out that property $\lambda_1 > 0$ is not true for all functions $p(\cdot)$. For instance, assuming that there exists an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) < p(x)$ (or $p(x_0) > p(x)$) for all $x \in \partial U$, then by [1, Theorem 3.1] we get $\lambda_1 = 0$. On the other hand, there are results establishing sufficient conditions on $p(\cdot)$ for satisfying $\lambda_1 > 0$. Indeed, it was proved in [1, Theorem 3.3] that assuming that there exists a vector $l \in \mathbb{R}^N \setminus \{0\}$ such that, for any $x \in \Omega$, the function $f(t) = p(x+tl)$ is monotone, for $t \in I_x := \{s; x + sl \in \Omega\}$ then $\lambda_1 > 0$. Furthermore, it was shown in [2, Theorem 1] that (1) holds provided that $p(\cdot) \in C^1(\Omega; \mathbb{R})$ and that there exists $\overrightarrow{a} \in C^1(\Omega; \mathbb{R}^N)$ such that $\operatorname{div}\overrightarrow{a}(x) \geq a_0 > 0$ and $\overrightarrow{a}(x) \cdot \nabla p(x) = 0$, for every $x \in \Omega$ (see also [3, Theorem 1] for similar results). Finally, we recall a very well-known fact that in the special case when $p(\cdot)$ is a constant function (defined on the interval $(1, \infty)$) then (1) holds.

Next, assume that $A : \Omega \to \mathbb{R}^{N^2}$ is a symmetric function matrix, i.e. $a_{ij} = a_{ji}$, such that $a_{ij} \in L^\infty(\Omega) \cap C^1(\Omega)$ and

$$\langle A\xi, \xi\rangle = \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^N, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathbb{R}^N$.

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In this paper we are concerned with the study of nonlinear and nonhomogeneous problems of the type

$$\begin{cases} -\operatorname{div}\left(\alpha(x,u)\,\langle A\nabla u,\nabla u\rangle^{\frac{p(x)-2}{2}}\,A\nabla u\right)=f & \text{for } x\in\Omega,\\ u=0 & \text{for } x\in\partial\Omega, \end{cases} \tag{3}$$

where $\alpha:\Omega\times\mathbb{R}\to(0,\infty)$ represents a Carathéodory function, which depends on the space variable x and on the state variable u and is far from zero and bounded above (see assumption (8)). This restriction was first introduced in the pioneering paper by Arcoya and Boccardo [4] but it was no longer assumed in subsequent papers by Filippucci [5,6]. On the other hand, $f:\Omega\to\mathbb{R}$ is a measurable function belonging to a suitable Lebesgue type space which will be specified later on in the paper.

The differential operator involved in Eq. (3) will be denoted by

$$\mathcal{A}_{p(\cdot)}u:=\operatorname{div}\left(\alpha(x,u)\,\langle A\nabla u,\nabla u\rangle^{\frac{p(x)-2}{2}}\,A\nabla u\right)$$

and will be called the $\mathcal{A}_{p(\cdot)}$-*Laplace operator*. It represents a generalization of the $p(\cdot)$-*Laplace operator*, i.e. $\Delta_{p(\cdot)}u=\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$, which is obtained in the case when $A=Id$ and $\alpha\equiv 1$. In the last few decades special attention has been paid to $p(\cdot)$-Laplace type operators since they can model with sufficient accuracy the phenomena arising from the study of electrorheological fluids [7,8], image restoration [9], mathematical biology [10], dielectric breakdown, electrical resistivity and polycrystal plasticity [11,12] and they arise in the study of some models for the growth of heterogeneous sandpiles [13]. In a similar context, we note that a collection of results obtained in the field of partial differential equations involving $p(\cdot)$-Laplace type operators can be found in the survey paper by Harjulehto et al. [14]. Finally, we recall that in the case when $p(\cdot)$ is a constant function, problems involving $\mathcal{A}_p$-Laplace type operators have been widely studied. In this regard we point out the papers by Reshetnyak [15], Alvino et al. [16] and El Khalil et al. [17] and the references therein.

2. A review on variable exponent spaces

In this section we provide a brief review of basic properties of the variable exponent Lebesgue–Sobolev spaces. For more details we refer the reader to the book by Diening et al. [18] and the paper by [19].

In this paper we reduce all our discussion to the special case when $\Omega\subset\mathbb{R}^N$ is an open bounded set. For any continuous function $p:\overline{\Omega}\to(1,\infty)$ we define

$$p^-:=\inf_{x\in\Omega}p(x)\quad\text{and}\quad p^+:=\sup_{x\in\Omega}p(x).$$

Next, we define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\to\mathbb{R}\text{ measurable}:\int_\Omega|u(x)|^{p(x)}\,dx<\infty\right\}.$$

Clearly, $L^{p(\cdot)}(\Omega)$ is a Banach space when endowed with the so-called *Luxemburg norm*, defined by

$$|u|_{p(\cdot)}:=\inf\left\{\mu>0:\int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx\le 1\right\}.$$

We note that the variable exponent Lebesgue space is a special case of an Orlicz–Musielak space. For constant functions p, $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)}:=\left(\int_\Omega|u(x)|^p dx\right)^{1/p}.$$

We recall that $L^{p(\cdot)}(\Omega)$ is separable and reflexive. Since Ω is bounded, if p_1, p_2 are variable exponents such that $p_1\le p_2$ in Ω, the embedding $L^{p_2(\cdot)}(\Omega)\hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous and its norm does not exceed $|\Omega|+1$.

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x)+1/p'(x)=1$. For any $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{4}$$

holds.

A key role in manipulating the variable exponent Lebesgue and Sobolev (see below) spaces is played by the *modular* of the space $L^{p(\cdot)}(\Omega)$, which is the mapping $\rho_{p(\cdot)}:L^{p(\cdot)}(\Omega)\to\mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u):=\int_\Omega|u(x)|^{p(x)}\,dx.$$

If $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold:

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^+}_{p(\cdot)}; \tag{5}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^-}_{p(\cdot)}; \tag{6}$$

$$|u|_{p(\cdot)} = 1 \Leftrightarrow \rho_{p(\cdot)}(u) = 1. \tag{7}$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On this space one can consider the following norm:

$$\|u\|_{p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

where, in the above definition, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$. We note that in the context of this discussion $W^{1,p(\cdot)}(\Omega)$ is also a separable and reflexive Banach space.

Finally, we define $W^{1,p(\cdot)}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(\cdot)}.$$

Note that $(W^{1,p(\cdot)}_0(\Omega), \|\cdot\|)$ is also a separable and reflexive Banach space. We remark that if $q : \overline{\Omega} \to (1, \infty)$ is a continuous function such that $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,p(\cdot)}_0(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geq N$.

3. The main result

The main result of this paper is given by the following theorem.

Theorem 1. *Assume that $\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for which there exist two positive constants $0 < \lambda \leq \Lambda$ such that*

$$0 < \lambda \leq \alpha(x, t) \leq \Lambda, \quad \text{a.e. } x \in \Omega, \ \forall\, t \in \mathbb{R}. \tag{8}$$

Assume that conditions (1) and (2) from Section 1 are satisfied. Then for each $f \in L^{p'(\cdot)}(\Omega)$ there exists a weak solution of problem (3), i.e. a function $u \in W^{1,p(\cdot)}_0(\Omega)$ such that

$$\int_\Omega \alpha(x, u)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u, \nabla \varphi\rangle\, dx = \int_\Omega f\varphi\, dx,$$

for all $\varphi \in W^{1,p(\cdot)}_0(\Omega)$.

4. Proof of the main result

Fix an arbitrary function $f \in L^{p'(\cdot)}(\Omega)$. The main ingredient of our proof of Theorem 1 will be Schauder's fixed point theorem (see [20, Theorem 3.21]):

Schauder's fixed point theorem. *Assume that K is a compact and convex subset of the Banach space B and $S : K \to K$ is a continuous map. Then S possesses a fixed point.*

We start by proving some auxiliary results which will be useful in establishing Theorem 1.

Lemma 1. *For each $v \in L^{p(\cdot)}(\Omega)$ the problem*

$$\begin{cases} -\operatorname{div}\left(\alpha(x, v)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} A\nabla u\right) = f & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{9}$$

has a weak solution $u \in W^{1,p(\cdot)}_0(\Omega)$, i.e.

$$\int_\Omega \alpha(x, v)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u, \nabla \varphi\rangle\, dx = \int_\Omega f\varphi\, dx, \tag{10}$$

for all $\varphi \in W^{1,p(\cdot)}_0(\Omega)$.

Proof. Fix $v \in L^{p(\cdot)}(\Omega)$. First, we note that condition (8) from Theorem 1 guarantees that $\alpha(x, v) \in L^\infty(\Omega)$.

Consider the energy functional associated with problem (9), $J : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$,

$$J(u) = \int_\Omega \frac{\alpha(x,v)}{p(x)} \langle A\nabla u, \nabla u\rangle^{p(x)/2}\, dx - \int_\Omega fu\, dx.$$

Standard arguments imply that $J \in C^1(W_0^{1,p(\cdot)}(\Omega); \mathbb{R})$ with the derivative given by

$$\langle J'(u), \varphi\rangle = \int_\Omega \alpha(x,v)\langle A\nabla u, \nabla u\rangle^{\frac{p(x)-2}{2}} \langle A\nabla u, \nabla\varphi\rangle\, dx = \int_\Omega f\varphi\, dx,$$

for all $u, \varphi \in W_0^{1,p(\cdot)}(\Omega)$. Thus, weak solutions of problem (9) are exactly the critical points of the functional J.

Since (2) and (8) are fulfilled it follows that for each $u \in W_0^{1,p(\cdot)}(\Omega)$ with $\|u\| > 1$ we have

$$\begin{aligned} J(u) &\geq \frac{\lambda}{p^+}\int_\Omega |\nabla u|^{p(x)}\, dx - \int_\Omega fu\, dx \\ &\geq \frac{\lambda}{p^+}\|u\|^{p^-} - c|f|_{p'(\cdot)}\|u\|, \end{aligned}$$

where c is a positive constant. The above estimate shows that J is coercive.

On the other hand, it was pointed out in [16, p. 449] that the following Clarkson's type inequality:

$$\frac{\langle A\xi_1, \xi_1\rangle^{s/2} + \langle A\xi_2, \xi_2\rangle^{s/2}}{2} \geq \left\langle A\left(\frac{\xi_1+\xi_2}{2}\right), \frac{\xi_1+\xi_2}{2}\right\rangle^{s/2} + \left\langle A\left(\frac{\xi_1-\xi_2}{2}\right), \frac{\xi_1-\xi_2}{2}\right\rangle^{s/2}, \tag{11}$$

holds for all $s \geq 2$ and $\xi_1, \xi_2 \in \mathbb{R}^N$. Thus, we deduce that J is convex and consequently weakly lower semi-continuous.

Since J is coercive and weakly lower semi-continuous we conclude via the Direct Method of the Calculus of Variations (see, e.g. [21, Theorem 1.2]) that there exists a global minimum point of J, $u \in W_0^{1,p(\cdot)}(\Omega)$, and consequently a weak solution of problem (9). The proof of Lemma 1 is thus complete. □

Next, for each $v \in L^{p(\cdot)}(\Omega)$ let $u = T(v) \in W_0^{1,p(\cdot)}(\Omega)$ be the weak solution of problem (9) given by Lemma 1. Thus, we can actually introduce an application $T : L^{p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ associating with each $v \in L^{p(\cdot)}(\Omega)$ the solution of problem (9), $T(v) \in W_0^{1,p(\cdot)}(\Omega)$.

Lemma 2. *There exists $C > 0$, a universal constant such that*

$$\int_\Omega |\nabla T(v)|^{p(x)}\, dx \leq C, \quad \forall\, v \in L^{p(\cdot)}(\Omega). \tag{12}$$

Proof. Taking $\varphi = T(v)$ in (10) we find

$$\int_\Omega \alpha(x,v)\langle A\nabla T(v), \nabla T(v)\rangle^{\frac{p(x)}{2}}\, dx = \int_\Omega fT(v)\, dx, \quad \forall\, v \in L^{p(\cdot)}(\Omega).$$

Taking into account relation (8) and condition (2), the above equality yields

$$\lambda\int_\Omega |\nabla T(v)|^{p(x)}\, dx \leq \int_\Omega fT(v)\, dx, \quad \forall\, v \in L^{p(\cdot)}(\Omega). \tag{13}$$

Let now $\epsilon > 0$ be such that $\epsilon < \min\{1, \lambda, \lambda/\lambda_1\}$. Then, by Young's inequality (see, e.g., [22, the footnote on p. 56]) we deduce

$$f(x)T(v(x)) \leq \frac{1}{\epsilon^{p(x)-1}}|f(x)|^{p'(x)} + \epsilon|T(v(x))|^{p(x)}, \quad \forall\, v \in L^{p(\cdot)}(\Omega),\ x \in \Omega,$$

or, since $\epsilon \in (0,1)$, there exists $C_\epsilon := \frac{1}{\epsilon^{p^+-1}}$ such that

$$f(x)T(v(x)) \leq C_\epsilon|f(x)|^{p'(x)} + \epsilon|T(v(x))|^{p(x)}, \quad \forall\, v \in L^{p(\cdot)}(\Omega),\ x \in \Omega.$$

Integrating the above estimate over Ω and taking into account that relations (1) and (13) hold we get

$$\lambda\int_\Omega |\nabla T(v)|^{p(x)}\, dx \leq C_\epsilon\int_\Omega |f|^{p'(x)}\, dx + \frac{\epsilon}{\lambda_1}\int_\Omega |\nabla T(v)|^{p(x)}\, dx, \quad \forall\, v \in L^{p(\cdot)}(\Omega).$$

Consequently, taking

$$C := \frac{C_\epsilon\int_\Omega |f|^{p'(x)}\, dx}{\lambda - \frac{\epsilon}{\lambda_1}},$$

we infer that relation (12) holds true. The proof of Lemma 2 is thus also complete. □

Remark 1. By Lemma 2 and relation (1) it clearly follows that there exists a universal constant $C_1 > 0$ such that

$$\int_\Omega |T(v)|^{p(x)}\,dx \le C_1, \quad \forall\, v \in L^{p(\cdot)}(\Omega).$$

Lemma 3. *The map $T : L^{p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ is continuous.*

Proof. Let (v_n), $v \subset L^{p(\cdot)}(\Omega)$ be such that v_n converges to v in $L^{p(\cdot)}(\Omega)$ as $n \to \infty$. Set

$$u_n := T(v_n), \quad \forall\, n.$$

By Lemma 2 we have

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx = \int_\Omega |\nabla T(v_n)|^{p(x)}\,dx \le C, \quad \forall\, n,$$

i.e. (u_n) is bounded on $W_0^{1,p(\cdot)}(\Omega)$. It follows that by eventually passing to a subsequence we can conclude that u_n converges weakly to u in $W_0^{1,p(\cdot)}(\Omega)$.

On the other hand, for each n we have

$$\int_\Omega \alpha(x, v_n)\langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u_n, \nabla\varphi\rangle\,dx = \int_\Omega f\varphi\,dx, \tag{14}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. Taking $\varphi = u_n - u$ in the above equality it follows that

$$\int_\Omega \alpha(x, v_n)\langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u_n, \nabla u_n - \nabla u\rangle\,dx = o(1).$$

This fact and relation (8) yield

$$\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u_n, \nabla u_n - \nabla u\rangle\,dx = o(1). \tag{15}$$

Next, by taking $\varphi = u_n$ in (14) we get

$$\int_\Omega \alpha(x, v_n)\langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx = \int_\Omega f u_n\,dx,$$

for each n. Relation (8), Hölder's inequality, the Poincaré's inequality and the fact that (u_n) is bounded on $W_0^{1,p(\cdot)}(\Omega)$ imply that there exist some constants C_2, C_3 $C_4 > 0$ such that

$$\lambda\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx \le \int_\Omega f u_n\,dx \le C_2|f|_{p'(\cdot)}|u_n|_{p(\cdot)} \le C_3\|u_n\| \le C_4, \quad \forall\, n.$$

The above estimates assure that sequence $(\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx)$ is bounded. Therefore we can deduce that there exists $b > 0$ such that, up to a subsequence,

$$\lim_{n\to\infty}\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx = b.$$

Furthermore, recalling relation (11) and the fact that $p(x) \ge 2$ for all $x \in \Omega$, we deduce that the map

$$W_0^{1,p(\cdot)}(\Omega) \ni w \to \int_\Omega \langle A\nabla w, \nabla w\rangle^{\frac{p(x)}{2}}\,dx \in \mathbb{R}, \tag{16}$$

is convex and consequently weakly lower semi-continuous. Thus, we deduce

$$\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\,dx \le \liminf_{n\to\infty}\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx = b.$$

On the other hand, using relation (A.2) in [16], i.e.

$$\langle A\xi_2, \xi_2\rangle^s \ge \langle A\xi_1, \xi_1\rangle^s + s\langle A\xi_1, \xi_1\rangle^{s-2}\langle A\xi_1, \xi_2 - \xi_1\rangle, \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^n,\ s \ge 2,$$

we obtain that

$$\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\,dx \ge \int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\,dx + p^-\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)-2}{2}}\langle A\nabla u_n, \nabla u - \nabla u_n\rangle\,dx, \quad \forall\, n.$$

The above pieces of information and relation (15) show that

$$\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\, dx = b.$$

Taking into account that $(\frac{u_n+u}{2})$ converges weakly to u in $W_0^{1,p(\cdot)}(\Omega)$ and again invoking the weak lower semi-continuity of the map defined in relation (16) we find

$$b = \int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\, dx \le \liminf_{n\to\infty} \int_\Omega \left\langle A\nabla \frac{u_n+u}{2}, \nabla \frac{u_n+u}{2}\right\rangle^{\frac{p(x)}{2}} dx. \tag{17}$$

Assume by contradiction that (u_n) does not converge (strongly) to u in $W_0^{1,p(\cdot)}(\Omega)$. Then there exist $\epsilon > 0$ and a subsequence of (u_n), still denoted by (u_n), such that

$$\int_\Omega \left|\frac{\nabla u_n - \nabla u}{2}\right|^{p(x)} dx \ge \epsilon, \quad \forall\, n.$$

On the other hand, relations (11) and (2) imply

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p(x)}{2}}\, dx + \frac{1}{2}\int_\Omega \langle A\nabla u_n, \nabla u_n\rangle^{\frac{p(x)}{2}}\, dx - \int_\Omega \left\langle A\nabla \frac{u_n+u}{2}, \nabla \frac{u_n+u}{2}\right\rangle^{\frac{p(x)}{2}} dx\\
&\quad \ge \int_\Omega \left\langle A\nabla \frac{u-u_n}{2}, \nabla \frac{u-u_n}{2}\right\rangle^{\frac{p(x)}{2}} dx\\
&\quad \ge \int_\Omega \left|\frac{\nabla u - \nabla u_n}{2}\right|^{p(x)} dx, \quad \forall\, n.
\end{aligned}$$

The last two estimates yield

$$b - \epsilon \ge \limsup_{n\to\infty} \int_\Omega \left\langle A\nabla \frac{u_n+u}{2}, \nabla \frac{u_n+u}{2}\right\rangle^{\frac{p(x)}{2}} dx,$$

which contradicts (17). Consequently, (u_n) converges (strongly) to u in $W_0^{1,p(\cdot)}(\Omega)$, or $T : L^{p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(\Omega)$ is continuous. The proof of Lemma 3 is complete. □

Remark 2. Since $W_0^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{p(\cdot)}(\Omega)$ (i.e. the inclusion operator $i : W_0^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$ is compact), it follows by Lemma 3 that the operator $S : L^{p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$, $S = i \circ T$, is compact.

Proof of Theorem 1. Let C_1 be the constant given in Remark 1, i.e.

$$\int_\Omega |S(v)|^{p(x)}\, dx \le C_1, \quad \forall\, v \in L^{p(\cdot)}(\Omega).$$

Consider the ball

$$B_{C_1}(0) := \left\{ v \in L^{p(\cdot)}(\Omega) : \int_\Omega |v|^{p(x)}\, dx \le C_1 \right\}.$$

Clearly, $B_{C_1}(0)$ is a convex closed subset of $L^{p(\cdot)}(\Omega)$ and $S(B_{C_1}(0)) \subset B_{C_1}(0)$. Moreover, by Remark 2, $S(B_{C_1}(0))$ is relatively compact in $B_{C_1}(0)$.

Finally, by Lemma 3 and Remark 2, $S : B_{C_1}(0) \to B_{C_1}(0)$ is a continuous map. Hence we can apply Schauder's fixed point theorem to obtain S with a fixed point. This gives us a weak solution to problem (3) and thus the proof of Theorem 1 is finally complete. □

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