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A topological characterization of *LF*-spaces ☆

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ABSTRACT

We present a topological characterization of *LF*-spaces and detect small box-products that are (locally) homeomorphic to *LF*-spaces.

1. Introduction

In this paper we shall present a simple criterion for recognizing topological spaces that are homeomorphic to (open subspaces of) *LF*-spaces. This criterion was applied in [3,4,7] for detecting topological groups that are homeomorphic to (open subspaces of) *LF*-spaces.

We recall that an *LF-space* is the direct limit $\text{lc-}\varinjlim X_n$ of an increasing sequence

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

of Fréchet (= locally convex complete linear metric) spaces in the category of locally convex spaces. The simplest example of a non-metrizable *LF*-space is the inductive limit $\mathbb{R}^\infty = \text{lc-}\varinjlim \mathbb{R}^n$ of the sequence

$$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots$$

of Euclidean spaces, where each space $\mathbb{R}^n$ is identified with the hyperplane $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. The space $\mathbb{R}^\infty$ is topologically isomorphic to the direct sum $\bigoplus_{n\in\omega} \mathbb{R}$ of one-dimensional Fréchet spaces in the category of locally convex spaces.

Mankiewicz [17] obtained a topological classification of *LF*-spaces and proved that each *LF*-space is homeomorphic to the direct sum $\bigoplus_{n\in\omega} l_2(\kappa_i)$ of Hilbert spaces for some sequence $(\kappa_i)_{i\in\omega}$ of cardinals. Here $l_2(\kappa)$ stands for the Hilbert space with

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orthonormal base of cardinality κ. In particular, $l_2(n) = \mathbb{R}^n$ for a finite cardinal n. A more precise version of Mankiewicz's classification says that the spaces

- $l_2(\kappa)$ for some cardinal $\kappa \geqslant 0$,
- $\mathbb{R}^\infty$,
- $l_2(\kappa) \times \mathbb{R}^\infty$ for some $\kappa \geqslant \omega$, and
- $\bigoplus_{n\in\omega} l_2(\kappa_i)$ for a strictly increasing sequence of infinite cardinals $(\kappa_i)_{i\in\omega}$

are pairwise non-homeomorphic and represent all possible topological types of *LF*-spaces. In particular, each infinite-dimensional separable *LF*-space is homeomorphic to one of the following spaces: l_2, $\mathbb{R}^\infty$ or $l_2 \times \mathbb{R}^\infty$.

The topological characterizations of the *LF*-spaces l_2 and $\mathbb{R}^\infty$ were given by Toruńczyk [23,24] and Sakai [21], respectively. These characterizations belong among the best achievements of the classical infinite-dimensional topology. In this paper we shall present a topological characterization of other *LF*-spaces, in particular, $l_2 \times \mathbb{R}^\infty$. First, we recall Sakai's topological characterization of the *LF*-space $\mathbb{R}^\infty$. This characterization is based on the observation that the *LF*-space $\mathbb{R}^\infty = \text{lc-}\varinjlim \mathbb{R}^n$ carries the topology of the topological direct limit of the tower $(\mathbb{R}^n)_{n\in\omega}$ of finite-dimensional Euclidean spaces.

By the *topological direct limit* $\text{t-}\varinjlim X_n$ of a tower

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

of topological spaces we understand the union $X = \bigcup_{n\in\omega} X_n$ endowed with the largest topology turning the identity inclusions $X_n \to X$, $n \in \omega$, into continuous maps.

Theorem 1.1 *(Sakai). A topological space X is homeomorphic to (an open subspace of) the space $\mathbb{R}^\infty$ if and only if*

(1) *X is homeomorphic to the topological direct limit $\text{t-}\varinjlim X_n$ of a tower $(X_n)_{n\in\omega}$ of finite-dimensional metrizable compacta and*
(2) *each embedding $f : B \to X$ of a closed subset $B \subset A$ of a finite-dimensional metrizable compact space A extends to an embedding of (an open neighborhood of B in) the space A into X.*

Deleting the adjective "finite-dimensional" from this characterization, we obtain a characterization of (open subspaces of) the space $Q \times \mathbb{R}^\infty$ where $Q = [0,1]^\omega$ is the Hilbert cube, see [21].

Replacing the class of finite-dimensional compact metrizable spaces in Theorem 1.1 by the class of Polish spaces, Pentsak [20] obtained a topological characterization of (open subspaces of) the topological direct limit $\text{t-}\varinjlim (l_2)^n$ of the tower of Hilbert spaces

$$l_2 \subset l_2 \times l_2 \subset \cdots \subset l_2^n \subset \cdots,$$

where each space l_2^n is identified with the subspace $l_2^n \times \{0\}$ of the Hilbert space l_2^{n+1}. However, the topology of the topological direct limit $\text{t-}\varinjlim l_2^n$ is strictly stronger than the topology of the direct limit $\text{lc-}\varinjlim l_2^n$ of that tower in the category of locally convex spaces. Moreover, $\text{t-}\varinjlim l_2^n$ is not even homeomorphic to a topological group, see [2]. In fact, an *LF*-space X is homeomorphic to the topological direct limit of a tower of metrizable spaces if and only if X is either metrizable or is topologically isomorphic to $\mathbb{R}^\infty$, see [1] and [8].

This means that topological direct limits cannot be used for describing the topology of non-metrizable *LF*-spaces which are different from $\mathbb{R}^\infty$. On the other hand, it was discovered in [6] that for any tower $(X_n)_{n\in\omega}$ of Fréchet spaces the topology of the *LF*-space $X = \text{lc-}\varinjlim X_n$ coincides with the topology of the direct limit $\text{u-}\varinjlim X_n$ of this tower in the category of uniform spaces!

By the *uniform direct limit* $\text{u-}\varinjlim X_n$ of a tower

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

of uniform spaces we understand the union $X = \bigcup_{n\in\omega} X_n$ endowed with the largest uniformity turning the identity inclusions $X_n \to X$ into uniformly continuous maps. Each linear topological space L carries the canonical uniformity generated by the entourages $\{(x,y) \in L\colon\ x - y \in U\}$ where $U = -U$ runs over all symmetric neighborhoods of the origin of L.

For any tower $(X_n)_{n\in\omega}$ of Fréchet spaces the identity map $\text{u-}\varinjlim X_n \to \text{lc-}\varinjlim X_n$ is continuous (because each continuous linear operator is uniformly continuous). A less trivial fact established in [6] is the continuity of the inverse map $\text{lc-}\varinjlim X_n \to \text{u-}\varinjlim X_n$. This means that we can identify *LF*-spaces with uniform direct limits of Fréchet spaces and reduce the problem of topological characterization of *LF*-spaces to the problem of recognizing uniform direct limits that are homeomorphic to *LF*-spaces. The answer to this problem will be given in Theorems 1.3 and 1.5 after some definitions.

All spaces considered in this paper are completely regular and all maps are continuous. On the other hand, functions need not be continuous. A *pointed space* is a space X with a distinguished point, which will be denoted by $*_X$.

The *small box-product* of a sequence of pointed topological spaces $(X_n)_{n\in\omega}$ is the subspace

$$\mathop{\boxdot}_{n\in\omega} X_n = \Big\{ (x_n)_{n\in\omega} \in \mathop{\square}_{n\in\omega} X_n\colon\ \exists m \in \omega\ \forall n \geqslant m\ x_n = *_{X_n} \Big\}$$

of the box-product $\square_{n\in\omega}X_n$. The latter space is the Tychonov product $\prod_{n\in\omega} X_n$ endowed with the topology generated by the products $\prod_{n\in\omega} U_n$ of open subsets $U_n \subset X_n$, $n \in \omega$. For a subset $A \subset \omega$ let

$$\underset{n\in A}{\boxdot} X_n = \Big\{(x_n)_{n\in\omega} \in \underset{n\in\omega}{\boxdot} X_n \colon \{n \in \omega\colon x_n \neq *_{X_n}\} \subset A\Big\} \subset \underset{n\in\omega}{\boxdot} X_n.$$

It follows that $\boxdot_{n\in\omega} X_n = \bigcup_{n\in\omega} \boxdot_{i\leqslant n} X_i$. By Proposition 5.3 of [6], for any sequence $(X_n)_{n\in\omega}$ of locally convex linear topological spaces the topology of the small box-product $\boxdot_{n\in\omega} X_n$ coincides with the topology of the direct sum $\bigoplus_{n\in\omega} X_n$ in the category of locally convex linear topological spaces.

For a uniform space X its uniformity will be denoted by $\mathcal{U}_X$. Elements of the uniformity $\mathcal{U}_X$ are called *entourages*. The Hausdorff property of X implies that $\bigcap \mathcal{U}_X = \{(x,x)\colon x \in X\}$. A uniform space X is called *metrizable* if its uniformity is generated by a metric. For a point $a \in X$, a subset $A \subset X$, and an entourage $U \in \mathcal{U}_X$, let $B(a,U) = \{x \in X\colon (x,a) \in U\}$ and $B(A,U) = \bigcup_{a\in A} B(a,U)$ be the U-neighborhoods of a and A, respectively. A neighborhood $O(A)$ of A in X is called *uniform* if $O(A)$ contains the U-neighborhood $B(A,U)$ for some entourage $U \in \mathcal{U}_X$.

Definition 1.2. Let C be a pointed topological space. A subset A of a uniform space X is called *C-complemented* in X if there is a homeomorphism $\gamma : A \times C \to X$ such that

(1) for any neighborhood $V \subset C$ of $*_C$ there is an entourage $U \in \mathcal{U}_X$ such that $B(A,U) \subset \gamma(A \times V)$;
(2) for any entourage $U \in \mathcal{U}_X$ there is a neighborhood $V \subset C$ of $*_C$ such that $\gamma(\{a\} \times V) \subset B(a,U)$ for each $a \in A$.

A subset A of X is called *locally C-complemented* if for some open neighborhood $V \subset C$ of $*_C$ the set A is V-complemented in some open uniform neighborhood $U(A)$ of A in X.

The following theorem shows that often uniform direct limits are (locally) homeomorphic to small box-products. We shall say that a topological space X is *locally homeomorphic* to a topological space Y if each point $x \in X$ has an open neighborhood $O_x \subset X$ which is homeomorphic to an open subspace of Y.

Theorem 1.3. *Let $(X_n)_{n\in\omega}$ be a tower of uniform spaces such that for every $n \in \omega$ the space X_n is (locally) C_n-complemented in X_{n+1} for some pointed topological space C_n and X_n is (locally) homeomorphic to the product $X_0 \times \boxdot_{i<n} C_i$. Then the uniform direct limit $\text{u-}\varinjlim X_n$ is (locally) homeomorphic to the small box-product $X_0 \times \boxdot_{n\in\omega} C_n$.*

In light of Theorem 1.3 it is important to recognize small box-products that are (locally) homeomorphic to *LF*-spaces. In this respect we have the following:

Conjecture 1.4. *The small box-product $\boxdot_{n\in\omega} X_n$ of pointed topological spaces is homeomorphic to (an open subset of) an LF-space if for every $n \in \omega$ the finite product $\prod_{i\leqslant n} X_i$ is homeomorphic to (an open subset of) a Hilbert space.*

We shall confirm this conjecture under an additional assumption that for infinitely many numbers $n \in \omega$ the space X_n is *lz*-pointed. The definition of an *lz*-pointed space involves the notion of a strong Z-set, well known in infinite-dimensional topology, see [5, §1.4], [10], and [14, §2.2].

We recall that a closed subset A of a topological space X is a (*strong*) *Z-set* in X if for any open cover $\mathcal{U}$ of X there is a map $f : X \to X$ such that f is $\mathcal{U}$-near to the identity map id_X of X and (the closure $\overline{f(X)}$ of) the set $f(X)$ does not intersect A. Two maps $f, g : X \to X$ are called *$\mathcal{U}$-near* if for each point $x \in X$ the doubleton $\{f(x), g(x)\}$ lies in some set $U \in \mathcal{U}$. It is clear that each strong Z-set is a Z-set. The converse is not true, see [11]. However, in Hilbert spaces each Z-set is a strong Z-set, see [11,24]. A point x_0 of a space X will be called a *strong Z-point* in X if the singleton $\{x_0\}$ is a strong Z-set in X.

A pointed space X will be called

- *l-pointed* if $*_X$ is not isolated and X is locally compact;
- *z-pointed* if $*_X$ is a strong Z-point in X;
- *lz-pointed* if X is *l*-pointed or *z*-pointed.

For example, each non-trivial Hilbert space H with distinguished point 0 is an *lz*-pointed space. More precisely, H is *l*-pointed if $0 < \dim(H) < \infty$ and H is *z*-pointed if $\dim(H) = \infty$.

The following theorem (that will be proved in Section 10) confirms Conjecture 1.4 for small box-products of *lz*-pointed spaces.

Theorem 1.5. *The small box-product $\boxdot_{n\in\omega} X_n$ of pointed topological spaces X_n is homeomorphic to (an open subset of) an LF-space if for every $n \in \omega$ the finite product $\prod_{i\leqslant n} X_i$ is homeomorphic to (an open subset of) a Hilbert space and for infinitely many numbers $n \in \omega$ the space X_n is lz-pointed.*

A subset A of a uniform space X will be called (*locally*) *lz-complemented* if A is (locally) C-complemented in X for some *lz*-pointed space C. By analogy we define (locally) z-complemented subsets of uniform spaces.

Theorems 1.3 and 1.5 imply the following criterion.

Theorem 1.6. *The uniform direct limit* $\text{u-}\varinjlim X_n$ *of a tower of uniform spaces* $(X_n)_{n\in\omega}$ *is*

(1) *homeomorphic to* (*an open subset of*) *an LF-space if for every* $n \in \omega$ *the space* X_n *is homeomorphic to* (*an open subset of*) *a Hilbert space and* X_n *is lz-complemented in* X_{n+1};
(2) (*locally*) *homeomorphic to an LF-space if for every* $n \in \omega$ *the space* X_n *is* (*locally*) *homeomorphic to a Hilbert space and* X_n *is* (*locally*) *lz-complemented in* X_{n+1};
(3) *homeomorphic to an LF-space if for every* $n \in \omega$ *the uniform space* X_n *is metrizable, is homeomorphic to a Hilbert space and* X_n *is locally z-complemented in* X_{n+1}.

The last statement of this theorem does not follow from Theorems 1.3 and 1.5. It will be proved in a more general context of typical model spaces in Theorem 9.1. Because of the lack of the Open Embedding Theorem for *LF*-manifolds, we distinguish between *LF*-manifolds and open subspaces of *LF*-spaces. That is why we included two separate items (1) and (2) in Theorem 1.6. It should be mentioned that the topological structure of open subspaces of *LF*-spaces is quite well understood, which cannot be said about *LF*-manifolds, see [18,19].

Theorem 1.3 will be proved in Section 3. In Section 2 we recall the necessary information on uniform direct limits. In Section 4 we study reduced products of pointed spaces and prove an important Lemma 4.1 on regular homeomorphisms of pairs. In Section 5 we introduce the notion of a typical model space so that manifolds modeled on such spaces have many common properties with Hilbert manifolds. In Section 6 we shall prove two lemmas about complemented subsets in metrizable uniform spaces that are homeomorphic to typical model spaces. In Section 7 we study small box-product of locally compact spaces and show that for any sequence $(X_i)_{i\in\omega}$ of locally compact ANR-spaces the small box-product $Q \times \boxdot_{i\in\omega} X_i$ is locally homeomorphic to $Q \times \mathbb{R}^\infty$ where $Q = [0,1]^\omega$ is the Hilbert cube. In Section 8 we apply this result to recognize the small box-products that are (locally) homeomorphic to small box-products $\boxdot_{n\in\omega} E_n$ of typical model spaces. In Section 9 we apply the results about small box-products and prove a criterion for recognizing uniform direct limits that are (locally) homeomorphic to small box-products $\boxdot_{n\in\omega} E_n$ of typical model spaces.

2. Uniform direct limits

In this section we recall the necessary information on uniform direct limits. By a *tower* of uniform spaces we shall understand an increasing sequence

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

of uniform spaces (so, the uniformity of each space X_n coincides with the uniformity inherited from the uniform space X_{n+1}).

The *uniform direct limit* $\text{u-}\varinjlim X_n$ of a tower of uniform spaces $(X_n)_{n\in\omega}$ is the union $X = \bigcup_{n\in\omega} X_n$ endowed with the largest uniformity making the identity inclusions $X_n \to X$, $n \in \omega$, uniformly continuous. The topology and the uniformity of the uniform direct limits $\text{u-}\varinjlim X_n$ has been described in [6].

By Proposition 5.4 of [6], for a tower $(X_n)_{n\in\omega}$ of locally compact uniform spaces the identity map $\text{t-}\varinjlim X_n \to \text{u-}\varinjlim X_n$ is a homeomorphism. This means that the topology of uniform direct limit on $\bigcup_{n\in\omega} X_n$ coincides with the topology of topological direct limit.

A map $f : X \to Y$ between uniform spaces is called *regular at a subset* $A \subset X$ if for any entourages $U \in \mathcal{U}_Y$ and $V \in \mathcal{U}_X$ there is an entourage $W \in \mathcal{U}_X$ such that for each point $x \in B(A, W)$ there is a point $a \in A$ such that $(x, a) \in V$ and $(f(x), f(a)) \in U$.

The following criterion for recognizing continuous maps between uniform direct limits was proved in Theorem 1.6 of [6].

Proposition 2.1. *Let* $(X_n)_{n\in\omega}$ *be a tower of uniform spaces and* $X = \text{u-}\varinjlim X_n$ *be its uniform direct limit. A function* $f : X \to Y$ *from* X *to a uniform space* Y *is continuous provided that for every* $n \in \mathbb{N}$ *the restriction* $f|X_n : X_n \to Y$ *is continuous and regular at* X_{n-1}.

Let X, Y be uniform spaces and $X_0 \subset X$, $Y_0 \subset X$ be subspaces. A homeomorphism of pairs $h : (X, X_0) \to (Y, Y_0)$ is called *regular* if $h(X_0) = Y_0$, h is regular at X_0 and h^{-1} is regular at Y_0.

Proposition 2.1 implies a simple criterion for recognizing homeomorphisms between uniform direct limits.

Corollary 2.2. *Let* $(X_n)_{n\in\omega}$ *and* $(Y_n)_{n\in\omega}$ *be towers of uniform spaces. A bijective function* $h : \text{u-}\varinjlim X_n \to \text{u-}\varinjlim Y_n$ *is a homeomorphism if for every* $n \in \omega$ *the restriction* $h|X_n$ *is a regular homeomorphism of the pairs* (X_{n+1}, X_n) *and* (Y_{n+1}, Y_n).

We shall often use the following fact established in Proposition 5.5 of [6].

Proposition 2.3. *For a sequence* $(X_n)_{n\in\omega}$ *of pointed uniform spaces the identity map* $\text{u-}\varinjlim \boxdot_{i\leqslant n} X_i \to \boxdot_{n\in\omega} X_n$ *is a homeomorphism.*

3. Proof of Theorem 1.3

We shall divide the proof of Theorem 1.3 into three lemmas.

Lemma 3.1. *Let $(X_n)_{n\in\omega}$ be a tower of uniform spaces such that for every $n \in \omega$ the space X_n is locally C_n-complemented in X_{n+1} for some pointed space C_n. Then the set X_0 has an open neighborhood $U \subset \text{u-}\varinjlim X_n$ that is homeomorphic to the small box-product $X_0 \times \boxdot_{n\in\omega} W_n$ for some open neighborhoods $W_n \subset C_n$ of the distinguished points $*_{C_n}$.*

Proof. For every $n \in \omega$ the set X_n is locally C_n-complemented in X_{n+1}. Consequently, for some open neighborhood $W_n \subset C_n$ of $*_{C_n}$ there is an open embedding $\gamma_n : X_n \times W_n \to X_{n+1}$ such that

(Γ_1) for any neighborhood $V \subset W_n$ of $*_C$ there is an entourage $U \in \mathcal{U}_{X_{n+1}}$ such that $B(X_n, U) \subset \gamma_n(X_n \times V)$;
(Γ_2) for each entourage $U \in \mathcal{U}_{X_{n+1}}$ there is a neighborhood $V \subset W_n$ of $*_{C_n}$ such that for any $x \in X_n$ and $c \in V$ we get $\gamma_n(x, c) \in B(x, U)$.

The condition (Γ_2) implies that $\gamma_n(x, *_{C_n}) = x$ for all $x \in X_n$.

If the set X_n is C_n-complemented in X_{n+1}, then we shall assume that $W_n = C_n$ and $\gamma_n(X_n \times W_n) = X_{n+1}$.

On each space C_n fix a uniformity that generates the topology of C_n and observe that the map γ_n determines a regular homeomorphism of the pairs $(X_n \times W_n, X_n \times \{*_{C_n}\})$ and $(\gamma_n(X_n \times W_n), X_n)$.

Let $U_0 = X_0$ and for every $n \in \omega$ define an open subset $U_{n+1} \subset X_{n+1}$ by the recursive formula $U_{n+1} = \gamma_n(U_n \times W_n)$.

Let $h_0 = \gamma_0 : X_0 \times W_0 \to U_1$. For every $n \in \mathbb{N}$ define a homeomorphism $h_n : X_0 \times \boxdot_{i\leqslant n} W_i \to U_{n+1}$ by the recursive formula $h_n(x, c) = \gamma_n(h_{n-1}(x), c)$ where $x \in X_0 \times \boxdot_{i<n} W_i$ and $c \in W_n$. It follows that $h_n|X_0 \times \boxdot_{i<n} W_i = h_{n-1}$ and

$$h_n : \Bigl(X_0 \times \mathop{\boxdot}_{i\leqslant n} W_i, X_0 \times \mathop{\boxdot}_{i<n} W_i\Bigr) \to (U_{n+1}, U_n)$$

is a regular homeomorphism of pairs. By Corollary 2.2 and Proposition 2.3, the bijective map

$$h = \bigcup_{n\in\omega} h_n : \bigcup_{n\in\omega}\Bigl(X_0 \times \mathop{\boxdot}_{i\leqslant n} W_i\Bigr) \to \bigcup_{n\in\omega} U_n$$

is a homeomorphism between the small box-product $X_0 \times \boxdot_{n\in\omega} W_k$ and the uniform direct limit $U = \text{u-}\varinjlim U_n$.

We claim that $U = \text{u-}\varinjlim U_n$ is an open subspace of $\text{u-}\varinjlim X_n$. First we show that the set $U = \bigcup_{n\in\omega} U_n$ is open in $X = \text{u-}\varinjlim X_n$.

Given any point $x \in U$, find $n \in \omega$ such that $x \in U_n$. Since the set U_n is open in the uniform space X_n, there is an entourage $\varepsilon_n \in \mathcal{U}_{X_n}$ such that $B(x, 2\varepsilon_n) \subset U_n$, where $2\varepsilon_n = \varepsilon_n \circ \varepsilon_n$. Let $B_n = B(x, \varepsilon_n)$.

Claim 3.2. *There is a sequence $(\varepsilon_k)_{k>n} \in \prod_{k>n} \mathcal{U}_{X_k}$ of entourages such that for every $k > n$ for the set $B_k = B(B_{k-1}, \varepsilon_k)$ we have the inclusion $B(B_k, \varepsilon_k) \subset U_k$.*

Proof. For $k = n$ the inclusion $B(B_n, \varepsilon_n) = B(x, 2\varepsilon_n) \subset U_n$ follows from the choice of ε_n. Assume that for some $k \geqslant n$ we have constructed an entourage $\varepsilon_k \in \mathcal{U}_{X_k}$ such that $B(B_k, \varepsilon_k) \subset U_k$.

Choose an entourage $\delta_{k+1} \in \mathcal{U}_{X_{k+1}}$ such that $X_k^2 \cap 3\delta_{k+1} \subset \varepsilon_k$. By the condition ($\Gamma_2$) there is a neighborhood $V_k \subset C_k$ of the distinguished point $*_{C_k}$ such that for every $c \in V_k$ and $x_k \in X_k$ we get $\gamma_k(x_k, c) \in B(x_k, \delta_{k+1})$. By the condition ($\Gamma_1$), there is an entourage $\varepsilon_{k+1} \in \mathcal{U}_{X_{k+1}}$ such that $B(X_k, 2\varepsilon_{k+1}) \subset \gamma_k(X_k \times V_k)$ and $\varepsilon_{k+1} \subset \delta_{k+1}$. Let us check that the entourage ε_{k+1} satisfies our requirements.

Consider the set $B_{k+1} = B(B_k, \varepsilon_{k+1})$ and its ε_{k+1}-neighborhood $B(B_{k+1}, \varepsilon_{k+1}) = B(B_k, 2\varepsilon_{k+1})$. We need to show that $B(B_k, 2\varepsilon_{k+1}) \subset U_{k+1}$. Take any point $y \in B(B_k, 2\varepsilon_{k+1}) \subset B(X_k, 2\varepsilon_{k+1}) \subset \gamma_k(X_k \times V_k)$ and find a pair $(x_k, c) \in X_k \times V_k$ such that $y = \gamma_k(x_k, c)$. The choice of the neighborhood V_k guarantees that $(y, x_k) \in \delta_{k+1}$. Then $x_k \in X_k \cap B(y, \delta_{k+1}) \subset X_k \cap (B_k, \delta_{k+1} \circ 2\varepsilon_{k+1}) \subset X_k \cap B(B_k, 3\delta_{k+1}) = B(B_k, X_k^2 \cap 3\delta_{k+1}) \subset B(B_k, \varepsilon_k) \subset U_k$ and hence $y = \gamma_k(x_k, c) \in \gamma_k(U_k \times W_k) = U_{k+1}$. □

Theorem 1.1 of [6] guarantees that the union $B_\infty = \bigcup_{k\geqslant n} B_k$ is a neighborhood of the point x in $\text{u-}\varinjlim X_n$. Since $B_\infty = \bigcup_{k\geqslant n} B_k \subset \bigcup_{k\geqslant n} U_k = U$, we see that the point x lies in the interior of U and hence the set U is open in $\text{u-}\varinjlim X_n$. Claim 3.2 and the description of the topology of uniform direct limits given in [6, 1.1] guarantee that the topology of the uniform direct limit on $\text{u-}\varinjlim U_n$ coincides with the subspace topology inherited from $\text{u-}\varinjlim X_n$. This completes the proof of the lemma. □

If each subset X_n is C_n-complemented in X_{n+1}, then $W_n = C_n$ and $\gamma_n(X_n \times C_n) = X_{n+1}$ for all $n \in \omega$. By induction we can prove that $U_n = X_n$ for all $n \in \omega$ and hence the uniform direct limit $\text{u-}\varinjlim X_n = \text{u-}\varinjlim U_n$ is homeomorphic to $X_0 \times \boxdot_{n\in\omega} C_n$. This argument yields the following version of Lemma 3.1.

Lemma 3.3. *Let $(X_n)_{n\in\omega}$ be a tower of uniform spaces such that for every $n \in \omega$ the space X_n is C_n-complemented in X_{n+1} for some pointed topological space C_n. Then the uniform direct limit $X = \text{u-}\varinjlim X_n$ is homeomorphic to the small box-product $X_0 \times \boxdot_{n\in\omega} C_n$.*

Our final lemma completes the proof of Theorem 1.3.

Lemma 3.4. *Let $(X_n)_{n\in\omega}$ be a tower of uniform spaces such that for every $n \in \omega$ the space X_n is locally C_n-complemented in X_{n+1} for some pointed space C_n and is locally homeomorphic to $X_0 \times \prod_{i<n} C_i$. Then the space $\text{u-}\varinjlim X_n$ is locally homeomorphic to $X_0 \times \boxdot_{n\in\omega} C_n$.*

Proof. Given any point $x \in \text{u-}\varinjlim X_n$, find a number $n \in \omega$ with $x \in X_n$. By Lemma 3.1, the point x has an open neighborhood $O(x)$ that is homeomorphic to the small box-product $X_n \times \boxdot_{i\geqslant n} W_i$ for some open neighborhoods $W_i \subset C_i$ of the distinguished points $*_{C_i}$. Since the space X_n is locally homeomorphic to $X_0 \times \prod_{i<n} C_i$, we conclude that $X_n \times \boxdot_{i\geqslant n} W_i$ is locally homeomorphic to $X_0 \times \prod_{i<n} C_i \times \boxdot_{i\geqslant n} C_i = X_0 \times \boxdot_{i\in\omega} C_i$. Consequently, x has an open neighborhood, homeomorphic to an open subset of $X_0 \times \boxdot_{i\in\omega} C_i$ witnessing that X is locally homeomorphic to $X_0 \times \boxdot_{i\in\omega} C_i$. □

4. Reduced products

In this section we collect the necessary information on reduced products of pointed spaces. This information will be used in the proofs of Theorems 1.5 and 8.1.

For a pointed space C with distinguished point $*_C$ we denote by $C^\circ = C \setminus \{*_C\}$ its complement in C.

The *reduced product* $C \rtimes E$ of a pointed topological space C and a topological space E is the space

$$\bigl(C^\circ \times E\bigr) \cup \{*_C\}$$

endowed with the smallest topology such that the identity inclusion $C^\circ \times E \to C \rtimes E$ is an open topological embedding and the natural projection $\mathrm{pr} : C \rtimes E \to C$ is continuous. The reduced product $C \rtimes E$ is a pointed space with the distinguished point $*_{C\rtimes E} = *_C$.

If C and E are uniform spaces, then their reduced product $C \rtimes E$ carries the smallest uniformity such that the projection $C \rtimes E \to C$ is uniformly continuous and for every closed subspace $F \subset C^\circ$ of C the embedding $F \times E \to C \rtimes E$ is a uniform homeomorphism.

A map $f : X \to Y$ between topological spaces is called a *near homeomorphism* if for any open cover $\mathcal{U}$ of Y there is a homeomorphism $h : X \to Y$ that is $\mathcal{U}$-*near* to f.

Lemma 4.1. *Let M, N, E be metric spaces and C be a pointed metric space. If the projection $\mathrm{pr} : M \times C^\circ \times E \to M \times C^\circ$, $\mathrm{pr} : (x, y, z) \mapsto (x, y)$, is a near homeomorphism, then for any homeomorphism $f : M \to N$ there is a regular homeomorphism of pairs*

$$\bar f : \bigl(M \times (C \rtimes E), M \times \{*_{C\rtimes E}\}\bigr) \to \bigl(N \times C, N \times \{*_C\}\bigr)$$

such that $\bar f(x, *_{C\rtimes E}) = (f(x), *_C)$ for all $x \in M$.

Proof. Let d_N and d_C be the metrics of the metric spaces N and C, respectively. These metrics determine the metric

$$d\bigl((x, y), (x', y')\bigr) = \max\bigl\{d_N(x, x'), d_C(y, y')\bigr\}$$

on the product $N \times C$. Since the projection $\mathrm{pr} : M \times C^\circ \times E \to M \times C^\circ$ is a near homeomorphism and M is homeomorphic to N, there exists a homeomorphism $h : N \times C^\circ \times E \to N \times C^\circ$ such that

$$d\bigl(h(x, c, e), \mathrm{pr}(x, c, e)\bigr) \leqslant \frac{1}{3} d_C(c, *_C) \quad \text{for all } (x, c, e) \in N \times C^\circ \times E.$$

Extend h to a homeomorphism $\bar h : N \times (C \rtimes E) \to N \times C$ by letting $\bar h | N \times C^\circ = h$ and $\bar h | N \times \{*_C\} = \mathrm{id}$.

The homeomorphism $f : M \to N$ induces a homeomorphism $f \times \mathrm{id} : M \times (C \rtimes E) \to N \times (C \rtimes E)$, $f \times \mathrm{id} : (x, y) \mapsto (f(x), y)$.

Now consider the homeomorphism $\bar f = \bar h \circ (f \times \mathrm{id}) : M \times (C \rtimes E) \to N \times C$ and observe that for each $(x, c, e) \in M \times C^\circ \times E \subset M \times (C \rtimes E)$ we get

$$\frac{2}{3} d_C(c, *_C) \leqslant d\bigl(\bar h\bigl(f(x), c, e\bigr), N \times \{*_C\}\bigr) \leqslant d\bigl(\bar h\bigl(f(x), c, e\bigr), \bigl(f(x), *_C\bigr)\bigr) \leqslant \frac{4}{3} d_C(c, *_C),$$

which implies that

$$\bar h : \bigl(M \times (C \rtimes E), M \times \{*_{C\rtimes E}\}\bigr) \to \bigl(N \times C, N \times \{*_C\}\bigr)$$

is a regular homeomorphism of pairs. □

5. Typical model spaces

Theorem 1.5 actually holds in a more general setting, with *LF*-spaces replaced by small box-products of typical model spaces.

Definition 5.1. A pointed topological space E is called a *typical model space* if

(1) E is a topologically homogeneous absolute retract containing a topological copy of the Hilbert cube $Q = [0,1]^\omega$;
(2) for any neighborhood $U \subset E$ of $*_E$ there are neighborhoods $V, W \subset U$ of $*_E$ such that W and $E \setminus V$ are homeomorphic to E and the boundary ∂V of V is a retract of $\overline{V}$ and a Z-set in $E \setminus V$;
(3) each contractible E-manifold is homeomorphic to E;
(4) each connected E-manifold M is homeomorphic to an open subset of E;
(5) any homeomorphism $h : A \to B$ between Z-sets $A, B \subset E$ extends to a homeomorphism $\bar{h} : E \to E$ of E;
(6) for any E-manifold M the projection $E \times M \to M$ is a near homeomorphism;
(7) for any retract X of an open subset of E the product $X \times E$ is homeomorphic to an open subset of E;
(8) for any retract X of an E-manifold and a strong Z-point $*_X \in X$ the reduced product $X \rtimes E$ is an E-manifold, homeomorphic to $X \times E$.

By an *ANR-space* we understand a metrizable space X, which is a neighborhood retract in each metric space that contains X as a closed subspace.

Theory of Hilbert manifolds developed in [9, §IX.7], [16,22–24] yields the following theorem.

Theorem 5.2. *Any infinite-dimensional Hilbert space is a typical model space.*

Remark 5.3. Many incomplete typical model spaces can be found among absorbing and coabsorbing spaces, see [5] and [10].

We finish this short section by a lemma that will be used in the proof of Theorem 8.1.

Lemma 5.4. *Let X be a pointed ANR-space and Y be a pointed topological space. If $*_X$ is a strong Z-point in X, then $(*_X, *_Y)$ is a strong Z-point in $X \times Y$.*

Proof. Given an open cover $\mathcal{U}$ of $X \times Y$, find a set $U \in \mathcal{U}$ that contains the point $(*_X, *_Y)$. Choose open sets $U_X \subset X$ and $U_Y \subset Y$ such that $(*_X, *_Y) \in U_X \times U_Y \subset U$ and find a neighborhood $V_X \subset X$ of $*_X$ such that $\overline{V}_X \subset U_X$. Since the space Y is completely regular, there is a continuous function $\lambda_Y : Y \to [0,1]$ such that $\lambda_Y^{-1}(0) \supset Y \setminus U_Y$ and $\lambda_Y^{-1}(1)$ is a neighborhood of $*_Y$. By the same reason, there is a continuous function $\lambda_X : X \to [0,1]$ such that $\lambda_X^{-1}(0) \supset X \setminus V_X$ and $\lambda_X^{-1}(1)$ is a neighborhood of $*_X$. Let Λ_X be the interior of $\lambda_X^{-1}(1)$ in X and W_X be a neighborhood of $*_X$ in X such that $\overline{W}_X \subset \Lambda_X$. Consider the open cover $\mathcal{V} = \{\Lambda_X, U_X \setminus \overline{W}_X, X \setminus \overline{V}_X\}$ of X. Since X is an ANR, there is an open cover $\mathcal{W}$ of X such that any two $\mathcal{W}$-near maps into X are $\mathcal{V}$-homotopic. Since $*_X$ is a strong Z-point, there is a map $f : X \to X$ such that f is $\mathcal{W}$-near to id_X and $*_X \notin \overline{f(X)}$. By the choice of the cover $\mathcal{W}$, the map f is $\mathcal{V}$-homotopic to id_X. Consequently, there is a homotopy $h : X \times [0,1] \to X$ such that for every $x \in X$ we get $h(x,0) = x$, $h(x,1) = f(x)$ and $h(\{x\} \times [0,1]) \subset V_x$ for some $V_x \in \mathcal{V}$.

Now consider the function $\lambda : X \times Y \to [0,1]$, $\lambda : (x,y) \mapsto \lambda_X(x) \cdot \lambda_Y(y)$, and the map

$$g : X \times Y \to X \times Y, \quad g : (x,y) \mapsto \big(h\big(x, \lambda(x,y)\big), y\big).$$

Claim 5.5. *The map g is $\mathcal{U}$-near to* $\mathrm{id}_{X \times Y}$.

Proof. Take any pair $(x,y) \in X \times Y$. If $(x,y) \notin \overline{V}_X \times U_Y$, then

$$g(x,y) = \big(h\big(x, \lambda(x,y)\big), y\big) = \big(h(x,0), y\big) = (x,y)$$

and hence the singleton $\{g(x,y), (x,y)\}$ lies in some element of the cover $\mathcal{U}$. Next, assume that $(x,y) \in \overline{V}_X \times U_Y$. Since h is a $\mathcal{V}$-homotopy, and $h(x,0) = x \notin X \setminus \overline{V}_X$, $h(\{x\} \times [0,1]) \subset U_X$. Then

$$g(x,y) = \big(h\big(x, \lambda(x,y)\big), y\big) \in U_X \times U_Y \subset U \in \mathcal{U}$$

and $(x,y) \in \overline{V}_X \times U_Y \subset U_X \times U_Y \in U \in \mathcal{U}$. $\square$

Consider the neighborhood $W'_X = W_X \cap (X \setminus \overline{f(X)})$ of $*_X$ and the neighborhood $W = W'_X \times \lambda_Y^{-1}(1)$ of $(*_X, *_Y)$ in $X \times Y$.

Claim 5.6. *The neighborhood W does not intersect $g(X \times Y)$.*

Proof. Fix any point $(x, y) \in X \times Y$. If $y \notin \lambda^{-1}(1)$, then $g(x, y) \in X \times \{y\} \subset X \times (Y \setminus \lambda^{-1}(1)) \subset (X \times Y) \setminus W$.

So, we assume that $y \in \lambda^{-1}(1)$. If $x \notin \overline{V}_X$, then $g(x, y) = (x, y) \notin W$. If $x = h(x, 0) \in \overline{V}_X \setminus \Lambda_X$, then $h(\{x\} \times [0, 1]) \subset U_X \setminus \overline{W_X}$ as h is a $\mathcal{V}$-homotopy. In this case $g(x, y) \in (X \setminus \overline{W}_X) \times Y \subset X \times Y \setminus W$.

If $x \in \Lambda_X$, then $\lambda(x, y) = \lambda_X(x) \cdot \lambda_Y(y) = 1$ and $g(x, y) = (f(x), y) \notin W$. $\square$

Thus the map g witnesses that $(*_X, *_Y)$ is a strong Z-point in $X \times Y$. $\square$

6. Complemented subsets in typical model spaces

In this section we prove two useful lemmas about complemented subsets in metrizable uniform spaces that are locally homeomorphic to typical model spaces.

Lemma 6.1. *Let C be a pointed topological space and A be a C-complemented subset of a metrizable uniform space X. If X is an E-manifold for some typical model space E, then A is $C \rtimes E$-complemented in X.*

Proof. Since A is C-complemented in X, there is a homeomorphism $\gamma : A \times C \to X$ satisfying the conditions (1) and (2) of Definition 1.2. By our hypothesis, the uniform space X is metrizable and hence its uniformity is generated by some bounded metric ρ_X. Since X is homeomorphic to the product $A \times C$, the space C is metrizable, so we can choose a metric $\rho_C \leqslant 1$ that generates the topology of C. Denote by $\mathcal{U}_C$ the uniformity on C generated by the metric ρ_C.

By induction construct a sequence of entourages $(U_n)_{n\in\omega} \in \mathcal{U}_X^\omega$ and $(V_n)_{n\in\omega} \in \mathcal{U}_C^\omega$ such that $U_0 = X \times X$, $V_0 = C \times C$ and for every $n \in \mathbb{N}$ the following conditions are satisfied:

(1) $B(A, U_n) \subset \gamma(A \times B(*_C, V_{n-1}))$;
(2) $\gamma(a, c) \in B(a, U_n)$ for each $a \in A$ and $c \in B(*_C, V_n)$;
(3) $U_n \circ U_n \circ U_n \subset U_{n-1} = U_{n-1}^{-1} \subset \{(x, x') \in X^2\colon\ \rho_X(x, x') \leqslant 2^{-n+1}\}$;
(4) $V_n \circ V_n \circ V_n \subset V_{n-1} = V_{n-1}^{-1} \subset \{(c, c') \in C^2\colon\ \rho_C(c, c') \leqslant 2^{-n+1}\}$.

By Theorem [15, 8.1.10], there are pseudometrics d_X and d_C on X and C, respectively, such that for every $n \in \omega$

(5) $\{(x, x') \in X^2\colon\ d_X(x, x') < 2^{-n}\} \subset U_n \subset \{(x, x') \in X^2\colon\ d_X(x, x') \leqslant 2^{-n}\}$ and
(6) $\{(c, c') \in C^2\colon\ d_C(c, c') < 2^{-n}\} \subset V_n \subset \{(c, c') \in C^2\colon\ d_C(c, c') \leqslant 2^{-n}\}$.

The conditions (3), (5) and (4), (6) imply that d_X and d_C are metrics generating the uniformities of the corresponding spaces.

For $\varepsilon > 0$ let $B_X(A, \varepsilon) = \{x \in X\colon\ d_X(x, A) < \varepsilon\}$ and $B_C(*_C, \varepsilon) = \{c \in C\colon\ d_C(c, *_C) < \varepsilon\}$. The conditions (1), (2) and (5), (6) imply that for every $\varepsilon > 0$ the following two conditions are satisfied:

(7) $B(A, \varepsilon) \subset \gamma(A \times B(*_C, 4\varepsilon))$;
(8) $\gamma(a, c) \in B(a, 4\varepsilon)$ for each $a \in A$ and $c \in B(*_C, \varepsilon)$.

The metrics d_X and d_C generate the metric

$$d_{AC}\big((a, c), (a', c')\big) = \max\big\{d_X(a, a'), d_C(c, c')\big\}$$

on the product $A \times C$.

Let $C^\circ = C \setminus \{*_C\}$. The space $A \times C^\circ$ is an E-manifold (being homeomorphic to the open subset $\gamma(A \times C^\circ)$ of the E-manifold X). Consequently, the projection $p : A \times C^\circ \times E \to A \times C^\circ$ is a near homeomorphism and we can choose a homeomorphism $h : A \times C^\circ \times E \to A \times C^\circ$ such that

$$d_{AC}\big(h(a, c, e), p(a, c, e)\big) \leqslant \frac{1}{3} d_C(c, *_C) \quad \text{and} \quad d_X\big(\gamma \circ h(a, c, e), \gamma \circ p(a, c, e)\big) \leqslant \frac{1}{2} d_C(c, *_C)$$

for all $(a, c, e) \in A \times C^\circ \times E$.

Extend h to a homeomorphism $\bar{h} : A \times C \rtimes E \to A \times C$ letting $\bar{h}|A \times C^\circ \times E = h$ and $\bar{h}|A \times \{*_C\} = \mathrm{id}$.

It can be shown that the homeomorphism $\tilde{\gamma} = \gamma \circ \bar{h} : A \times C \rtimes E \to X$ witnesses that A is $C \rtimes E$-complemented in X. $\square$

Our second lemma reduces the local z-complementability in uniform spaces homeomorphic to typical model spaces to the E-complementability.

Lemma 6.2. *Let A be a retract of a metrizable uniform space X such that X is homeomorphic to some typical model space E. If A is locally z-complemented in X, then A is E-complemented in X.*

Proof. Assuming that A is locally z-complemented in X, find a z-pointed space C, an open neighborhood $V \subset C$ of $*_C$ and an open uniform neighborhood $U \subset X$ of X such that A is V-complemented in U. It follows that $A \times V$ is homeomorphic to U and hence $A \times V$ and V are ANRs. The ANR-property of V can be used to show that the distinguished point $*_C$ is a strong Z-point not only in C but also in V. So, we lose no generality assuming that $V = C$. By Lemma 6.1, the set A is $C \rtimes E$-complemented in U. By the condition (8) of Definition 5.1, the reduced product $C \rtimes E$ is an E-manifold. Consequently, the distinguished point $*_{C \rtimes E}$ has a neighborhood in $C \rtimes E$, homeomorphic an open neighborhood $U \subset E$ of the distinguished point $*_E$ of E. It follows that the set A is locally U-complemented in X. Hence, we can find an open embedding $\gamma : A \times U \to X$ satisfying the condition (1), (2) of Definition 1.2. By the condition (2) of Definition 5.1 the distinguished point $*_E$ of U has a neighborhood $V \subset E$ such that $\overline{V} \subset U$, the complement $E \setminus V$ is homeomorphic to E and the boundary $\partial V = \overline{V} \setminus V$ of V is a retract of $\overline{V}$ and a Z-set in $E \setminus V$.

Since A is a retract of the space X and the spaces X and $E \setminus V$ are homeomorphic to E, the product $A \times (E \setminus V)$ is homeomorphic to E, being a contractible E-manifold. Since ∂V is a Z-set in $E \setminus V$, the product $A \times \partial V$ is a Z-set in $A \times (E \setminus V)$.

We claim that the complement $M = X \setminus \gamma(A \times V)$ is homeomorphic to E and $\gamma(A \times \partial V)$ is a Z-set in M. First we show that M is contractible. Since ∂V is a retract of $\overline{V}$, $A \times \partial V$ is a retract of $A \times \overline{V}$ and hence $\gamma(A \times \partial V)$ is a retract of $\gamma(A \times \overline{V})$. Then M is a retract of $X = M \cup \gamma(A \times \overline{V})$. Since X is contractible, so is its retract M. To see that M is an E-manifold, observe that M is the union of two open subsets $X \setminus \gamma(A \times \overline{V})$ and $\gamma(A \times (U \setminus V))$, the first of which is open in the E-manifold X while the second is a topological copy of the E-manifold $A \times (U \setminus V) \subset A \times (E \setminus V)$. Being a contractible E-manifold, the space M is homeomorphic to E.

Since ∂V is a Z-set in $E \setminus V$, it is a Z-set in $U \setminus V$. Then $A \times \partial V$ is a Z-set in $A \times (U \setminus V)$ and $\gamma(A \times \partial V)$ is a Z-set in the open subset $\gamma(A \times (U \setminus V))$ of M and hence a Z-set in M.

Then $\gamma|A \times \partial V$ is a homeomorphism between the Z-sets $A \times \partial V$ and $\gamma(A \times \partial V)$ is the spaces $A \times (E \setminus V)$ and M which are homeomorphic to E. By the condition (5) of Definition 5.1, there is a homeomorphism $h : A \times (E \setminus V) \to M$ such that $h(x) = \gamma(x)$ for all $x \in A \times \partial V$. Extend the homeomorphism h to a homeomorphism $\tilde{\gamma} : A \times E \to X$ letting $\tilde{\gamma}|A \times \overline{V} = \gamma|A \times \overline{V}$ and $\tilde{\gamma}|A \times (E \setminus V) = h$. The homeomorphism $\tilde{\gamma}$ witnesses that the set A is E-complemented in X. □

7. Small box-products of locally compact spaces

In this section we study the topological structure of small box-products of pointed locally compact ANR-spaces. By $Q = [0,1]^\omega$ we denote the Hilbert cube. A pointed space X is called *non-isolated* if its distinguished point $*_X$ is not isolated in X. By a *polyhedron* we understand a topological space, homeomorphic to the geometric realization of some simplicial complex.

The following theorem is the main result of this section.

Theorem 7.1. *For any sequence $(X_n)_{n\in\mathbb{N}}$ of non-isolated pointed locally compact ANR-spaces the small box-product $Q \times \boxdot_{n\in\mathbb{N}} X_n$ is homeomorphic to the product $K \times Q \times \mathbb{R}^\infty$ for some locally compact polyhedron K. If each space X_n, $n \in \mathbb{N}$, is contractible, then the small box-product $Q \times \boxdot_{n\in\mathbb{N}} X_n$ is homeomorphic to $Q \times \mathbb{R}^\infty$.*

Proof. Put $X_0 = Q$ and instead of the product $Q \times \boxdot_{n\in\mathbb{N}} X_n$ consider the small box-product $\boxdot_{n\in\omega} X_n$. In order to prove that $\boxdot_{n\in\omega} X_n$ is a $Q \times \mathbb{R}^\infty$-manifold, we shall apply Sakai's characterization [21] of open subspaces of $Q \times \mathbb{R}^\infty$, mentioned in the Introduction.

We can assume that each space X_n carries a uniformity that generates its topology. Since each ANR-space is locally connected, it suffices to prove the theorem in the case of connected locally compact ANR-spaces X_n. In this case each space X_n is σ-compact and so is each finite product $\boxdot_{i<n} X_i$. Then $(\boxdot_{i\leqslant n} X_i)$ is a tower of locally compact σ-compact uniform spaces. By Propositions 5.4 and 5.5 of [6] the identity maps

$$\text{t-}\varinjlim \boxdot_{i\leqslant n} X_i \to \text{u-}\varinjlim \boxdot_{i\leqslant n} X_i \to \boxdot_{n\in\omega} X_n$$

are homeomorphisms.

Taking into account that each finite product $\boxdot_{i\leqslant n} X_i$ is locally compact and σ-compact, we can show that the topological direct limit $\text{t-}\varinjlim \boxdot_{i\leqslant n} X_i$ is a k_ω-space, which means that it can be written as a topological direct limit of a tower of compact metrizable spaces. Now the Sakai's characterization [21] will imply that $\boxdot_{n\in\omega} X_i$ is homeomorphic to an open subset of the space $Q \times \mathbb{R}^\infty$ as soon as we show that each embedding $f : B \to \boxdot_{n\in\omega} X_i$ defined on a closed subset B of a compact metrizable space A extends to an embedding $\bar{f} : O(B) \to \boxdot_{n\in\omega} X_i$ of some neighborhood $O(B)$ of B in A.

Since $f(B)$ is a compact subset of the topological direct limit $\text{t-}\varinjlim \boxdot_{i\leqslant n} X_i$, there is $n \in \mathbb{N}$ such that $f(B) \subset \boxdot_{i<n} X_i$. Since $\boxdot_{i<n} X_i$ is an ANR, the map $f : B \to \boxdot_{i<n} X_i$ admits a continuous extension $\bar{f} : O(B) \to \boxdot_{i<n} X_i$ to some closed neighborhood $O(B)$ of B in A.

By the ANR-Theorem for Q-manifolds [13, 44.1], the product $\boxdot_{i<n} X_i = Q \times \boxdot_{1\leqslant i<n} X_i$ is a Q-manifold and so is the product $[0,1] \times \boxdot_{i<n} X_i$. Identify $\boxdot_{i<n} X_i$ with the Z-set $\{0\} \times \boxdot_{i<n} X_i$ in $[0,1] \times \boxdot_{i\leqslant n} X_i$. By Theorem 18.2 of [13], the map $\bar{f} : O(B) \to \boxdot_{i<n} X_i$ can be approximated by an embedding $\tilde{f} : O(B) \to [0,1] \times \boxdot_{i<n} X_i$ such that $\tilde{f}|B = f$.

Since X_n is a non-isolated pointed space, there is an embedding $\gamma : [0,1] \to X_n$ such that $\gamma(0) = *_X$. The embedding γ induces the embedding

$$\tilde{\gamma} : [0,1] \times \mathop{\boxdot}_{i<n} X_i \to \mathop{\boxdot}_{i\leqslant n} X_i, \quad \tilde{\gamma} : (t, \vec{x}) \mapsto \big(\vec{x}, \gamma(t)\big).$$

Then $g = \tilde{\gamma} \circ \tilde{f} : O(B) \to \boxdot_{i\leqslant n} X_i \subset \boxdot_{i\in\omega} X_i$ is a required embedding that extends the embedding f. By Sakai's characterization of $Q \times \mathbb{R}^\infty$-manifolds [21], $\boxdot_{i\in\omega} X_i$ is a $Q \times \mathbb{R}^\infty$-manifold and by the Triangulation Theorem [21] for $Q \times \mathbb{R}^\infty$-manifolds, the $Q \times \mathbb{R}^\infty$-manifold $\boxdot_{i\in\omega} X_i$ is homeomorphic to $K \times Q \times \mathbb{R}^\infty$ for some locally compact polyhedron K.

If each space X_n is contractible, the product $\boxdot_{i<n} X_i$ is an absolute retract. In this case, we can assume that $O(B) = A$ and then the embedding $f : B \to \boxdot_{i\in\omega} X_i$ extends to an embedding $\bar{f} : A \to \boxdot_{i\in\omega} X_i$. By Sakai's characterization [21] of the space $Q \times \mathbb{R}^\infty$, the k_ω-space $\boxdot_{i\in\omega} X_i$ is homeomorphic to $Q \times \mathbb{R}^\infty$. □

8. The topological structure of some small box-products

In this section we prove a “typical” version of Theorem 1.5.

Theorem 8.1. *Let $(X_n)_{n\in\omega}$ be a sequence of pointed topological spaces such that for every $n \in \omega$ the finite product $\prod_{i\leqslant n} X_n$ is homeomorphic to (an open subspace of) some typical model space E_n. Assume that for infinitely many numbers $n \in \omega$ the space X_n is lz-pointed. Then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to (an open subset of) the small box-product $\boxdot_{n\in\omega} E_n$.*

Proof. Let

$$L = \{n \in \mathbb{N}:\ X_n \text{ is } l\text{-pointed}\} \quad \text{and} \quad Z = \{n \in \mathbb{N}:\ X_n \text{ is } z\text{-pointed}\}.$$

Assume that for every $n \in \omega$ the finite product $\prod_{i\leqslant n} X_n$ is homeomorphic to an open subspace of some typical model space E_n. Then each space X_n is metrizable, so its topology is generated by some metrizable uniformity $\mathcal{U}_{X_n}$.

Claim 8.2. *For every $n \in \mathbb{N}$ the product $E_{n-1} \times E_n$ is homeomorphic to E_n.*

Proof. By our assumption, the product $\prod_{i\leqslant n} X_i$ is locally homeomorphic to the model space E_n. Consequently, there are non-empty open sets $U \subset \prod_{i<n} X_i$ and $V \subset X_n$ whose product $U \times V$ is homeomorphic to an open subset of E_n. Since $\prod_{i<n} X_i$ is locally homeomorphic to E_{n-1}, the open set U contains a non-empty open set W that is homeomorphic to an open subset of the model space E_{n-1}. Since each non-empty open set of E_{n-1} contains an open subset homeomorphic to E_{n-1} we lose no generality assuming that the set W is homeomorphic to E_{n-1}. Then $W \times V$ is homeomorphic to an open subset of E_n and hence E_{n-1} is homeomorphic to the retract W of the E_n-manifold $W \times V$. By the conditions (3) and (7) of Definition 5.1, the product $E_{n-1} \times E_n$, being a contractible E_n-manifold, is homeomorphic to E_n. □

Claim 8.3. *The small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to $\boxdot_{n\in\omega} X_n \times \boxdot_{n\in Z} E_n$.*

Proof. Let $Y_0 = X_0$ and for every $n \in \mathbb{N}$ let

$$Y_n = \begin{cases} X_n & \text{if } n \notin Z, \\ X_n \rtimes E_n & \text{if } n \in Z. \end{cases}$$

By Proposition 5.5 of [6], the small box-products $\boxdot_{n\in\omega} X_n$ and $\boxdot_{n\in\omega} Y_n$ can be identified with the uniform direct limits of the towers $(\boxdot_{i<n} X_i)_{n\in\omega}$ and $(\boxdot_{i<n} Y_i)_{n\in\omega}$, respectively.

For every $n \in Z$ let $X_n^\circ = X_n \setminus \{*_{X_n}\}$. By our assumption, the finite product $\boxdot_{i\leqslant n} X_i$ is homeomorphic to an open subset of the typical model space E_n. Then the space $X_n^\circ \times \boxdot_{i<n} X_i$, being an open subset of $\boxdot_{i\leqslant n} X_i$ also is homeomorphic to an open subset of E_n. Since E_n is a typical model space, the projection

$$\mathrm{pr} : E_n \times X_n^\circ \times \mathop{\boxdot}_{i<n} X_i \to X_n^\circ \times \mathop{\boxdot}_{i<n} X_i, \quad \mathrm{pr} : (e, x, \vec{x}) \mapsto (x, \vec{x}),$$

is a near homeomorphism.

Let $h_0 : X_0 \to Y_0$ be the identity homeomorphism. Using Lemma 4.1, by induction we can construct a sequence of regular homeomorphisms of pairs

$$h_n : \Big(\mathop{\boxdot}_{i\leqslant n} X_i, \mathop{\boxdot}_{i<n} X_i\Big) \to \Big(\mathop{\boxdot}_{i\leqslant n} Y_i, \mathop{\boxdot}_{i<n} Y_i\Big)$$

such that $h_n | \boxdot_{i<n} X_i = h_{n-1}$. By Corollary 2.2, the map $h : \boxdot_{n\in\omega} X_n \to \boxdot_{n\in\mathbb{N}} Y_n$ defined by $h | \boxdot_{i\leqslant n} X_i = h_n$ is a homeomorphism.

By the condition (8) of Definition 5.1, for every $n \in Z$ the reduced product $X_n \rtimes E_n$ is homeomorphic to $X_n \times E_n$. Consequently, we get the following chain of homeomorphisms:

$$\begin{aligned}\boxdot_{n\in\omega} X_n &\cong \boxdot_{n\in\omega} Y_n = \boxdot_{n\in\omega\setminus Z} Y_n \times \boxdot_{n\in Z} Y_n = \boxdot_{n\in\omega\setminus Z} X_n \times \boxdot_{n\in Z}(X_n \rtimes E_n)\\ &\cong \boxdot_{n\in\omega\setminus Z} X_n \times \boxdot_{n\in\omega}(X_n \times E_n) = \boxdot_{n\in\omega} X_n \times \boxdot_{n\in Z} E_n. \qquad \square\end{aligned}$$

Claim 8.4. *If the set Z is infinite, then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to an open subspace of the small box-product $\boxdot_{n\in\omega} E_n$.*

Proof. Let $Z = \{n_k\colon k \in \omega\}$ be the increasing enumeration of the infinite set Z. It will be convenient to assume that $n_{-1} = -1$. For every $k \in \omega$ let $Y_k = \prod_{n_{k-1}<i\leqslant n_k} X_i$. By Claim 8.3, the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to $\boxdot_{k\in\omega}(Y_k \times E_{n_k})$. Since the finite product $\prod_{i\leqslant n_k} X_i$ is homeomorphic to an open subset of E_{n_k}, the space Y_k is a retract of an open subset of the typical model space E_{n_k} and hence the product $Y_k \times E_{n_k}$ is homeomorphic to an open subset U_{n_k} of E_{n_k}. If the space Y_k is contractible, then U_{n_k}, being a contractible E_{n_k}-manifold, is homeomorphic to E_{n_k}. In this case we can assume that $U_{n_k} = E_{n_k}$. Claim 8.2 implies that the product $\prod_{n_{k-1}<i\leqslant n_k} E_i$ is homeomorphic to E_{n_k} and hence the open set U_{n_k} is homeomorphic to some open set W_{n_k} in $\prod_{n_{k-1}<i\leqslant n_k} E_i$ (which coincides with $\prod_{n_{k-1}<i\leqslant n_k} E_i$ if $U_{n_k} = E_{n_k}$). The space $Y_k \times E_{n_k}$, being homeomorphic to an open subset U_{n_k} of E_{n_k}, is homeomorphic to the open subset W_{n_k} of the product $\prod_{n_{k-1}<i\leqslant n_k} E_i$.

Now we see that the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to the small box-product $\boxdot_{k\in\omega}(Y_k \times E_{n_k})$ and the latter small box-product is homeomorphic to the small box-product $W = \boxdot_{k\in\omega} W_{n_k}$, which is an open subset of the small box-product $\boxdot_{n\in\omega} E_n$. This finishes the proof of Claim 8.4. $\square$

If each finite product $\prod_{i\leqslant n} X_n$ is homeomorphic to E_n, then each space X_i, $i \in \omega$, is contractible and so are the spaces Y_k, $k \in \omega$. In this case $U_k = E_{n_k}$ and $W = \boxdot_{n\in\omega} E_n$. Therefore we have proved the following modification of Claim 8.4.

Claim 8.5. *If the set Z is infinite, and each finite product $\prod_{i\leqslant n} X_i$ is homeomorphic to the model space E_n, then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to $\boxdot_{n\in\omega} E_n$.*

Claims 8.4 and 8.5 prove Theorem 8.1 in case of infinite set Z. If the set L is infinite, then Theorem 8.1 follows from Claims 8.7 and 8.8 proved below.

Claim 8.6. *If the set L is infinite and each space X_n, $n \in \omega$, is connected, then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to an open subspace of $\boxdot_{n\in\omega} E_n$.*

Proof. Let $A \subset L$ be an infinite subset such that for each $n \in A$ we get $0 < n-1 \notin A$. This condition implies that the complement $\mathbb{N} \setminus A$ is infinite.

By Theorem 7.1, the small box-product $Q \times \boxdot_{n\in A} X_n$ is homeomorphic to $K \times Q \times \mathbb{R}^\infty$ for some connected locally compact polyhedron K. If each space X_n is contractible, then we can assume that K is a singleton. Theorem 7.1 also implies that the space $Q \times \mathbb{R}^\infty$ is homeomorphic to the small box-product $Q \times \boxdot_{n\in\omega} \mathbb{I}$ where $\mathbb{I} = [0,1]$ is the closed interval with the distinguished point 0.

First we show that for every $n \in \omega$ the product $K \times E_n$ is homeomorphic to an open subset of E_n. Indeed, by the condition (1), (3) and (7) of Definition 5.1, the Hilbert cube Q is a retract of the typical model space E_n and the product $E_n \times Q \times [0,1]$ is homeomorphic to E_n. The locally compact polyhedron K is connected and hence admits a closed embedding into $Q \times [0,1)$. Then K is a neighborhood retract of the space $E_n \times Q \times [0,1)$, which is homeomorphic to an open subspace of E_n. By the condition (7) of Definition 5.1, $E_n \times K$ is homeomorphic to an open subset of E_n. By Definition 5.1(7), for every open subset $U \subset E_n$ the product $U \times E_n$ is homeomorphic to U and hence $U \times K$ is homeomorphic to $U \times E_n \times K$ and the latter space is homeomorphic to an open subset of the square $E_n \times E_n$, which is homeomorphic to E_n (being a contractible E_n-manifold).

Since X_0 is homeomorphic to an open subset of E_0, the product $X_0 \times K$ is homeomorphic to an open subset of the model space E_0. Since X_0 is homeomorphic to $X_0 \times E_0$ and E_0 is homeomorphic to $E_0 \times Q$, the space X_0 is homeomorphic to $X_0 \times Q$. So, we get the following chain of homeomorphisms

$$\begin{aligned}\boxdot_{n\in\omega} X_n &\cong X_0 \times Q \times \boxdot_{n\in\mathbb{N}} X_n \cong X_0 \times \boxdot_{n\in\mathbb{N}\setminus A} X_n \times \Big(Q \times \boxdot_{n\in A} X_n\Big)\\ &\cong X_0 \times \boxdot_{n\in\mathbb{N}\setminus A} X_n \times \Big(K \times Q \times \boxdot_{n\in\omega} \mathbb{I}\Big) \cong X_0 \times K \times Q \times \boxdot_{n\in\mathbb{N}\setminus A}(X_n \times \mathbb{I})\\ &\cong X_0 \times K \times \boxdot_{n\in\mathbb{N}\setminus A}(X_n \times \mathbb{I}).\end{aligned}$$

By Lemma 5.4, for every $n \in \mathbb{N} \setminus A$ the distinguished point $(*_{X_n}, 0)$ of the pointed space $X_n \times \mathbb{I}$ is a strong Z-point.

Then by Claim 8.4, the small box-product $\boxdot_{n\in\omega} X_n \cong K \times X_0 \times \boxdot_{n\in\mathbb{N}\setminus A}(X_n \times \mathbb{I})$ is homeomorphic to an open subset of $\boxdot_{n\in\omega\setminus A} E_n$. By Claim 8.2, for every $n \in \mathbb{N} \setminus A$ the space E_n is homeomorphic to $E_n \times E_{n-1}$. Consequently, the small box-product $\boxdot_{n\in\omega\setminus A} E_n$ is homeomorphic to $\boxdot_{n\in\omega} E_n$ and thus $\boxdot_{n\in\omega} X_n$ is homeomorphic to an open subspace of $\boxdot_{n\in\omega} E_n$. $\square$

By analogy we can prove:

Claim 8.7. *If the set L is infinite, and every finite product $\prod_{i\leqslant n} X_n$ is contractible, then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to $\boxdot_{n\in\omega} E_n$.*

Our final claim finishes the proof of Theorem 8.1.

Claim 8.8. *If the set L is infinite, then the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to an open subspace of the small box-product $\boxdot_{n\in\omega} E_n$.*

Proof. For every $n \in \omega$ denote by κ_n the number of connected components of the space X_n. It is clear that the finite product $\prod_{i\leqslant n} X_i$ has $\prod_{i\leqslant n} \kappa_i$ many connected components. Since $\prod_{i\leqslant n} X_i$ is homeomorphic to an open subset of E_n, the model space E_n contains a family $\mathcal{U}_n$ consisting of $\prod_{i\leqslant n} \kappa_i$ many pairwise disjoint non-empty open subsets. Since each non-empty open subset of E_n contains an open subset homeomorphic to E_n, we can assume that each set $U \in \mathcal{U}_n$ is homeomorphic to E_n. Since E_n is topologically homogeneous, we can assume that its distinguished point lies in the union $U_n = \bigcup \mathcal{U}_n$. It follows that $U = \boxdot_{n\in\omega} U_n$ is an open subset of $\boxdot_{n\in\omega} E_n$ and each connected component of U is homeomorphic to $\boxdot_{n\in\omega} E_n$. Observe that the spaces $\boxdot_{n\in\omega} X_n$ and $\boxdot_{n\in\omega} U_n$ consist of $\kappa = \sup_{n\in\omega} \prod_{i\leqslant n} \kappa_i$ many connected components. So, we can choose a bijective map γ assigning to each connected component of $\boxdot_{n\in\omega} X_n$ a connected component of the space $\boxdot_{n\in\omega} U_n$. By Claim 8.6, each connected component C of $\boxdot_{n\in\omega} X_n$ is homeomorphic to an open subset of $\boxdot_{n\in\omega} E_n$. So, we can define an open topological embedding $f_C : C \to \gamma(C)$ of C into the connected component $\gamma(C)$ of the small box-product $\boxdot_{n\in\omega} U_n$. Then the union $f = \bigcup_C f_C : \boxdot_{n\in\omega} X_n \to \boxdot_{n\in\omega} U_n \subset \boxdot_{n\in\omega} E_n$ is a required open embedding of $\boxdot_{n\in\omega} X_n$ into $\boxdot_{n\in\omega} E_n$. $\square$

9. Recognizing the topology of some uniform direct limits

In this section we prove a "typical" version of Theorem 1.6.

Theorem 9.1. *Let $(E_n)_{n\in\omega}$ be a sequence of typical model spaces. The uniform direct limit $\text{u-}\varinjlim X_n$ of a tower of uniform spaces $(X_n)_{n\in\omega}$ is*

(1) *homeomorphic to (an open subset of) $\boxdot_{n\in\omega} E_n$ if each space X_n is lz-complemented in X_{n+1} and X_n is homeomorphic to (an open subset of) the model space E_n;*
(2) *(locally) homeomorphic to $\boxdot_{n\in\omega} E_n$ if each space X_n (locally) lz-complemented in X_{n+1} and X_n is (locally) homeomorphic to E_n;*
(3) *homeomorphic to $\boxdot_{n\in\omega} E_n$ if each uniform space X_n is metrizable, homeomorphic to E_n and is locally z-complemented in X_{n+1}.*

Proof. (1) Assume that for every $n \in \omega$ the space X_n is *lz*-complemented in X_{n+1} and is homeomorphic to (an open subset of) the model space E_{n+1}. Then X_n is C_n-complemented in X_{n+1} for some *lz*-pointed space C_n. By Theorem 1.3, the uniform direct limit $\text{u-}\varinjlim X_n$ is homeomorphic to the small box-product $X_0 \times \boxdot_{n\in\omega} C_n$.

The C_n-complementedness of X_n in X_{n+1} implies that the space X_{n+1} is homeomorphic to the product $X_n \times C_n$. Continuing by induction we can prove that X_{n+1} is homeomorphic to $X_0 \times \prod_{i\leqslant n} C_n$. Then the latter product is homeomorphic to (an open subset of) E_{n+1} and we can apply Theorem 8.1 to prove that $X_0 \times \boxdot_{n\in\omega} X_n$ is homeomorphic to (an open subset of) the small box-product $\boxdot_{n\in\omega} E_n$.

(2) Now assume that for every $n \in \omega$ the space X_n is locally *lz*-complemented in X_{n+1} and is locally homeomorphic to E_n. We need to show that each point $x \in \text{u-}\varinjlim X_n$ has an open neighborhood homeomorphic to an open subset of $\boxdot_{n\in\omega} E_n$. Find a number $n \in \omega$ with $x \in X_n$. By Lemma 3.1, the point x has an open neighborhood which is homeomorphic to the small box-product $X_n \times \boxdot_{i\geqslant n} W_i$ for some open neighborhoods $W_i \subset C_i$ of the distinguished points $*_{C_i}$. It follows from the proof of Lemma 3.1 that for every $m \geqslant n$ the product $X_n \times \prod_{n\leqslant i\leqslant m} W_i$ is homeomorphic to an open subset of X_{n+1} and hence is locally homeomorphic to E_{n+1}. Then Theorem 8.1 guarantees that the small box-product $X_n \times \boxdot_{i\geqslant n} W_i$ is locally homeomorphic to $\boxdot_{i\geqslant n} E_i$. Consequently, the point x has a neighborhood $O(x)$ that is homeomorphic to an open subset of $\boxdot_{i\geqslant n} E_i$.

Repeating the argument of Claim 8.2, we can prove that for every $k \in \mathbb{N}$ the model space E_k is homeomorphic to $E_{k-1} \times E_k$. Then E_{n-1} is homeomorphic to $\prod_{i<n} E_i$ and $\boxdot_{i\geqslant n} E_i$ is homeomorphic to $\boxdot_{n\in\omega} E_n$. Now we see that $O(x_0)$ is homeomorphic to an open subset of $\boxdot_{n\in\omega} E_n$ and hence the uniform direct limit $\text{u-}\varinjlim X_n$ is locally homeomorphic to $\boxdot_{n\in\omega} E_n$.

(3) Assume that each uniform space X_n is metrizable, homeomorphic to E_n and is locally z-complemented in X_{n+1}. By Lemma 6.2, the set X_n is E_{n+1}-complemented in X_{n+1} and by the statement (1), $X = \text{u-}\varinjlim X_n$ is homeomorphic to the small box-product $\boxdot_{n\in\omega} E_n$. $\square$

10. Proof of Theorem 1.5

Let $(X_n)_{n\in\omega}$ be a sequence of pointed topological spaces such that for every $n\in\omega$ the finite product $\prod_{i\leqslant n} X_i$ is homeomorphic to (an open subset of) a Hilbert space E_n and for infinitely many numbers $n\in\omega$ the space X_n is *lz*-pointed.

We consider three cases.

I. For some $m\in\omega$ the Hilbert space E_m is infinite-dimensional. Then for all $n\geqslant m$ the Hilbert spaces E_n are infinite-dimensional. In this case we can apply Theorem 8.1 to conclude that the small box-product $\boxdot_{n>m} X_n$ is homeomorphic to (an open subset of) $\boxdot_{n>m} E_n$. Since the product $\prod_{i\leqslant m} X_i$ is homeomorphic to (an open subset of) the Hilbert space E_m, the small box-product $\boxdot_{n\in\omega} X_n$ is homeomorphic to (an open subset of) the small box-product $\boxdot_{n\geqslant m} E_n$ of Hilbert spaces. The latter box-product can be identified with the *LF*-space $\bigoplus_{n\geqslant m} E_n$.

II. All Hilbert spaces E_n are finite-dimensional and $\sup_{n\in\omega}\dim(E_n)=\infty$. In this case for every $n\in\omega$ the finite product $\prod_{i\leqslant n} X_n$, being homeomorphic to an open subset of the finite-dimensional Hilbert space E_n, is a locally compact σ-compact finite-dimensional ANR. Then the topological direct limit $\text{t-}\varinjlim \boxdot_{i\leqslant m} X_i$ of the tower $(\boxdot_{i\leqslant n} X_i)_{n\in\omega}$ can be written as the topological direct limit of a tower of finite-dimensional metrizable compacta. By Propositions 5.4 and 5.5 of [6], the identity map

$$\text{t-}\varinjlim \mathop{\boxdot}_{i\leqslant n} X_n \to \mathop{\boxdot}_{n\in\omega} X_n$$

is a homeomorphism. Now by Theorem 1.1, the topological equivalence of $\boxdot_{n\in\omega} X_n$ to (an open subset of) the *LF*-space $\mathbb{R}^\infty$ will follow as soon as we prove that each embedding $f: B\to\boxdot_{n\in\omega} X_n$ of a closed subset B of a finite-dimensional metrizable compact space A can be extended to an embedding $\bar f$ of (some neighborhood of B in) the space A.

Since $f(B)$ is a compact subset of the topological direct limit $\boxdot_{i\in\omega} X_i=\text{t-}\varinjlim\boxdot_{i\leqslant n} X_i$, there is $n\in\omega$ such that $f(B)\subset \boxdot_{i\leqslant n} X_i$. Since $\boxdot_{i\leqslant n} X_i$ is an ANR-space, the map f admits a continuous extension $\bar f: O(B)\to\boxdot_{i\leqslant n} X_i$ defined on a closed neighborhood $O(B)$ in A. Since $\boxdot_{i\leqslant n} X_i$ is homeomorphic to the Hilbert space E_n, then it is an absolute retract and we can additionally assume that $O(B)=A$.

Now consider the quotient space $O(B)/B$ and the corresponding quotient map $\pi: O(B)\to O(B)/B$. Being metrizable and finite-dimensional, the compact space $O(B)/B$ admits an embedding $e: O(B)/B\to\mathbb{I}^k$ for some k such that the distinguished point B of $O(B)/B$ maps onto the distinguished point $(0,\dots,0)$ of the cube $\mathbb{I}^k$. Then the map

$$\tilde f: O(B)\to\mathbb{I}^k\times\prod_{i\leqslant n} X_i,\quad \tilde f: x\mapsto \big(e\circ\pi(x),\bar f(x)\big),$$

is a topological embedding.

Consider the set M of all numbers m for which the point $*_{X_m}$ is not isolated in the ANR-space X_n. It follows from $\sup_{m\in\omega}\dim(E_m)=\infty$ that the set N is infinite. For every $m\in M$ we can find an embedding $\gamma_m:\mathbb{I}\to X_m$ such that $\gamma_m(0)=*_{X_m}$. Since the set M is infinite, we can choose a sequence of numbers $m_1<m_2<\dots<m_k$ in M such that $m_1>n$. Consider the embedding $\gamma:\mathbb{I}^k\to\prod_{i=1}^k X_{m_k}$ defined by $\gamma:(t_1,\dots,t_k)\mapsto(\gamma_{m_1}(t_1),\dots,\gamma_{m_k}(t_k))$.

Identify the product $\prod_{i=1}^k X_{m_i}$ with the subset

$$\left\{(x_i)_{i=n+1}^{m_k}\in\prod_{i=n+1}^{m_k} X_i\colon\ i\notin\{m_1,\dots,m_k\}\Rightarrow x_i=*_{X_i}\right\}$$

of the product $\prod_{i=n+1}^{m_k} X_i$ and consider the embedding

$$\delta:\left(\prod_{i\leqslant n} X_i\right)\times\mathbb{I}^k\to\prod_{i\leqslant m_k} X_i=\prod_{i\leqslant n} X_i\times\prod_{i=n+1}^{m_k} X_i,\quad \delta:(x,t)\mapsto\big(x,\gamma(t)\big).$$

Then the composition $\delta\circ\tilde f: O(B)\to\prod_{i\leqslant m_k} X_i\subset\boxdot_{i\in\omega} X_i$ is a required embedding of $O(B)$ that extends the embedding f. Now it is legal to apply Theorem 1.1 and conclude that the small box-product $\boxdot_{i\in\omega} X_i$ is homeomorphic to (an open subspace of) the *LF*-space $\mathbb{R}^\infty$.

III. $k=\sup_{n\in\omega}\dim(E_n)$ is finite. Then there is $m\in\omega$ such that $\dim(E_n)=k$ for all $n\geqslant m$. For every $n<m$ the finite products $\boxdot_{i<n} X_i$ and $\boxdot_{i\leqslant n} X_i$ are homeomorphic to open subsets of the Euclidean space $\mathbb{R}^k$. By the Brouwer Domain Preservation Principle [12], the space $\boxdot_{i<n} X_i$ is open in $\boxdot_{i\leqslant n} X_i$. This implies that the space X_n is discrete and at most countable. Then the small box-product $\boxdot_{n>m} X_n$ is discrete and at most countable. Since the product $\boxdot_{i\leqslant m} X_i$ is homeomorphic to an open subset of $\mathbb{R}^k$ and $\mathbb{R}^k$ contains an open subspace homeomorphic to $\mathbb{R}^k\times\omega$, the small box-product $\boxdot_{i\in\omega} X_i=\boxdot_{i\leqslant m} X_i\times\boxdot_{i>m} X_i$ is homeomorphic to an open subset of E_m.

If each finite product $\prod_{i\leqslant n} X_i$ is homeomorphic to the Hilbert space E_n, then for every $n>m$ the space X_n is a singleton. Consequently, the small box-product $\boxdot_{n\in\omega} X_i=\boxdot_{i\leqslant m} X_i$ is homeomorphic to the finite-dimensional *LF*-space E_m.

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